# Mathematical Aspects of the Power Control Problem in Mobile Communication Systems *

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## 1 Introduction

Wireless networking is evolving as a major component of modern communication infrastructure. Today, mobile phones are becoming a mainstream voice communication medium in our daily lives. It can be envisaged that the wide deployment of wireless network in the next century will revolutionize the concept of communication. As the proliferation of the demand for wireless services can be anticipated, it is of paramount importance to speed up the rate of technological advance in maximizing the system capacity.

To design a high-capacity wireless system, cochannel interference, the most restraining factor on the system capacity, must be properly managed. There are several ways to control the interference; for example, well-designed channel allocation plan, efficient power control, interference cancellation techniques, and orthgonal signalling (time, frequency, or code). There is a vast amount of literature on these topics. In this article, we focus on power control.

Traditionally, there are two kinds of power control. The first one is called open-loop power control. This kind of power control estimates the channel gain based on the pilot channel and adjusts the transmit power accordingly. This is not very accurate, but has the advantage that it can respond quickly in case of a sudden change in the channel, such as a a mobile user travels into a region shadowed by a building. The other kind, called close-loop power control, adjusts the transmit power based on feedback information. This is more accurate as the decision is based on the actual performance metric, for example, the received power, the signal-to-interference ratio (SIR) or the bit error rate (BER).

For instance, in IS-95 CDMA standard, both open-loop and close-loop power control are implemented. Basically, the algorithm strives to maintain the received power at a constant level [21]. However, it was shown in [40] that algorithms based on the received SIR outperforms those that based on the received power. It is expected that SIR-based algorithms will be employed in the next-generation wireless standards [1].

To evaluate the performance of power-controlled systems, simulation studies with realistic but complex models are usually involved. Unfortunately, the large variety of model assumptions renders comparisons difficult. To circumvent this difficulty, we present an analytical framework for the power control problem, which was based on the earlier work by Aein [2]. Associated are many interesting optimization issues such

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as power minimization and throughput maximization. Our aim is to provide a mathematical perspective on this problem by describing recent research developments, presenting new solution approaches and identifying key future research directions.

The organization of this article is as follows. In Section 2, we give a classification of the power control problem. In Section 3, we describe the system model and define the power control problem in its primitive form. In Section 4, an early approach called *power balancing*, in which the system aims at optimizing the worst-case signal quality, is presented. In Section 5, we describe a new paradigm called *QoS tracking*, in which powers are allocated in a way such that each user maintains a connection with acceptable link quality. In Section 6, a more realistic model, which restricts the power to discrete levels, are considered. In Section 7, the power control problem is extended to integrate the base station assignment problem. In Section 8, we present a framework which unifies results found for different variations of the power control problem. It is further generalized to an asynchronous model in Section 9. In Section 10, we turn our attention to time varying channels. Preliminary results for stochastic link gains are presented. In Section 11, we consider the power control problem for data networks. In Section 12, a game theoretic framework is presented. A well studied class of noncooperative game called *supermodular game* is described in Section 13. Finally, we give our conclusion in Section 14.

## 2 Classification of Power Control Problem

In this section, we outline the different aspects of the power control problem. We follow the classification in [36].

In a wireless multimedia system, the applications can broadly be divided into two classes. One class requires real-time delivery of the message. A representative is the traditional mobile phone services. Another class requires high reliability but can tolerate a larger delay. A typical example is the transmission of computer data. These two classes of services have very different objective functions. In the literature, the power control problem is formulated for the second-generation digital cellular systems. The target service is mobile phone. For computer data, there is no widely accepted formulation of the problem. Therefore, from Section 3 to 10, we restrict ourselves to the power control problem for mobile phone services. Afterwards, we will present a new paradigm on power control for data traffic.

Power control refers to the adjustment of transmit power levels to compensate for the fading effect of both the mobile and the interferers. The aim is to achieve an acceptable *quality of service* (QoS) for a mobile user without causing unnecessary interference to other users. QoS can be measured in terms of the signal-to-interference ratio (SIR), the bit error rate (BER) or other quantities. Usually, for voice traffic, the QoS is specified in terms of SIR. For example, the analog AMPS system requires an SIR of 18 dB for acceptable reception. We denote the minimum acceptable SIR by $\gamma_0$, which is determined by the required error performance and the coding/modulation method.

The power control problem can be separated into two sub-problems. In the first sub-problem, one concentrates on the scenario of a single user who already achieves a SIR that is close to the target. However, due to the time varying nature of a fading channel, the mobile needs to make fine adjustments from time to time to track the target. This is called the *single user model*. In this model, the interference from other users is assumed to be constant. The second sub-problem concerns with how to coordinate the power levels among a group of users. In this *multiuser model*, the interference effect among the users is examined. Based on this model, there are two paradigms established in the literature. One is called *power balancing* and the other is *QoS tracking*. Each of them uses a different objective function in its mathematical formulation. Netherthe-less, both paradigms assume the channel is *static*. There are few research results on multiuser model with

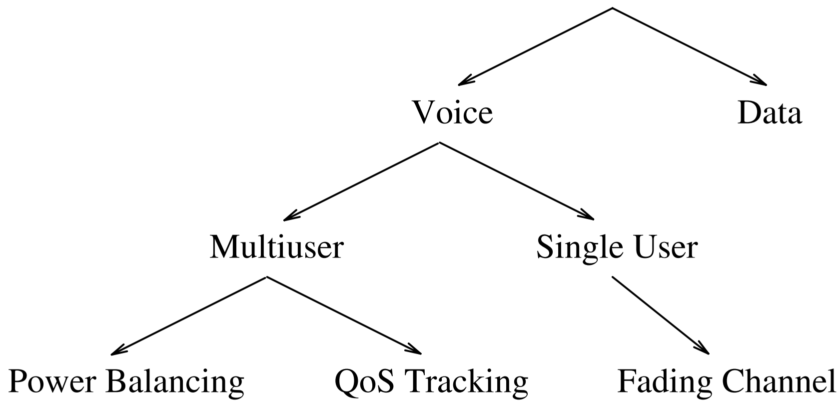


Figure 1: Classification of power control problem

fading channel. A first attempt which considers the special case of a $2 \times 2$ stochastic link gain matrix will be presented in Section 10.

Figure 1 shows the classification of the power control problem. A detailed description of the multiuser model will be presented in the next section.

# 3 Multiuser Power Control Problem

The multiuser model can be further divided into single cell or multicell. Single cell models occur only in CDMA systems. Multicell models occur in FDMA, TDMA, as well as CDMA systems.

In this section, we consider a multicell model for a FDMA/TDMA system. This model embraces the single cell CDMA model as a special case and it can be extended to a multicell CDMA model.

In a cellular FDMA/TDMA system, a pair of channels is assigned to each communication link. One is for the mobile-to-base (uplink) direction and the other is for the base-to-mobile (downlink) direction. If the frequencies for the uplink and the downlink channel are different, the technique is called *frequency division duplex* (FDD). If the same frequency is used but the two channels utilize different time slots for transmission, it is called *time division duplex* (TDD). No matter which technique is used, there is no interference between the uplink and the downlink channel. Hereafter, we consider the power control issue for the uplink channel. However, the results are equally applicable to the downlink.

Now consider the uplink scenario. A channel is assigned to each active mobile terminal. Due to the scarcity of the radio bandwidth, the same channel may be shared by different terminals. As a result, interference arises. Our aim is to choose a suitable transmit power of each mobile terminal such that the effect of cochannel interference can be reduced.

We assume that there is no interference between different channels. We focus on a particular channel. Assume that tere are $M$ mobile terminals currently using it. Note that in FDMA/TDMA system, there is at most one terminal using this particular channel in each cell. We denote the *link gain* on the path between the terminal in cell $i$ and the base station in cell $j$ at some given moment by $G_{ij}$ (see Figure 2). Note that $G_{ij}(t)$, in general, is a stochastic process. However, assuming that the power control iterations are much faster than the change of the environment, we consider a snapshot of the system. Thus, $G_{ij}$ is treated as a random variable. Its magnitude reflects the effect of path loss and shadow fading.

In wireless communication, the link quality is usually measured by the SIR. Under our model, the SIR of

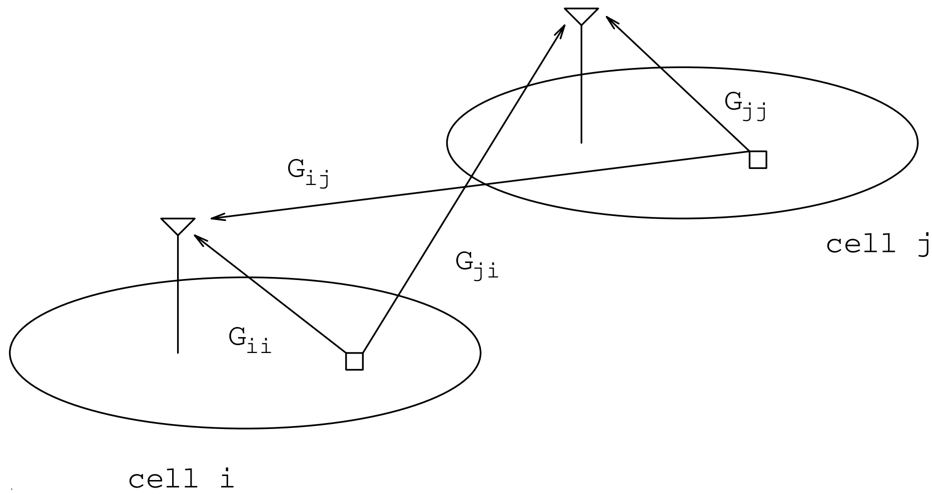


Figure 2: The link gain model.

mobile $i$, $\Gamma_i$, can be written as

$$\Gamma_i = \frac{G_{ii}P_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i} \tag{1}$$

where $\eta_i$ is the receiver noise at base station $i$. We let $\mathbf{n}$ denote the noise vector $[\eta_1, \eta_2, \cdots, \eta_M]^\top$.

For voice applications, the received SIR is usually required to be greater than a certain threshold, $\gamma_0$. Thus, the power control problem is to find a non-negative power vector $\mathbf{P} = (P_1, P_2, \cdots, P_M)$ such that

$$\Gamma_i \geq \gamma_0 \tag{2}$$

for all $i$.

In most circumstances, the solution to a given power control problem, if exists, is not unique. To find a favorable one among the solution set, either of the following criteria may be used.

1. Maximizing the worst-case signal quality.
2. Minimizing the power consumption.

In the literature, two paradigms based on these two different criteria have been established. They are termed *power balancing* and *QoS tracking* respectively.

# 4 Power Balancing

In mobile cellular systems, the interference power is typically much greater than the receiver noise. If we neglect the noise term $\eta_i$, the SIR at base station $i$ becomes

$$\Gamma_i = \frac{P_i}{\sum_{j\neq i} Z_{ij}P_j} \tag{3}$$

where $\mathbf{Z} = [Z_{ij}] = [\frac{G_{ij}}{G_{ii}}]$ is the normalized link gain matrix.

Without losing much generality, we assume that $\mathbf{Z}$ is an *irreducible* matrix, which is defined as follows [25].

**Definition 1** *A square non-negative matrix* $\mathbf{T}$ *is irreducible if for every pair* $i, j$ *of its index set, there exists a positive integer* $m \equiv m(i,j)$ *such that* $t_{ij}^{(m)} > 0$.

For example, consider the following three matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.2 & 0.2 & 1 & 0.5 \\ 0.2 & 0.2 & 0.5 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0.2 & 0 & 0 \\ 0 & 1 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0.2 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

If matrix $\mathbf{A}$ is regarded as a normalized link gain matrix, we can see that the first two mobiles form a class and the remaining two form another class. There is no mutual interference between these two classes. In matrix theory, we said that $\mathbf{A}$ is reducible. Similarly, concerning matrix $\mathbf{B}$, we can divide the mobiles into the same two classes. While the second class generates interference to the first one, there is no interference from the first class to the second. Still $\mathbf{B}$ is a reducible matrix. Regarding matrix $\mathbf{C}$, the interference effect arisen from any mobile will eventually propagate to all other mobiles. Thus, $\mathbf{C}$ is an irreducible matrix.

Before proceeding, we introduce one more concept called *primitive matrix* [25].

**Definition 2** *A square non-negative matrix $\mathbf{T}$ is primitive if there exists a positive integer $k$ such that $\mathbf{T}^k > \mathbf{0}$.*

It should be noted that the class of primitive matrices is a subclass of irreducible matrices. If we assume that $\mathbf{Z}$ is irreducible, by the fact that all its diagonal elements are ones, we can show that $\mathbf{Z}$ is also primitive.

To maximize the worst-case signal quality, one would like to find a power vector, $\mathbf{P}^*$, which maximizes the minimum SIR in the system:

$$\mathbf{P}^* = \arg \max_{\mathbf{P}>\mathbf{0}} \min_i \Gamma_i \tag{5}$$

Let $\Gamma_i^*$ be the resulting SIR of mobile $i$ at the optimal solution. If $\Gamma_i^* \geq \gamma_0$ for all $i$, we call $\mathbf{P}^*$ a *feasible* solution.

Since $\mathbf{Z}$ is irreducible, it is easy to see that the optimal solution is obtained when all the SIRs are equal, for otherwise, if we keep on decreasing the power of mobiles which achieve a higher SIR than the minimum, the minimum SIR will eventually increase. Thus, we have

$$\Gamma_i^* = \Gamma_j^* \tag{6}$$

for all $i$ and $j$.

As the resulting SIR of each user is *balanced* at the optimal solution, this formulation is often referred to as the power balancing problem. Denote the balanced SIR by $\gamma^*$. Then we have

$$\frac{P_i^*}{\sum_{j\neq i} Z_{ij} P_j^*} = \gamma^* \tag{7}$$

$$P_i^* = \gamma^* \left( \sum_{j=1}^{M} Z_{ij} P_j^* - P_i^* \right) \tag{8}$$

$$(1 + 1/\gamma^*) P_i^* = \sum_{j=1}^{M} Z_{ij} P_j^* \tag{9}$$

This equation can also be written in matrix form as follows.

$$(1 + 1/\gamma^*)\mathbf{P} = \mathbf{ZP} \tag{10}$$

Therefore, the optimal power vector, $\mathbf{P}^*$, is an eigenvector of $\mathbf{Z}$. By Perron-Frobenius Theorem for irreducible matrices, $\mathbf{Z}$ has a positive real eigenvalue, $\lambda_Z$, which can be associated with strictly positive eigenvectors. We call $\lambda_Z$ *Perron-Frobenius* eigenvalue. Furthermore, $\lambda_Z \geq |\lambda|$ for any eigenvalue $\lambda \neq \lambda_Z$

[25]. Since $\mathbf{Z}$ is also primitive, this statement can be replaced by a stronger one: $\lambda_Z > |\lambda|$ for any eigenvalue $\lambda \neq \lambda_Z$ [25].

We have shown that the power balancing problem is solved by finding the Perron-Frobenius eigenvalue of $\mathbf{Z}$, $\lambda_Z$, and its corresponding eigenvector. From equation (10), $\gamma^*$ is related to $\lambda_Z$ by

$$\gamma^* = \frac{1}{\lambda_Z - 1} \tag{11}$$

If $\gamma^* \geq \gamma_0$, the solution is feasible. Thus, we have

**Theorem 1** *A feasible solution* $\mathbf{P}^*$ *exists if and only if the Perron-Frobenius eigenvalue of* $\mathbf{Z}$, $\lambda_Z$, *satisfies*

$$\lambda_Z < 1 + \frac{1}{\gamma_0} \tag{12}$$

If a feasible solution does not exist, the eigenvector $\mathbf{P}^*$ will cause the SIR of all the connections below the required threshold. This is of course unacceptable and some of the users should be dropped from the system. To minimize the outage probability, we would like to drop as few users as possible. Mathematically, we need to find the largest submatrix of $\mathbf{Z}$ for which $\gamma_0$ is achievable. This is called the *user removal problem* [40]. It was shown in [3] that this problem is NP-complete. Some heuristic algorithms were proposed in [4, 17, 40].

## 4.1 Distributed Algorithms Based on Power Method

Assume that feasible solutions to a given power control problem exist. The solution described above requires the matrix $\mathbf{Z}$ to be completely known. However, measuring all the path gains in real time is a formidable task in large cellular systems. For practical implementation, we would like to have a power control scheme which requires far less measurements and allows each mobile user to compute his own power level. The following distributed algorithm was proposed by Zander [41].

**Zander's Algorthm**

$$\mathbf{P}^{(0)} = \mathbf{P}_0, \quad \mathbf{P}_0 > \mathbf{0} \tag{13}$$

$$P_i^{(n+1)} = \beta^{(n)} P_i^{(n)} \left(1 + \frac{1}{\Gamma_i^{(n)}}\right), \quad \beta^{(n)} > 0 \tag{14}$$

In matrix form, this can be expressed as

$$\mathbf{P}^{(n+1)} = \beta^{(n)} \mathbf{Z} \mathbf{P}^{(n)} \tag{15}$$

This algorithm is essentially the *power method* for finding the dominant eigenvalue and its corresponding eigenvector of a matrix [6]. A necessary condition for the convergence is that the dominant eigenvalue must be strictly greater than all other eigenvalues. This condition is satisfied in our case since $\mathbf{Z}$ is primitive. Thus we have

$$\lim_{n\to\infty} \mathbf{P}^{(n)} = \mathbf{P}^* \tag{16}$$

$$\lim_{n\to\infty} \Gamma_i^{(n)} = \gamma^* \ \forall i \tag{17}$$

The sequence $\beta^{(n)}$ has no effect on the resulting SIR. However, if it is not chosen properly, the power levels may become too large or too small. A possible choice for $\beta^{(n)}$ is $M/(\sum P_i^{(n)})$. However, this inevitably requires some sorts of communication among the base stations.

Zander's algorithm was later improved by Grandhi *et. al.* [11]. Equation (10) is rewritten as

$$(1/\gamma^*)\mathbf{P} = \mathbf{AP} \tag{18}$$

where $\mathbf{A} = \mathbf{Z} - \mathbf{I}$.

Since $\mathbf{Z}$ is irreducible, it can be shown that $\mathbf{A}$ is also irreducible. However, $\mathbf{A}$ is not necessarily primitive. Let $\lambda_A$ be the Perron-Frobenius eigenvalue of $\mathbf{A}$. We have

$$\lambda_A = 1/\gamma^* \tag{19}$$

Theorem 1 can be restated as follows.

**Theorem 2** *A feasible solution* $\mathbf{P}^*$ *exists if and only if the Perron-Frobenius eigenvalue of* $\mathbf{A}$*,* $\lambda_A$*, satisfies*

$$\lambda_A < \frac{1}{\gamma_0} \tag{20}$$

The optimal power vector can be obtained by finding the eigenvector corresponding to $\lambda_A$. Again the power method can be used.

**Grandhi et. al. 's Algorthm**

$$\mathbf{P}^{(n+1)} = \beta^{(n)}\mathbf{AP}^{(n)} \tag{21}$$

Similar to Zander's algorithm, this algorithm also requires global information for a proper setting of $\beta^{(n)}$. It was shown numerically that this algorithm, in average, converges faster than Zander's. However, a subtle point worth noting is that $\mathbf{A}$ is irreducible, but not necessarily primitive. Thus it is possible to construct an example in which the algorithm does not converge. For example, consider the following problem.

$$\mathbf{Z} = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0.5 \\ 0.5 & 0 & 1 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \quad \mathbf{P}^{(0)} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{22}$$

In this example, Zander's algorithm converges, but Grandhi *et. al.* 's does not. The reason is that the eigenvalues of $\mathbf{A}$, in this example, are 0.5 and $-0.25 \pm 0.433i$, which have the same magnitudes. Thus the power method fails to converge.

Zander's and Grandhi *et. al.* 's algorithms can be considered special cases of a class of distributed algorithms given as follows [17].

$$\mathbf{P}^{(n+1)} = \beta^{(n)}(\mathbf{Z} - \alpha\mathbf{I})\mathbf{P}^{(n)} \quad (0 \leq \alpha \leq 1) \tag{23}$$

Notice that $\mathbf{Z} - \alpha\mathbf{I}$ is an irreducible nonnegative matrix. It is clear that $\alpha = 0$ and $\alpha = 1$ correspond, respectively, to Zander's and Grandhi *et. al.* 's algorithms. If we exclude the case $\alpha = 1$, then $\mathbf{Z} - \alpha\mathbf{I}$ is guaranteed to be a primitive matrix. The degenerate case as shown in the above example will not occur, and the convergence is guaranteed for any irreducible matrix $\mathbf{Z}$.

## 4.2 Cooperative Algorithm

As we discussed before, the class of distributed algorithms which bases on the power method needs a normalization factor to scale the power vector to a desired range. In addition, the computation of such a factor requires global user information, thus weakening the distributed property of these algorithms. To remedy this problem, the idea of allowing limited information flow among base stations was proposed. Algorithms which based on this idea is collectively termed *Cooperative Algorithm*.

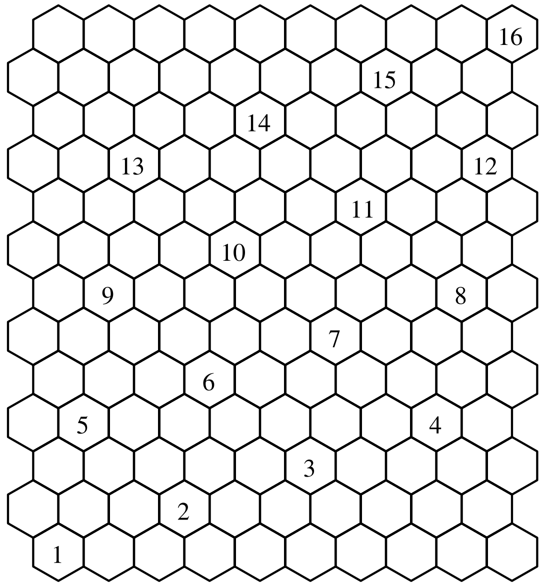


Figure 3: Layout of sixteen interfering cells.

In practice, the base stations in a cellular network are wired via a backbone network. Consequently, control data can be sent from one base station to another. Obviously, there is a cost associated with this kind of information flow. To minimize this cost, the transmission of control data from a base station should be restricted to its *network neighbors* as much as possible. By network neighbors we refer to those base stations between which the data communication cost is small. This is determined by the topology of the wired backbone network.

To define the way how the power control data are passed among the base stations, we use the *control data flow structure*. This structure is a directed graph where each node represents a base station. If there is a directed arc from node $A$ to node $B$, then control data are passed from base station $A$ to $B$. In addition, we assume that the control data flow structure satisfies the following:

*Reachability Condition:* Given any pair of nodes $(A, B)$, there is a chain of directed arcs starting from $A$ and terminating at $B$.

For example, we consider sixteen cochannel cells as shown in Figure 3. There are many different ways to define a control data flow structure. Two possible ways are given in Figure 4. Note that the reachability condition is satisfied in both cases.

According to the control data flow structure, we define $\mathcal{N}_i$ as the set of indices of base stations that send control data to base station $i$. One form of the Cooperative Algorithm is given as follows [16].

$$\begin{cases} P_i^{(0)} & = & M_i \\ P_i^{(n+1)} & = & \alpha_i^{(n)} P_i^{(n)} \\ \alpha_i^{(n)} & = & \left[\min(\Gamma_i^{(n)}, \min_{j \in \mathcal{N}_i} \Gamma_j^{(n)})/\Gamma_i^{(n)}\right]^{\epsilon} \quad \text{where } 0 \le \epsilon \le 1 \end{cases} \tag{24}$$

Under this algorithm, there is no oscillation of power levels during the adjustment period. The power of mobile $i$ decreases monotonically from $M_i$. Furthermore, the SIR of each user is shown to converge to the balanced SIR, $\gamma^*$. The rate of convergence is governed by the control data flow structure. Roughly speaking, the sparser the directed edges in the control data flow structure, the lower the convergence rate. For example,

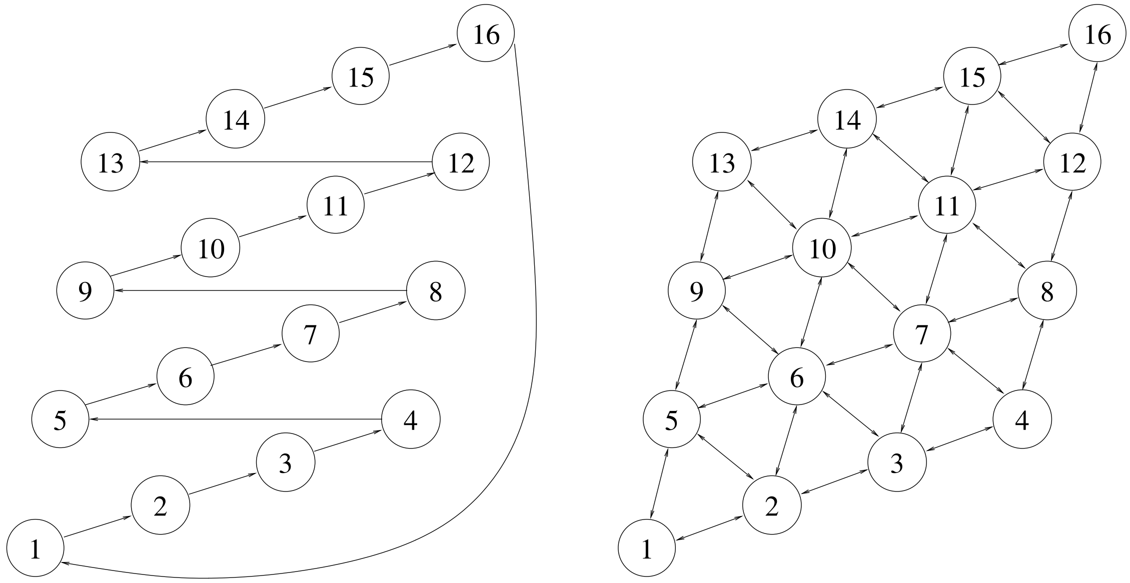


Figure 4: Two examples of control data flow structure.

for the sixteen cells shown in Figure 3, the algorithm converges faster if it uses the control data flow structure on the right of Figure 4, instead of using the one on the left.

Equation (24) gives one form of the Cooperative Algorithm. A dual version which adjusts the power levels monotonically upward is stated as follows [29].

$$\begin{cases} P_i^{(0)} & = & m_i \\ P_i^{(n+1)} & = & \alpha_i^{(n)} P_i^{(n)} \\ \alpha_i^{(n)} & = & \left[\max(\Gamma_i^{(n)}, \max_{j \in \mathcal{N}_i} \Gamma_j^{(n)})/\Gamma_i^{(n)}\right]^{\epsilon} \quad \text{where } 0 \leq \epsilon \leq 1 \end{cases} \tag{25}$$

Similarly, this algorithm is proved to converge to a solution at which power balancing is achieved.

## 4.3 Uplink-Downlink Equivalence

For voice applications in a cellular system, a duplex connection is required. Thus it is important to provide each user an acceptable link quality for both uplink and downlink simultaneously. In this subsection, we will show that the same SIR can be achieved in both links [42].

We assume that the uplink channel and the downlink channel have the same link gain. If $\mathbf{G}$ is the uplink gain matrix, the downlink gain matrix is given by $\tilde{\mathbf{G}} = \mathbf{G}^\top$. We write the normalized uplink gain matrix, $\mathbf{Z}$, in the following form.

$$\mathbf{Z} = \mathbf{D}\mathbf{G} \tag{26}$$

where $\mathbf{D}$ is a diagonal matrix with $D_{ii} = 1/G_{ii}$. Similarly, we define the normalized downlink gain matrix, $\mathbf{W}$, by

$$\mathbf{W} = \mathbf{D}\mathbf{G}^\top \tag{27}$$

The characteristic equation for $\mathbf{Z}$ is

$$|\mathbf{Z} - \lambda\mathbf{I}| = |\mathbf{D}||\mathbf{G} - \lambda\mathbf{D}^{-1}| = 0 \tag{28}$$

where we have used the standard result that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ for any diagonal matrix $\mathbf{A}$.

Using the fact that $\mathbf{D}^{-1}$ is a diagonal matrix as well as $|\mathbf{D}^\top| = |\mathbf{D}|$, we can rewrite equation (28) as

$$
\begin{aligned}
|\mathbf{Z} - \lambda \mathbf{I}| &= |\mathbf{D}||(\mathbf{G} - \lambda \mathbf{D}^{-1})^\top| && (29)\\
&= |\mathbf{D}||(\mathbf{G}^\top - \lambda \mathbf{D}^{-1})| && (30)\\
&= |\mathbf{D}\mathbf{G}^\top - \lambda \mathbf{I}| && (31)\\
&= |\mathbf{W} - \lambda \mathbf{I}| && (32)
\end{aligned}
$$

Hence, $\mathbf{Z}$ and $\mathbf{W}$ have the same characteristic equation, and thus, identical eigenvalues. Thus, we have proved

**Theorem 3** *The balanced SIRs in the uplink and downlink are identical.*

# 5 QoS tracking

In the power balancing approach, the receiver noise is excluded from the model. The resulting solution gives a relative magnitude on the power levels among the users. In Foschini and Miljanic's model [9], the receiver noise is included. Furthermore, instead of optimizing the worst-case signal quality, they aim at finding a feasible solution which minimizes the power consumption. Their approach can be generalized such that every user has a different QoS requirement.

$$\Gamma_i \geq \gamma_i \tag{33}$$

With this generalization and the inclusion of the receiver noise, the power control problem can be written in matrix form as follows.

$$[\mathbf{I} - \mathbf{B}]\mathbf{P} \geq \mathbf{u} \tag{34}$$

where $\mathbf{I}$ is the $M \times M$ identity matrix, $\mathbf{B}$ is an $M \times M$ non-negative matrix defined as

$$B_{ij} = \begin{cases} 0 & i = j \\ \gamma_i G_{ij}/G_{ii} & i \neq j \end{cases} \tag{35}$$

and $\mathbf{u}$ is the vector with elements

$$u_i = \gamma_i \eta_i / G_{ii} \tag{36}$$

We call $\mathbf{B}$ the *normalized interference matrix* and $\mathbf{u}$ the *normalized noise vector*. We denote the set of feasible power vectors by $\mathcal{P}$. If we assume that $\mathbf{Z}$ is irreducible, it can be shown that $\mathbf{B}$ is also irreducible. Let $\lambda_B$ be the Perron-Frobenius eigenvalue of $\mathbf{B}$. Using standard results from the theory of non-negative matrices [25], we have the following theorem.

**Theorem 4** *A nonnegative solution* $\mathbf{P}$ *to the equation*

$$(\mathbf{I} - \mathbf{B})\mathbf{P} = \mathbf{u}$$

*exists for any* $\mathbf{u} \geq \mathbf{0}, \neq \mathbf{0}$*, if and only if* $\lambda_B < 1$*. In this case, there is only one solution* $\mathbf{P}^*$*, which is strictly positive and given by*

$$\mathbf{P}^* = (\mathbf{I} - \mathbf{B})^{-1}\mathbf{u}$$

At this solution, the SIR of user $i$ is equal to $\gamma_i$, and it has the following property.

**Theorem 5** *The solution* $\mathbf{P}^*$ *is Pareto optimal in the sense that for any* $\mathbf{P} \in \mathcal{P}$*, we have*

$$\mathbf{P} \geq \mathbf{P}^*$$

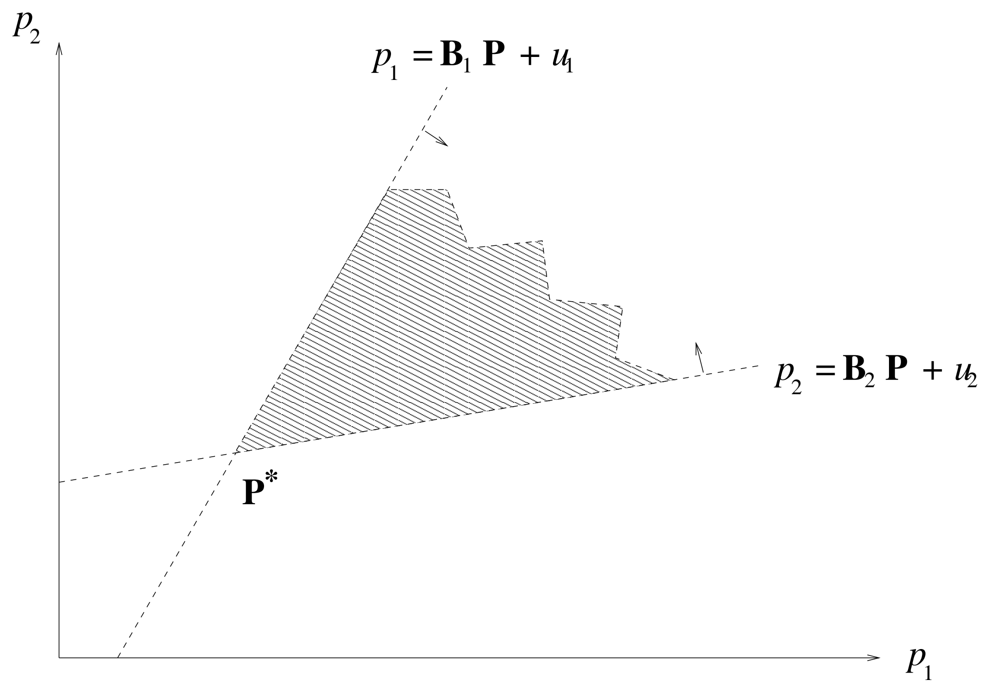


Figure 5: The feasible region of the QoS tracking problem for two users.

**Proof:**

Given $\mathbf{P} \in \mathcal{P}$, let $\hat{\mathbf{P}} = \mathbf{P}^* + \alpha(\mathbf{P} - \mathbf{P}^*)$. Since $\mathbf{P}^* = \mathbf{B}\mathbf{P}^* + \mathbf{u}$ and $\mathbf{P} \geq \mathbf{B}\mathbf{P} + \mathbf{u}$,

$$\hat{\mathbf{P}} - (\mathbf{B}\hat{\mathbf{P}} + \mathbf{u}) = \alpha(\mathbf{P} - \mathbf{B}\mathbf{P} - \mathbf{u}) \tag{37}$$

$$\geq 0 \tag{38}$$

Hence, $\hat{\mathbf{P}} \in \mathcal{P}$ for all nonnegative $\alpha$. Suppose that $P_i < P_i^*$ for some $i$. In this case, we can choose $\alpha$ such that for some $i$, $\hat{P}_i = 0$ and $\hat{P}_j \geq 0$ for all $j \neq i$. For this choice of $\alpha$,

$$0 = \hat{P}_i < \mathbf{B}_i \hat{\mathbf{P}} + u_i \tag{39}$$

where $\mathbf{B}_i$ is the $i$th row of $\mathbf{B}$. Thus, it contradicts with the fact $\hat{\mathbf{P}} \in \mathcal{P}$.

□

Geometrically, the set of feasible power vectors, $\mathcal{P}$, is a cone whose vertex is the solution $\mathbf{P}^*$. The feasible region for a system of two users is depicted in Figure 5. The normalized interference matrix $\mathbf{B}$ dictates the shape of the cone while the normalized noise vector $\mathbf{u}$ displaces the cone from the origin.

By Theorem 4, the feasibility of a QoS tracking problem is governed only by the matrix $\mathbf{B}$. By Perron-Frobenius Theorem, given any pair of irreducible matrices $\mathbf{B}$ and $\tilde{\mathbf{B}}$, if $\mathbf{B} \geq \tilde{\mathbf{B}}$, then $\lambda_B \geq \lambda_{\tilde{B}}$ [25]. Thus, we have the following proposition.

**Proposition 1** *Given two QoS tracking problem associated with matrix* $\mathbf{B}$ *and* $\tilde{\mathbf{B}}$, *where*

$$B_{ij} = \begin{cases} 0 & i = j \\ \gamma_i G_{ij}/G_{ii} & i \neq j \end{cases} \quad \text{and} \quad \tilde{B}_{ij} = \begin{cases} 0 & i = j \\ \tilde{\gamma}_i \tilde{G}_{ij}/\tilde{G}_{ii} & i \neq j, \end{cases}$$

*if* $G_{ii} \geq \tilde{G}_{ii}$, $G_{ij} \leq \tilde{G}_{ij} (j \neq i)$ *and* $\gamma_i \geq \tilde{\gamma}_i$ *for all* $i$ *and* $j$, *then the feasiblity of the problem associated with* $\mathbf{B}$ *implies the feasibility of the problem associated with* $\tilde{\mathbf{B}}$.

## 5.1 Distributed Algorithm

For the tracking problem, a distributed algorithm which converges to the solution $\mathbf{P}^*$ has been proposed by Foschini and Miljanic [9].

**Foschini-Miljanic's Algorithm**

$$P_i^{(n+1)} = \frac{\gamma_i}{\Gamma_i^{(n)}} P_i^{(n)} \quad \forall i \tag{40}$$

It can also be expressed as

$$P_i^{(n+1)} = \frac{\gamma_i}{G_{ii}} (\sum_{j \neq i} G_{ij} P_j^{(n)}) + u_i \tag{41}$$

In matrix form, it becomes

$$\mathbf{P}^{(n+1)} = \mathbf{B}\mathbf{P}^{(n)} + \mathbf{u} \tag{42}$$

The general solution of this difference equation is

$$\mathbf{P}^{(n)} = \mathbf{B}^n \mathbf{P}^{(0)} + \left( \sum_{i=1}^{n-1} \mathbf{B}^i \right) \mathbf{u} \tag{43}$$

A lemma in [25] states that

**Lemma 1** *If $\mathbf{A}$ is a square matrix such that $\mathbf{A}^k \to 0$ elementwise as $k \to \infty$, then $(\mathbf{I} - \mathbf{A})^{-1}$ exists and*

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k$$

*convergence being elementwise. ($\mathbf{A}^0 = I$ by definition.)*

Since a necessary and sufficient condition for $\mathbf{B}^i \to 0$ is $\lambda_B < 1$ [25], $\mathbf{P}^{(n)}$ converges elementwise to the solution

$$\mathbf{P} = (\mathbf{I} - \mathbf{B})^{-1} \mathbf{u} \tag{44}$$

provided a feasible solution exists.

## 5.2 Uplink-Downlink Equivalence

In Subsection 4.3, we have shown that the same balanced SIR is achieved by the uplink and the downlink. For QoS tracking, we assume that for each user, the QoS requirements in the uplink and the downlink are identical.

The feasibility of a QoS tracking problem for the uplink depends only on the normalized interference matrix, $\mathbf{B}$, which can be expressed as

$$\mathbf{B} = \mathbf{D}\mathbf{F} \tag{45}$$

where $\mathbf{D}$ is a diagonal matrix defined as $D_i = \gamma_i / G_{ii}$. $F_{ij}$ equals $G_{ij}$ for $i \neq j$ and equals 0 for $i = j$.

Let $\tilde{\mathbf{n}} = [\tilde{\eta}_1 \cdots \tilde{\eta}_M]^\top$ be the receiver noise vector in the downlink. Then the QoS tracking problem for the downlink becomes

$$[\mathbf{I} - \tilde{\mathbf{B}}]\mathbf{P} \geq \tilde{\mathbf{u}} \tag{46}$$

where $\tilde{\mathbf{B}} = \mathbf{D}\mathbf{F}^\top$ and $\tilde{u}_i = \gamma_i \tilde{\eta}_i / G_{ii}$.

By the same method used in Subsection 4.3, it can be shown that $\mathbf{B}$ and $\tilde{\mathbf{B}}$ have identical eigenvalues. By Theorem 4, we have

**Theorem 6** *Assuming that for each user, the QoS requirements in the uplink and the downlink are the same, the QoS tracking problem for the uplink is feasible if and only if the problem for the downlink is feasible.*

Although the feasibility of the uplink problem and the downlink problem are the same, there is no simple relation between the optimal power vectors for them. However, we have the following result [10].

**Theorem 7** *Assuming that the noise vectors in the uplink and the downlink are the same up to a constant, i.e.* $\mathbf{n} = c\tilde{\mathbf{n}}$*, the optimal uplink and downlink power vectors,* $\mathbf{P}^*$ *and* $\tilde{\mathbf{P}}^*$ *respectively, if exist, are related as*

$$\mathbf{n}^\top \mathbf{P}^* = c^2 \tilde{\mathbf{n}}^\top \tilde{\mathbf{P}}^*$$

**Proof:**

By Theorem 4, the optimal uplink and downlink power vectors can be expressed as follows.

$$\mathbf{P}^* = (\mathbf{I} - \mathbf{B})^{-1}\mathbf{D}\mathbf{n} \tag{47}$$
$$\tilde{\mathbf{P}}^* = (\mathbf{I} - \tilde{\mathbf{B}})^{-1}\mathbf{D}\tilde{\mathbf{n}} \tag{48}$$

By Lemma 1, we have

$$\begin{aligned}
(\mathbf{I} - \mathbf{B})^{-1}\mathbf{D} &= \left(\sum_{k=1}^{\infty} \mathbf{B}^k\right)\mathbf{D} && (49)\\
&= \left[\mathbf{D}\sum_{k=1}^{\infty}(\mathbf{B}^\top)^k\right]^\top && (50)\\
&= \left[\mathbf{D}\sum_{k=1}^{\infty}(\mathbf{F}^\top\mathbf{D})^k\right]^\top && (51)\\
&= \left[\sum_{k=1}^{\infty}(\mathbf{D}\mathbf{F}^\top)^k\mathbf{D}\right]^\top && (52)\\
&= \left[(\mathbf{I} - \tilde{\mathbf{B}})^{-1}\mathbf{D}\right]^\top && (53)
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{n}^\top\mathbf{P}^* &= \mathbf{n}^\top(\mathbf{I} - \mathbf{B})^{-1}\mathbf{D}\mathbf{n} && (54)\\
&= \mathbf{n}^\top\left[(\mathbf{I} - \tilde{\mathbf{B}})^{-1}\mathbf{D}\right]^\top\mathbf{n} && (55)\\
&= \left[\mathbf{n}^\top(\mathbf{I} - \tilde{\mathbf{B}})^{-1}\mathbf{D}\mathbf{n}\right]^\top && (56)\\
&= c^2\left[\tilde{\mathbf{n}}^\top\tilde{\mathbf{P}}^*\right]^\top && (57)\\
&= c^2\tilde{\mathbf{n}}^\top\tilde{\mathbf{P}}^* && (58)
\end{aligned}$$

□

Note that the noise power depends only on the bandwidth and the noise figure of a receiver. Thus, it is reasonable to assume that $\eta_i$ are the same for all base stations and $\tilde{\eta}_i$ are the same for all mobiles. In this case, by putting $\mathbf{n} = [1\ 1\ \cdots\ 1]^\top$, we have the following result.

**Corollary 1** *If the noise power is the same for all base stations and the noise power is the same for all mobiles, then the minimal sum of powers in the uplink is always the same as that in the downlink up to a constant.*

This result is useful for the integrated power control and base station assignment problem to be discussed later.

## 6 Discrete Power Control Model

In the original power control model, the transmit power can assume any positive value in a *continuous range*. However, in practice, the transmit power is limited to discrete levels. This limitation follows from the fact that in practical implementation, the power is controlled by a bit sequence, which needs to be converted into an amplifying signal by a Digital-to-Analog (D/A) converter. Furthermore, the power level grid is not very dense as a high-precision D/A converter is expensive. For example, the power levels in GSM [19] and IS-95 [23] are equally spaced by 2 dB and 1 dB respectively.

Due to this limitation, it may be impossible for users to achieve the *exact* target QoS values. Therefore, a discrete version of the power control model [28] is needed.

We assume that the power level is quantized in logarithmic scale. The difference between two consecutive power level is $\delta^{(dB)}(>0)$ dB [1] Under this discrete model, the achievable QoS is related to that in the original continuous model as follows [28].

**Theorem 8** *If there exists a power vector $\mathbf{P}^*$ such that $\Gamma_i(\mathbf{P}^*) = \gamma_i$ for all $i$, then there exists a discrete power vector $\hat{\mathbf{P}}$ such that $\delta^{-1}\gamma_i \le \Gamma_i(\hat{\mathbf{P}}) \le \delta\gamma_i$ for all $i$.*

**Proof:**

Given $P_i^*$, we can always find one and only one discrete power level $\hat{P}_i$ such that $\delta^{-1/2}P_i^* \le \hat{P}_i < \delta^{1/2}P_i^*$. Assume that $P_i^*$ is quantized to $\hat{P}_i$. Let $\hat{\mathbf{P}}$ be the quantized power vector corresponding to the given vector $\mathbf{P}^*$.

$$\begin{aligned}
\Gamma_i(\hat{\mathbf{P}}) &= \frac{G_{ii}\hat{P}_i}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i} \\
&\ge \frac{G_{ii}\delta^{-1/2}P_i^*}{\sum_{j\neq i} G_{ij}\delta^{1/2}P_j^* + \eta_i} \\
&\ge \delta^{-1}\Gamma_i(\mathbf{P}^*) \\
&= \delta^{-1}\gamma_i
\end{aligned}$$

The upper bound can be derived similarly.

□

In QoS tracking, if we change the target QoS value, $\gamma_i$, into a target QoS range, $[\delta^{-1}\gamma_i, \delta\gamma_i]$, by Theorem 8, the existence of solution in the continuous model implies the existence of solution in the discrete one. This result motivates our design of the following fixed-step algorithm [28]. We assume that each user adjusts his power in fixed step $\delta^{(dB)} > 0$ dB. The control rule is stated as follows.

**Fixed-step Power Control Algorithm**

[1] We use $x^{(dB)}$ to denote the decibel value of $x$, i.e. $x^{(dB)} = 10\log_{10} x$.

$$P_i^{(n+1)} = \begin{cases} \delta P_i^{(n)} & \text{if } \Gamma_i^{(n)} < \delta^{-1}\gamma_i \\ \delta^{-1} P_i^{(n)} & \text{if } \Gamma_i^{(n)} > \delta\gamma_i \\ P_i^{(n)} & \text{otherwise} \end{cases} \tag{59}$$

The most notable feature of this algorithm is its simplicity. At each iteration, a user adjusts his transmit power upward or downward by one step, or keeps his power constant. It can be shown that the algorithm converges provided that a feasible solution exists. The proof is composed of two parts. First, it will be established that the power level of each user has a lower bound and an upper bound. Then, it will be shown that the power levels do not oscillate.

**Proposition 2** *If a feasible solution* $\hat{\mathbf{P}}$ *exists, then under the fixed-step algorithm, the power vector,* $\mathbf{P}^{(n)}$*, at any iteration* $n$ *has an upper bound and a lower bound which depend only on the gain matrix and the initial power vector.*

**Proof:**

We let $\mathbf{P}^{(0)}$ be the initial power vector. $P_i^{(0)}$ differs from $\hat{P}_i$ by a multiple of $\delta^{(dB)}$ dB, i.e.

$$P_i^{(0)} = \hat{P}_i \delta^{a(i,0)} \tag{60}$$

where $a(i,0)$ is an integer. In general, we define $a(i,n)$ by

$$P_i^{(n)} = \hat{P}_i \delta^{a(i,n)} \tag{61}$$

Note that $a(i,n)$ is an integer and $|a(i,n+1) - a(i,n)| \leq 1$.

Now, define

$$K(n) = \max_i \{a(i,n), 0\} \tag{62}$$

For mobile $i$ where $a(i,n) = K(n)$, we have

$$\begin{aligned} \Gamma_i^{(n)} &= \frac{G_{ii} P_i^{(n)}}{\sum_{j \neq i} G_{ij} P_j^{(n)} + \eta_i} && (63) \\ &= \frac{G_{ii} \hat{P}_i \delta^{K(n)}}{\sum_{j \neq i} G_{ij} \hat{P}_j \delta^{a(j,n)} + \eta_i} && (64) \\ &\geq \frac{G_{ii} \hat{P}_i \delta^{K(n)}}{\sum_{j \neq i} G_{ij} \hat{P}_j \delta^{K(n)} + \eta_i} && (65) \\ &\geq \Gamma_i(\hat{\mathbf{P}}) && (66) \\ &\geq \delta^{-1}\gamma_i && (67) \end{aligned}$$

Therefore, those mobiles which achieve $K(n)$ will not increase its power at iteration $n+1$. In other words, $a(i,n+1) \leq a(i,n) = K(n)$.

For mobile $i$ where $a(i,n) < K(n)$, we have

$$a(i,n+1) \leq a(i,n) + 1 \leq K(n) \tag{68}$$

Hence, $K(n)$ is a non-increasing sequence. As a result, for every mobile $i$, we have

$$P_i^{(n)} \leq \hat{P}_i \delta^{K(0)} \tag{69}$$

The existence of lower bound can be shown in the same way.

□

The states of the fixed-step algorithm are represented by the sequence $\mathbf{P}^{(n)}$. The sequence is said to be *asymptotically periodic* if there exists integers, $N > 0$ and $T > 1$ such that for all $n > N$,

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n+T)}$$

The transition of the fixed-step algorithm depends only on the current state and is deterministic. By Proposition 2, the number of states is finite. Thus, if the power state sequence does not converge, it must be asymptotically periodic. However, we can show that it is not asymptotically periodic. Before proving that, we need the following lemma.

**Lemma 2** *If $P_j^{(m)} \geq \delta^x P_j^{(n)}$ and $\Gamma_j < \delta^{-1}\gamma_j$, where $r < m < n$ and $x \geq 1$, then there exists $k \neq j$ such that $P_k^{(s)} \geq \delta^{x+1} P_k^{(t)}$, where $r \leq s < m \leq t < n$.*

**Proof:**

If $P_j^{(r)} < \delta^{-1} P_j^{(m)}$, then there exists $s$ such that $r < s < m$ and $P_j^{(s)} = \delta^{-1} P_j^{(m)}$ and $\Gamma_j < \delta^{-1}\gamma_j$. If $P_j^{(r)} \geq \delta^{-1} P_j^{(m)}$, we let $s = r$.

Since $P_j^{(m)} \geq \delta^x P_j^{(n)}$, there exists $t$, where $m \leq t < n$, such that $P_j^{(t)} = \delta P_j^{(n)}$ and $\Gamma_j > \delta\gamma_j$.

Therefore,

$$P_j^{(s)} \geq \delta^{-1} P_j^{(m)} \geq \delta^{x-1} P_j^{(n)} = \delta^{x-2} P_j^{(t)} \tag{70}$$

Denote the interference at base station $j$ by $I_j^{(n)}$, i.e.

$$I_j^{(n)} \equiv \sum_{k \neq j} G_{jk} P_k^{(n)} + \eta_j \tag{71}$$

Since $\Gamma_j < \delta^{-1}\gamma_j$ and $\Gamma_j > \delta\gamma_j$, by equation (70), we have

$$\begin{aligned} \delta\gamma_j &< \frac{G_{jj} P_j^{(t)}}{I_j^{(t)}} && (72) \\ &\leq \frac{G_{jj} P_j^{(s)}}{I_j^{(t)} \delta^{x-2}} && (73) \\ &< \frac{I_j^{(s)} \gamma_j}{I_j^{(t)} \delta^{x-1}} && (74) \\ \delta^x I_j^{(t)} &< I_j^{(s)} && (75) \end{aligned}$$

It implies that there exists $k$ such that

$$P_k^{(s)} > \delta^x P_k^{(t)} \tag{76}$$

Since the power level is quantized into discrete levels with step $\delta$, we have

$$P_k^{(s)} \geq \delta^{x+1} P_k^{(t)} \tag{77}$$

□

**Proposition 3** *If the fixed-step algorithm does not converge, the power vector is not aymptotically periodic.*

**Proof:**

Assume that the power vector oscillates with period $T$, where $T > 1$, i.e. $\mathbf{P}^{(n)} = \mathbf{P}^{(n+T)}$ for large enough $n$.

Since the algorithm does not converge, one can find a mobile $i$ such that $P_i^{(m)} = \delta P_i^{(n)}$, where $m < n$ and $n - m < T$.

Note that $P_i^{(m)} = \delta P_i^{(n-T)}$. Therefore, there exists $r$ such that $\Gamma_i < \delta^{-1}\gamma_i$, where $n - T \leq r < m$.

By Lemma 2, there exists a mobile $j$ $(j \neq i)$ such that $P_j^{(s)} = \delta^2 P_j^{(t)}$ where $r \leq s < m \leq t < n$. Note that $t - s < T$.

By repeating the argument, one can find a mobile $k$ such that $P_k^{(p)} = \delta^x P_k^{(q)}$ for any integer $x$ where $q > p$ and $q - p < T$. Since at each step, the power level can change by an amount bounded by $\delta$, $x$ is upper bounded by $T$. Hence, this leads to a contradiction.

□

The convergence of the fixed-step algorithm then follows directly from Proposition 2 and 3.

Thus far, all the power control algorithms we consider are driven by the SIR. In practice, however, the SIR is difficult to estimate accurately in real time. It was shown in [31] that the fixed-step algorithm, with slight modification, can be driven by any quality measure, for example, the bit error rate. A detailed description of the modified algorithm can be found in [31].

# 7 Integrated Power Control and Base Station Assignment

In the previous sections, we assume that the assignment of users to base stations is specified by outside means. To attain higher capacity and reduce power consumption, it is beneficial to integrate the tasks of base station assignment and power allocation. The goal, same as before, is to establish connections which meet the QoS requirement of all users. First, we describe the problem for uplink. This problem was independently investigated by Hanly [14], and Yates and Huang [38]. Our treatment basically follows the approach of Yates and Huang. Afterwards, we describe the problem for downlink [8].

## 7.1 Uplink Scenario

We assume $N$ users, $M$ base stations and a common radio channel. The link gain between user $i$ and base station $k$ is denoted by $G_{ki}$, and the receiver noise at base station $k$ is denoted by $\eta_k$. We define $A_i = k$ if user $i$ is assigned to base station $k$. The other notations, such as power and SIR, are the same as that described in Section 3. We want to determine the base station assignment $\mathbf{A} = [A_1, A_2, \cdots, A_N]^\top$ and the power vector $\mathbf{P} = [P_1, P_2, \cdots, P_N]^\top$ which minimizes the sum of powers, $\sum_i P_i$, subject to the QoS constraint, $\Gamma_i \geq \gamma_i$ for all $i$.

When user $i$ is assigned to base station $k$, the QoS constraint for user $i$ is

$$P_i \geq \mathbf{H}_i^{(k)}\mathbf{P} + s_i^{(k)} \tag{78}$$

where $s_i^{(k)} = \gamma_i \eta_k / G_{ki}$ and $\mathbf{H}_i^{(k)}$ is a row vector with $j$-th component defined as

$$H_{ij}^{(k)} = \begin{cases} 0 & j = i \\ \gamma_i G_{kj}/G_{ki} & j \neq i \end{cases} \tag{79}$$

Since there are $N$ users and $M$ base stations, the number of possible base station assignments is $M^N$. We can enumerate all the possible assignments and denote the $l$-th one by $\mathbf{A}(l) = [A_1(l), \cdots, A_N(l)]$. We let $\mathbf{B}^{(l)}$ be the normalized interference matrix under assignment $l$, whose element is defined by $B_{ij}^{(l)} = H_{ij}^{A_i(l)}$. Similarly,

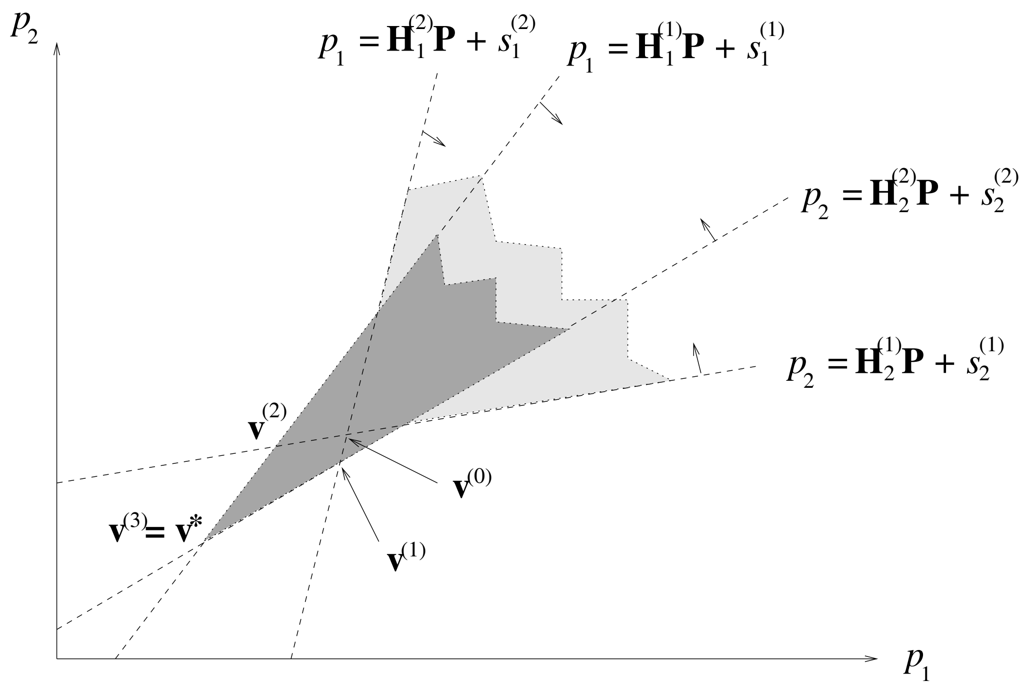


Figure 6: The feasible region of the integrated power control and base station assignment problem for two users and two base stations.

we let $\mathbf{u}^{(l)}$ be the normalized noise vector under assignment $l$, whose element is defined by $u_i^{(l)} = s_i^{A_i(l)}$. Therefore, the set of feasible power vectors under assignment $l$ is

$$\mathcal{P}^{(l)} = \{\mathbf{P} \geq \mathbf{0} | \mathbf{P} \geq \mathbf{B}^{(l)}\mathbf{P} + \mathbf{u}^{(l)}\} \tag{80}$$

The integrated power control and base station assignment problem can be viewed as the minimization of total transmitted power over the set of feasible power vectors, $\cup_l \mathcal{P}^{(l)}$. Figure 6 depicts the set of feasible power vectors for a system of two users and two base stations [38]. Totally, there are four different combinations of base station assignment. They correspond to the four cones with vertices labelled as $\mathbf{v}^{(0)}$, $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$. The union of the shaded regions depicts the set of feasible power vectors, $\cup_l \mathcal{P}^{(l)}$. Note that this set is typically not a convex set.

As discussed in Section 5, a vertex, $\mathbf{v}^{(l)}$, is the unique solution to

$$\mathbf{v}^{(l)} = \mathbf{B}^{(l)}\mathbf{v}^{(l)} + \mathbf{u}^{(l)} \tag{81}$$

Among all these vertices, we can find one which is a lower bound of all others.

**Theorem 9** *There exists a vertex $\mathbf{v}^*$ such that $\mathbf{v}^* \leq \mathbf{v}^{(l)}$ for all $l$.*

**Proof:**

Among all feasible assignments $l$, let $l_i$ be the one that minimizes $v_i^{(l)}$. Let $\mathbf{P} = [v_1^{(l_1)} \cdots v_N^{(l_N)}]^\top$. Thus, we have

$$\begin{aligned} P_i = v_i^{(l_i)} &= \mathbf{B}_i^{(l_i)}\mathbf{v}^{(l_i)} + \mathbf{u}^{(l_i)} && (82) \\ &\geq \mathbf{B}_i^{(l_i)}\mathbf{P} + \mathbf{u}^{(l_i)} && (83) \end{aligned}$$

That is, $\mathbf{P}$ is feasible with respect to the $i$th constraint of cone $\mathcal{P}^{(l_i)}$. Let $\mathbf{B}^*$ denote the normalized interference matrix with $i$th row $\mathbf{B}_i^{(l_i)}$, and let $\mathbf{u}^*$ denote the normalized noise vector with $i$th element $u_i^* = u_i^{(l_i)}$. The pair $\mathbf{B}^*$ and $\mathbf{u}^*$ describes a cone of feasible vectors, $\mathcal{P}^*$. Call its vertex $\mathbf{v}^*$. Since $\mathbf{P} \in \mathcal{P}^*$, Theorem 5 implies $\mathbf{P} \geq \mathbf{v}^*$. By construction, $\mathbf{P} \leq \mathbf{v}^{(l)}$ for all $l$. Hence $\mathbf{P} = \mathbf{v}^* \leq \mathbf{v}^{(l)}$ for all $l$.

□

The vertex $\mathbf{v}^*$ and its corresponding base station assignment is the solution to the integrated power control and base station problem. Although $\mathbf{v}^*$ is unique, it may be the vertex of more than one cone. Thus there may be more than one optimal base station assignment.

This optimal solution can be obtained iteratively by the following algorithm.

$$A_i^{(n+1)} = \arg\min_k \mathbf{H}_i^{(k)} \mathbf{P}^{(n)} + v_i^{(k)} \tag{84}$$

$$P_i^{(n+1)} = \mathbf{H}_i^{(A_i^{(n+1)})} \mathbf{P}^{(n)} + v_i^{(A_i^{(n+1)})} \tag{85}$$

At each iteration, the algorithm selects the base station to which the mobile can use minimum power to transmit. After the base station is chosen, Foschini-Miljanic algorithm is used to determine the power level. It was proved in [38] that it converges to the desired solution.

## 7.2 Downlink scenario

In the QoS tracking problem, with predetermined base station assignment, the feasibility of uplink and downlink are equivalent. Furthermore, the same power control algorithm can be applied to both links. However, the integrated problems for the uplink and the downlink are substantially different. While in the uplink, there is a Pareto optimal solution, this is not the case for the downlink. This fact will be demonstrated through a counterexample.

In [8], an iterative algorithm for the integrated downlink problem was proposed. It can be stated as follows.

- Base station assignment:
  $$\tilde{\mathbf{A}}^{(n+1)} = \mathbf{A}^{(n+1)} \tag{86}$$
  where $A_i^{(n+1)}$ is determined by equation (84).
- Power assignemnt:
  $$\tilde{\mathbf{P}}^{(n+1)} = \tilde{\mathbf{B}}^{(\tilde{\mathbf{A}}^{(n+1)})} \tilde{\mathbf{P}}^{(n)} + \tilde{\mathbf{u}}^{(\tilde{\mathbf{A}}^{(n+1)})} \tag{87}$$
  where $\tilde{\mathbf{B}}^{(\tilde{\mathbf{A}}^{(n+1)})}$ and $\tilde{\mathbf{u}}^{(\tilde{\mathbf{A}}^{(n+1)})}$ are respectively the normalized downlink interference matrix and normalized noise vector under base station assignment $\tilde{\mathbf{A}}^{(n+1)}$.

The idea of this algorithm is that it uses the same base station assignment for uplink, determined by equation (84), to that for downlink. If a feasible solution for the uplink exists, equation (84) has been proved to converge. By the uplink-downlink equivalence, a feasible solution for the downlink exists also, provided the same base station assignment is used. Thus Foschini-Miljanic algorithm can be applied and is guaranteed to converge to a feasible solution, $\tilde{\mathbf{P}}^*$.

**Theorem 10** *If the noise power is the same for all base stations and the noise power is the same for all mobiles, then the solution* $\tilde{\mathbf{P}}^*$ *minimizes the sum of downlink powers.*

**Proof:**

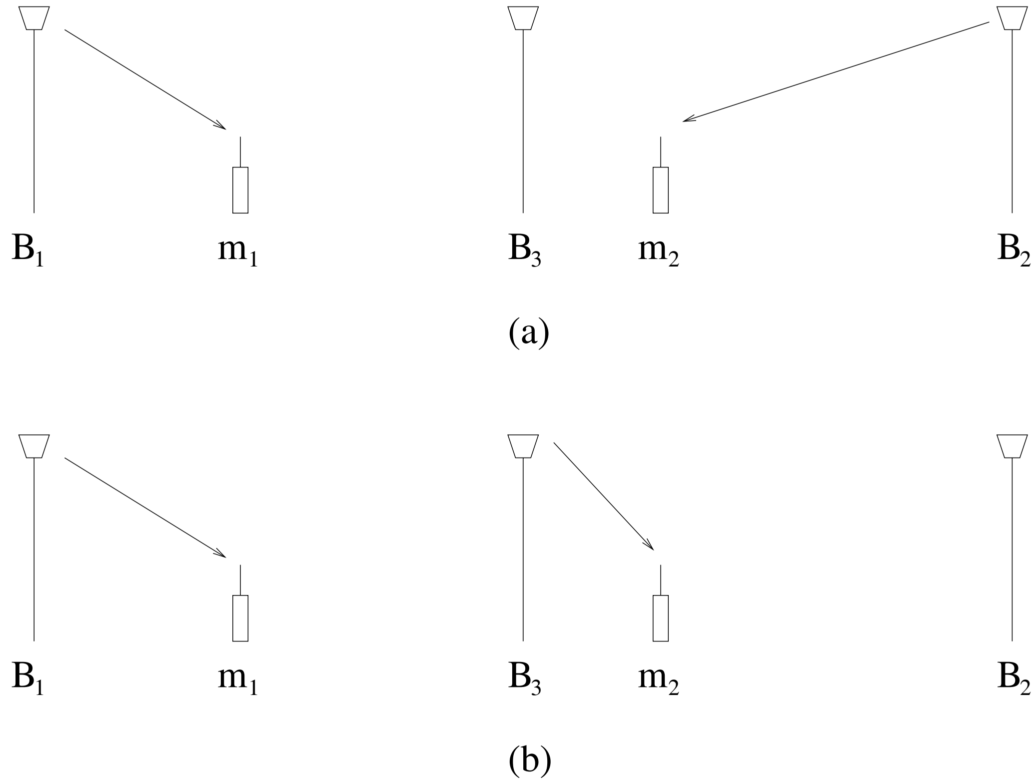


Figure 7: Two base station assignments for downlink.

The base station assignment determined by equation (84) is optimal in the sense that under this assignment we can find a power vector which minimizes the sum of uplink powers among all feasible solutions. Since by Corollary 1, the sum of uplink and downlink powers are the same up to a constant, $\tilde{\mathbf{P}}^*$ minimizes the sum of downlink powers.

□

In the following, we show by an example that, in general, there is no Pareto optimal solution to the integrated power control and base station assignment for the downlink. The example is taken from [8]. We consider two mobiles and three base stations as shown in Figure 7. Figure 7(a) shows one base station assignment, $\mathbf{A}^1$ and Figure 7(a) shows another one, $\mathbf{A}^2$. We ignore the case of assignment $m_1$ to $B_2$ or $B_3$, since it is obvious worse than assigning to $B_1$.

We let the noise power at both mobiles be $\tilde{\eta}$ and the SIR target be $\gamma$. The optimal power levels under assignment $\mathbf{A}^1$ are

$$\tilde{P}_1^{\mathbf{A}^1} = \alpha\left(\frac{\gamma G_{12}}{G_{11}G_{22}} + \frac{1}{G_{11}}\right) \tag{88}$$

$$\tilde{P}_2^{\mathbf{A}^1} = \alpha\left(\frac{\gamma G_{21}}{G_{11}G_{22}} + \frac{1}{G_{22}}\right) \tag{89}$$

where $\alpha = \gamma\tilde{\eta}/(1-\gamma^2 G_{12}G_{21}/(G_{11}G_{22}))$.

In Figure 7(a), since $m_1$ is closer to its assigned base station, we assume that $G_{11} > G_{22}$ and $G_{12} < G_{21}$. In this case, $\tilde{P}_1^{\mathbf{A}^1} < \tilde{P}_2^{\mathbf{A}^1}$.

Similarly, under assignment $\mathbf{A}^2$, the optimal power levels are

$$\tilde{P}_1^{\mathbf{A}^2} = \beta\left(\frac{\gamma G_{13}}{G_{11}G_{23}} + \frac{1}{G_{11}}\right) \tag{90}$$

$$\tilde{P}_2^{\mathbf{A}^2} = \beta\left(\frac{\gamma G_{21}}{G_{11}G_{23}} + \frac{1}{G_{23}}\right) \tag{91}$$

where $\beta = \gamma\tilde{\eta}/(1 - \gamma^2 G_{13}G_{21}/(G_{11}G_{23}))$.

In Figure 7(a), since $m_2$ is closer to its assigned base station, we assume that $G_{11} < G_{23}$ and $G_{13} > G_{21}$. Thus, $\tilde{P}_2^{\mathbf{A}^2} < \tilde{P}_1^{\mathbf{A}^2}$.

If we further assume that the location of $B_3$ is close enough to $m_2$ such that $G_{13}/G_{23} > G_{12}/G_{22}$, then $\tilde{P}_1^{\mathbf{A}^1} < \tilde{P}_1^{\mathbf{A}^2}$. In other words, if we change the assignment from $\mathbf{A}^1$ to $\mathbf{A}^2$, $\tilde{P}_1$ will increase while $\tilde{P}_2$ will decrease. Thus, in this example, there is no Pareto optimal solution.

# 8 Standard Interference Function

In [37], Yates presented a framework for the uplink power control problem. The framework unifies results found for different variations of the power control problem. He identified that for a broad class of power controlled systems, the users' QoS requirements can be described by a vector inequality written in the form

$$\mathbf{P} \geq \mathbf{I}(\mathbf{P}) \tag{92}$$

where $I_j(\mathbf{P})$ denotes the effective interference that user $j$ must overcome.

For example, equation (92) can be applied to the following systems.

- *QoS tracking with fixed base station assignment*:

  As we have described before, the SIR of user $j$ at base station $k$ is $P_j\mu_{kj}(\mathbf{P})$ where

  $$\mu_{kj}(\mathbf{P}) = \frac{G_{kj}}{\sum_{i\neq j} G_{ki}P_i + \eta_k} \tag{93}$$

  The QoS constraint can be written as

  $$P_j \geq I_j^{QoS}(\mathbf{P}) = \frac{\gamma_j}{\mu_{jj}(\mathbf{P})} \tag{94}$$

- *Integrated power control and base station assignment for uplink*:

  For the uplink channel, user $j$ should be assigned to the base station which yields the highest SIR. Thus, the SIR constraint of user $j$ is $\max_k P_j\mu_{kj}(\mathbf{P}) \geq \gamma_j$, which can be written as

  $$P_j \geq I_j^{IPB}(\mathbf{P}) = \min_k \frac{\gamma_j}{\mu_{kj}(\mathbf{P})} \tag{95}$$

- *Macro diversity*:

  To improve the quality of communication, the received signal of user $j$ at all base stations can be combined [13]. If maximal ratio combining is used, the SIR of the resultant signal is simply the sum of the SIR of all individual signals [23]. Thus the QoS constraint is of the form

  $$P_j \sum_k \mu_{kj}(\mathbf{P}) \geq \gamma_j \tag{96}$$

  In this case, we have

  $$P_j \geq I_j^{MD}(\mathbf{P}) = \frac{\gamma_j}{\sum_k \mu_{kj}(\mathbf{P})} \tag{97}$$

- *MMSE interference suppression*:

  In CDMA system, a conventional receiver consists of filters that are matched to the signature sequences of the users. To suppress part of the interference, a Minimum Mean Squared Error (MMSE) detector can be employed. If we let $\mathbf{s}_i$ be the signature sequence of user $i$ and $\mathbf{c}_i$ be the code sequence used at receiver $i$, the SIR can be expressed as [34]

$$\Gamma_j = P_j \zeta_j(\mathbf{P}) \tag{98}$$

  where

$$\zeta_j(\mathbf{P}) = \frac{G_{jj}(\mathbf{c}_j^\top \mathbf{s}_j)^2}{\sum_{k \neq j} P_k G_{jk}(\mathbf{c}_j^\top \mathbf{s}_k)^2 + \eta_j(\mathbf{c}_j^\top \mathbf{c}_j)} \tag{99}$$

  Thus, the QoS constraint becomes

$$P_j \geq I_j^{MMSE}(\mathbf{P}) = \frac{\gamma_j}{\zeta_j(\mathbf{P})} \tag{100}$$

For systems which can be expressed in the form of equation (92), we now examine the iterative power control algorithm

$$\mathbf{P}^{(n+1)} = \mathbf{I}(\mathbf{P}^{(n)}) \tag{101}$$

The function $\mathbf{I}(\mathbf{P})$ is called a *standard interference function* if it satisfies the following:

**Definition 3** *Interference function $\mathbf{I}(\mathbf{P})$ is standard if for all $\mathbf{P} \geq \mathbf{0}$, the following properties are satisfied.*

- *Positivity: $\mathbf{I}(\mathbf{P}) > \mathbf{0}$.*
- *Monotonicity: If $\mathbf{P} \geq \mathbf{P}'$, then $\mathbf{I}(\mathbf{P}) \geq \mathbf{I}(\mathbf{P}')$.*
- *Scalability: For all $\alpha > 1$, $\alpha \mathbf{I}(\mathbf{P}) > \mathbf{I}(\alpha \mathbf{P})$.*

When $\mathbf{I}(\mathbf{P})$ is standard, the algorithm in equation (101) will be called the *standard power control algorithm*. It was proved in [37] that for any initial power vector $\mathbf{P}$, the standard power control algorithm converges to a unique fixed point whenever a feasible solution exists. We now present the convergence result for $\mathbf{I}^n(\mathbf{P})$, the power vector produced by the standard power control algorithm at the $n$th iteration [37].

**Theorem 11** *If the standard power control algorithm has a fixed point, then that fixed point is unique.*

**Proof:**

Let $\mathbf{P}$ and $\mathbf{P}'$ be distinct fixed point. By the positivity property, $P_j > 0$ and $P_j' > 0$ for all $j$. Without loss of generality, we assume that there exists $j$ such that $P_j < P_j'$. Hence, there exists $\alpha > 1$ such that $\alpha \mathbf{P} \geq \mathbf{P}'$ and that for some $j$, $\alpha P_j = P_j'$. The monotonicty and scalability properties imply

$$P_j' = I_j(\mathbf{P}') \leq I_j(\alpha \mathbf{P}) < \alpha I_j(\mathbf{P}) = \alpha P_j \tag{102}$$

which contradicts with the fact $\alpha P_j = P_j'$.

□

**Lemma 3** *If $\mathbf{P}$ is a feasible power vector, then $\mathbf{I}^n(\mathbf{P})$ is a monotone decreasing sequence of feasible power vectors that converges to the unique fixed point $\mathbf{P}^*$.*

**Proof:**

Let $\mathbf{P}^{(k)}$ be a feasible power vector. Feasibility of $\mathbf{P}^{(k)}$ implies that $\mathbf{P}^{(k)} \geq \mathbf{I}(\mathbf{P}^{(k)}) = \mathbf{P}^{(k+1)}$. $\mathbf{P}^{(k+1)}$ is also a feasible vector because monotonicity implies $\mathbf{I}(\mathbf{P}^{(k)}) \geq \mathbf{I}(\mathbf{P}^{(k+1)})$. By mathematical induction, if $\mathbf{P}^{(0)}$ is feasible, $\mathbf{P}^{(n)}$ is a monotone decreasing sequence of feasible power vectors. Since $\mathbf{P}^{(n)}$ is bounded below by the zero vector, Theorem 11 implies the sequence must converge to the unique fixed point $\mathbf{P}^*$.

□

**Lemma 4** *If a feasible power vector exists, then starting from the zero vector,* $\mathbf{z}$*,* $\mathbf{I}^n(\mathbf{P})$ *is a monotone increasing sequence of power vectors that converges to the unique fixed point* $\mathbf{P}^*$*.*

**Proof:**

Let $\mathbf{z}^{(n)} = \mathbf{I}^n(\mathbf{z})$. Since $\mathbf{P}^* > \mathbf{z}$, monotonicity implies

$$\mathbf{P}^* = \mathbf{I}(\mathbf{P}^*) \geq \mathbf{I}(\mathbf{z}^{(0)}) = \mathbf{z}^{(1)} \geq \mathbf{z} \tag{103}$$

Suppose $\mathbf{z}^{(n-1)} \leq \mathbf{z}^{(n)} \leq \mathbf{P}^*$, monotonicty implies

$$\mathbf{P}^* = \mathbf{I}(\mathbf{P}^*) \geq \mathbf{I}(\mathbf{z}^{(n)}) \geq \mathbf{I}(\mathbf{z}^{(n-1)}) = \mathbf{z}^{(n)} \tag{104}$$

That is, $\mathbf{P}^* \geq \mathbf{z}^{(n+1)} \geq \mathbf{z}^{(n)}$.

By mathematical induction, the sequence $\mathbf{z}^{(n)}$ is nondecreasing and is bounded above by $\mathbf{P}^*$. By Theorem 11, $\mathbf{z}^{(n)}$ must converge to $\mathbf{P}^*$.

□

**Theorem 12** *If a feasible power vector exists, then for any initial power vector* $\mathbf{P}$*, the standard power control algorithm converges to a unique fixed point* $\mathbf{P}^*$*.*

**Proof:**

Feasibility of the problem implies the existence of the unique fixed point $\mathbf{P}^*$ (Theorem 11). For any initial vector $\mathbf{P}$, we can find $\alpha \geq 1$ such that $\alpha\mathbf{P}^* \geq \mathbf{P}$. By scalability, $\alpha\mathbf{P}^* \geq \alpha\mathbf{I}(\mathbf{P}^*) > \mathbf{I}(\alpha P^*)$. Hence, $\alpha\mathbf{P}^*$ is also a feasible vector.

Since $\mathbf{z} \leq \mathbf{P} \leq \alpha\mathbf{P}^*$, monotonicity implies

$$\mathbf{I}^n(\mathbf{z}) \leq \mathbf{I}^n(\mathbf{P}) \leq \mathbf{I}^n(\alpha\mathbf{P}^*) \tag{105}$$

Lemmas 3 and 4 imply $\lim_{n\to\infty} \mathbf{I}^n(\mathbf{z}) = \lim_{n\to\infty} \mathbf{I}^n(\alpha\mathbf{P}^*) = \mathbf{P}^*$ and the claim follows.

□

It should be noted that while Foschini-Miljanic algorithm is a standard power control algorithm, the fixed-step algorithm does not belong to this class.

# 9 Asynchronous Convergence

In this section, we examine a totally asynchronous model for distributed computation [5]. This model can be applied to distributed power control algorithms. It allows the users to update their powers at different rates and different times. In addition, each user may update his power based on outdated information on the interference caused by other users.

Define $\mathbf{P}^{(t)} = (P_1^{(t)}, P_2^{(t)}, \ldots, P_K^{(t)})$ where $P_i^{(t)}$ is the power of mobile $i$ at time $t$.

We assume that there is a set of times $\mathcal{T} = \{0, 1, 2, \cdots\}$ at which one or more components $P_i$ of $\mathbf{P}$ are updated. Let $\mathcal{T}^i$ be the set of times at which $P_i$ is updated. We assume that the base station to which mobile $i$ belongs may not have access to the most recent value of the components of $\mathbf{P}$. At time $t$, let $\tau_j^i(t)$ be the most recent time for which $P_j$ is known to user $i$. Note that $0 \le \tau_j^i(t) \le t$. Thus, the iterative power control algorithm can be expressed as

$$P_i^{(t+1)} = f_i(P_1^{(\tau_1^i(t))}, P_2^{(\tau_2^i(t))}, \cdots, P_K^{(\tau_K^i(t))}) \qquad \forall t \in \mathcal{T}^i \tag{106}$$

At all times $t \notin \mathcal{T}^i$, $P_i$ is left unchanged.

$$P_i^{(t+1)} = P_i^{(t)} \qquad \forall t \notin \mathcal{T}^i \tag{107}$$

**Definition 4** *(Total Asynchronism) The sets $\mathcal{T}^i$ are infinite, and if $\{t_k\}$ is a sequence of elements of $\mathcal{T}^i$ that tends to infinity, then $\lim_{k\to\infty} \tau_j^i(t_k) = \infty$ for every $j$.*

This condition guarantees that each component is updated infinitely often, and that old information is eventually purged from the system. More precisely, given any time $t_1$, there exists a time $t_2 > t_1$ such that $\tau_j^i(t) \ge t_1$ $\forall i, j$, and $t \ge t_2$.

In words, given any time $t_1$, values of components generated prior to $t_1$ will not be used in updates after a sufficiently long time $t_2$. On the other hand, the amount $t - \tau_j^i(t)$ by which the variables used in iterations are outdated can become unbounded as $t$ increases. This is the main difference between total asynchronism and another model called partial asynchronism [5].

Using the Asychronous Convergence Theorem in [5], it can be shown that the standard power control algorithm converges under total asynchronism. In addition, it was shown in [18] that the fixed-step power control algorithm also converges under this model.

# 10 Time Varying Channel

## 10.1 Correlated Fading Model

Previously, we have assumed that the link gain matrix $\mathbf{G}$=$[G_{ij}]$ is constant during the power control process. However, in reality, the communication channel suffers from fading. Thus, the matrix $\mathbf{G}$ is in effect time-varying and should be more appropriately denoted by $\mathbf{G}(t)$. In this section, we assume that the link gains vary in accordance with the shadow effect. The effect of multipath fading is assumed averaged out and is not considered.

Now let us consider a particular time instant $t$. Let $d_{ij}(t)$ be the distance between mobile $j$ and base station $i$. Let $A_{ij}(t)$ be the dB attenuation due to shadow fading and $\alpha$ be the path loss exponent. The link gain $G_{ij}(t)$ at time $t$ can be written as

$$G_{ij}(t) = \frac{10^{-A_{ij}(t)/10}}{d_{ij}(t)^\alpha} \tag{108}$$

Note that at a particular time instant, $A_{ij}$ is usually modeled as a Gaussian random variable with mean zero and variance $\sigma^2$. Empirical data show that $\alpha$ is around 4 and $\sigma$ lies between 4 and 8.

Equation (108) represents the link gain at a particular time instant. In [12], Gudmundson shows that the dB attenuation due to shadow fading, $A$, exhibits an exponential spatial correlation. Consider two points separated by a distance $d$. The correlation between the shadow fading factor $A_1$ and $A_2$ is

$$E[A_1 A_2] = \sigma^2 \exp(-d/D_0) \tag{109}$$

where $D_0$ is the correlation distance. Typical values of $D_0$ for urban areas are around 20 meters. In suburban areas, $D_0$ may be ten times larger.

Assume that a mobile travels at speed $v$. The spatial correlation can be expressed as a time correlation, that is,

$$E[A(t)A(t+\tau)] = \sigma^2 \exp(-v\tau/D_0) \tag{110}$$

We assume that the power level of each mobile is adjusted every $\tau$ units of time. In second generation cellular systems, $\tau$ is in the order of 1 to 100 milliseconds. For example, in GSM system, $\tau$ is 480 msec [19], while in IS-95 CDMA system, $\tau$ is 1.25 msec [23].

During the power control process, the change in distance between base station $i$ and mobile $j$, $d_{ij}(t)$, is usually insignificant. Therefore, to simplify our model, we assume that $d_{ij}$ is a constant.

Now we introduce a discrete-time model for the link gain matrix. Since the distance change is assumed negligible, the link gains depend only on the dynamics of the shadow fading. Let $G_{ij}^{(n)} = G_{ij}(t_0 + n\tau)$ and $A_{ij}^{(n)} = A_{ij}(t_0 + n\tau)$, where $t_0$ is a reference time instant. Equation (108) and (110) then become

$$G_{ij}^{(n)} = \frac{10^{-A_{ij}^{(n)}/10}}{d_{ij}^{\alpha}} \tag{111}$$

and

$$E[A_{ij}^{(n)} A_{ij}^{(n+1)}] = \rho\sigma^2 \tag{112}$$

where $\rho = \exp(-v\tau/D_0)$ is called the correlation coefficient. For example, in GSM system, a user travelling with speed 100 km/h in urban area has $\rho = 0.51$. For pedestrians with speed 10 km/h, $\rho = 0.94$. Thus, we are most interested in the region where $0.5 < \rho < 1$.

The random sequence $\{A_{ij}^{(n)}\}$ can be represented by the following model based on Gauss-Markov process (see e.g. [27]).

$$A_{ij}^{(n+1)} = \rho A_{ij}^{(n)} + \sqrt{1-\rho^2} W^{(n)} \tag{113}$$

where $W^{(n)}$ is a Gaussian random variable with mean zero and variance $\sigma^2$. Note that the sequence $\{W^{(n)}, n = 1, 2, 3 \cdots\}$ is white, that is, $W^{(m)}$ and $W^{(n)}$ are uncorrelated when $m \neq n$.

For any power control algorithm, the SIR of mobile $i$ at time $n$ can be expressed as

$$\Gamma_i^{(n)} = \frac{G_{ii}^{(n)} P_i^{(n)}}{\sum_{j \neq i} G_{ij}^{(n)} P_j^{(n)} + \eta_i} \tag{114}$$

Note that $\{\Gamma_i^{(n)} : n = 1, 2, \cdots\}$ as defined by (114) is a stochastic sequence. We define the distribution of $\Gamma_i^{(n)}$ as

$$F_i^{(n)}(\gamma) = \Pr\{\Gamma_i^{(n)} \leq \gamma\} \tag{115}$$

It is natural to ask whether the distribution function $F_i^{(n)}(\gamma)$ eventually settles down as $n$ gets large. If the limiting distribution exists, it is legitimate to define an useful performance measure, *outage probability*, in the following way.

For a given minimum required SIR, $\gamma_0$, the outage probability, $\delta$, is defined as the probability that the SIR of a randomly chosen mobile falls below $\gamma_0$ when $n \to \infty$.

$$\delta = \frac{1}{M} \sum_{i=1}^{M} \lim_{n \to \infty} F_i^{(n)}(\gamma_0) \tag{116}$$

This measure will be used in the sequel for the evaluation of different power control schemes. Due to the complexity of the problem, we will consider only two special cases. The first one assumes a time-varying channel for single user. The second one assumes two users interfering with each other.

## 10.2 Special Case: Noise Only

First we consider the special case where the receiver noise dominates as if there is only one single mobile terminal using that channel. Our treatment basically follows that in [39].

In the single-user scenario, the SIR can be simplified as

$$\Gamma^{(n)} = \frac{G^{(n)} P^{(n)}}{\eta} \tag{117}$$

If measured in decibels, it becomes

$$\Gamma^{(n)(dB)} = A^{(n)} + P^{(n)(dB)} - L - \eta^{(dB)} \tag{118}$$

where $L = 10\alpha \log_{10} d$.

For clarity, from now on, we omit the superscript (dB), with the understanding that all the power levels, SIRs and noise levels are measured in decibels.

### 10.2.1 Path Loss Compensation

To maintain a target SIR, $\gamma_0$, one can simply let

$$P^{(n)} = \gamma_0 + L + \eta \tag{119}$$

for all $n$. This method is called *path loss compensation*, since it compensates only for the path loss effect.

With this power setting, the SIR becomes

$$\Gamma^{(n)} = A^{(n)} + \gamma_0 \tag{120}$$

Since $A^{(n)}$ is a Gaussian random variable with mean zero, it is easy to see that the outage probability is 0.5, which is unacceptably high for reliable communication. Thus it is necessary to deliberately raise the target SIR by a certain amount. This concept is called *fade margin*. We denote it by $F$ (also measured in decibels). The power setting and the resulting SIR become

$$P^{(n)} = \gamma_0 + L + \eta + F \tag{121}$$

$$\Gamma^{(n)} = A^{(n)} + \gamma_0 + F \tag{122}$$

As a consequence, $\Gamma^{(n)}$ is a Gaussian random variable with mean $\gamma_0 + F$ and variance $\sigma^2$. Thus, we have the following relation between the outage probability, $\delta$, and the fade margin, $F$.

$$\delta_{PL} = \frac{1}{2} \text{erfc}\left(\frac{F}{\sqrt{2}\sigma}\right) \tag{123}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{124}$$

For a given outage probability, we would like to minimize the fade margin such that the power consumption can be reduced. We can see that the fade margin depends on the width of the probability density function of the received SIR. By employing a more effective power control scheme, it may be possible to reduce the variation of the SIR.

### 10.2.2 Foschini-Miljanic algorithm

Now let us consider the application of Foschini-Miljanic algorithm to this situation. With fade margin $F$, by equation (40), the power level at iteration $n$ is given by

$$P^{(n)} = \gamma_0 + F - \Gamma^{(n-1)} + P^{(n-1)} \tag{125}$$

$$= \gamma_0 + F - A^{(n-1)} + L + \eta \tag{126}$$

The resulting SIR at iteration $n$ is

$$\Gamma^{(n)} = \gamma_0 + F - A^{(n-1)} + A^{(n)} \tag{127}$$

The outage probability can be written as follows.

$$\delta_{FM} = \Pr\{A^{(n-1)} - A^{(n)} > F\} \tag{128}$$

Note that $A^{(n-1)} - A^{(n)}$ is Gaussian distributed. The mean and variance are

$$E[A^{(n-1)} - A^{(n)}] = 0 \tag{129}$$

$$E[(A^{(n-1)} - A^{(n)})^2] = 2(1-\rho)\sigma^2 \tag{130}$$

Thus, the variance of the SIR is changed from $2\sigma^2$ to $2(1-\rho)\sigma^2$. It implies that Foschini-Miljanic algorithm is effective only if $\rho > 0.5$.

Furthermore, equation (128) becomes

$$\delta = \frac{1}{2}\operatorname{erfc}\left(\frac{F}{\sqrt{4(1-\rho)}\sigma}\right) \tag{131}$$

### 10.2.3 MMSE estimator

It is worth noting that the occurrence of outages is due to the variation of the received SIR. In our model, this variation arises from the attenuation factor, $A^{(n)}$, due to shadow fading. The path loss compensation method has got nothing to do with this factor. For Foschini-Miljanic algorithm, it tries to compensate this factor, in essence, by using $A^{(n-1)}$ as an estimate of $A^{(n)}$. In fact, we can further reduce the SIR variation by means of an Minimum Mean Square Error (MMSE) estimator.

It is well known that the MMSE estimator of a random variable $Y$ based on observing the random variable $X$ is the conditional mean $E[Y|X]$ (see e.g. [27]. In our problem, we need to estimate $A^{(n)}$ based on the past observations, $A^{(n-1)}, A^{(n-2)}, \cdots$. Due to the Markovian property, the MMSE estimator is

$$E[A^{(n)}|A^{(n-1)}, A^{(n-2)}, \cdots] = E[A^{(n)}|A^{(n-1)}] \tag{132}$$

$$= \rho A^{(n-1)} \tag{133}$$

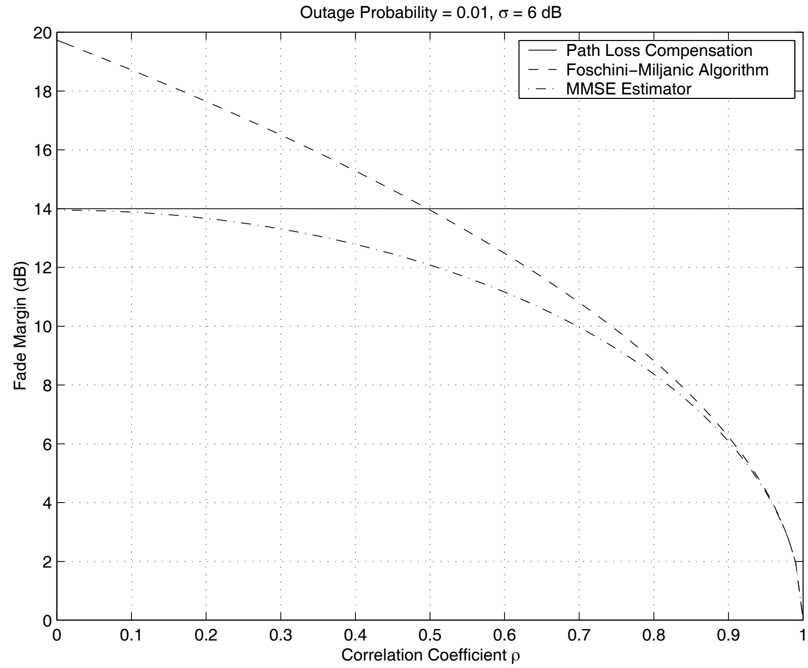


Figure 8: The relation between fade margin and correlation coefficient for single user.

This is equivalent to setting the power

$$P^{(n)} = \gamma_0 + F - \rho A^{(n-1)} + L + \eta \tag{134}$$

which assumes that correlation coefficient, $\rho$, is known.

The resulting SIR is

$$\Gamma^{(n)} = \gamma_0 + F + A^{(n)} - \rho A^{(n-1)} \tag{135}$$

Again $A^{(n)} - \rho A^{(n-1)}$ is Gaussian distributed. The mean and variance are

$$E[A^{(n-1)} - \rho A^{(n)}] = 0 \tag{136}$$

$$E[(A^{(n-1)} - \rho A^{(n)})^2] = (1-\rho^2)\sigma^2 \tag{137}$$

From the variance, we can see that the MMSE estimator always perform better than the other two methods. The outage probability is given by

$$\delta_{MMSE} = \frac{1}{2}\text{erfc}\left(\frac{F}{\sqrt{2(1-\rho^2)}\sigma}\right) \tag{138}$$

Figure 8 shows the fade margin required for the three power control schemes with different correlation coefficients.

## 10.3 Special Case: Two Users

In this subsection, we analyse another special case where there are two users interfering with each other [30]. We further assume that the interference dominates the noise term. Thus the SIR of mobile 1 at time $n$ can be written as

$$\Gamma_1^{(n)} = \frac{G_{11}^{(n)} P_1^{(n)}}{G_{12}^{(n)} P_2^{(n)}} \tag{139}$$

From now on, we assume that the SIR and the power levels are all measured in decibels. As before, we omit the superscript for clarity. Equation (139) can then be rewritten as

$$\Gamma_1^{(n)} = P_1^{(n)} - P_2^{(n)} + B_1^{(n)} + L_1 \tag{140}$$

where

$$B_1^{(n)} = A_{11}^{(n)} - A_{12}^{(n)}$$

and

$$L_1 = 10\alpha \log_{10}\left(\frac{d_{12}}{d_{11}}\right)$$

Note that $B_1^{(n)}$ is a Gaussian random variable with mean zero and variance $\sigma_B^2 = 2\sigma^2$ ($\sigma^2$ is the variance of the shadow fading factor, $A_{ij}$). Moreover, $\{B_1^{(n)}\}$ is a correlated sequence with correlation coefficient $\rho$.

Similarly, we have

$$\Gamma_2^{(n)} = P_2^{(n)} - P_1^{(n)} + B_2^{(n)} + L_2 \tag{141}$$

where $B_2^{(n)}$ and $L_2$ are defined in a similar way. Note that $B_1^{(n)}$ and $B_2^{(n)}$ are independent.

#### 10.3.1 Path Loss Compensation

To compensate for the path loss effect, we assume that $P_1 = \frac{L_2}{2}$ and $P_2 = \frac{L_1}{2}$. Then

$$\Gamma_1^{(n)} = \frac{L_1 + L_2}{2} + B_1^{(n)} \tag{142}$$

$$\Gamma_2^{(n)} = \frac{L_1 + L_2}{2} + B_2^{(n)} \tag{143}$$

Thus, both $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$ are Gaussian distributed with mean and variance given by

$$E[\Gamma_i^{(n)}] = \frac{L_1 + L_2}{2} \tag{144}$$

$$Var[\Gamma_i^{(n)}] = 2\sigma^2 \tag{145}$$

for $i = 1, 2$.

For an SIR requirement of $\gamma_0$, the outage probability, $\delta_{PL}$, is given by

$$\delta_{pl} = \text{Prob}[\Gamma_i \leq \gamma_0] = \frac{1}{2}\text{erfc}\left(\frac{L_1 + L_2 - 2\gamma_0}{4\sigma}\right) \tag{146}$$

#### 10.3.2 Instantaneous SIR Balancing

Next we investigate the case where power control mechanism is fast enough to achieve SIR balancing at every time instant.

To *balance* the SIR of the two mobiles, we require that $\Gamma_1^{(n)} = \Gamma_2^{(n)}$. It is easy to verify that the balanced SIR at time $n$ is given by

$$\gamma^{*(n)} = \frac{1}{2}(B_1^{(n)} + B_2^{(n)} + L_1 + L_2) \tag{147}$$

That is, $\Gamma_1^{(n)} = \Gamma_2^{(n)} = \gamma^{*(n)}$.

By equation (113), $B_i^{(n)} (i = 1, 2)$ can be written as

$$B_i^{(n)} = \sum_{k=0}^{n-1} \rho^{n-k-1}\sqrt{1-\rho^2}W^{(k)} + \rho^n B_i^{(0)} \tag{148}$$

where $W^{(k)}$ is Gaussian distributed with mean zero and variance $\sigma_B^2$.

The expectation and variance are found to be

$$E[B_i^{(n)}] = \rho^n B_i^{(0)} \quad (149)$$

$$Var[B_i^{(n)}] = (1-\rho^{2n-2})\sigma_B^2 \quad (150)$$

By equation (147), (149) and (150), $\gamma^{*(n)}$ is a Gaussian random variable with mean and variance given by

$$E[\gamma^{*(n)}] = \frac{1}{2}\left(\rho^n(B_1^{(0)}+B_2^{(0)})+L_1+L_2\right)$$

$$Var[\gamma^{*(n)}] = \frac{1}{2}(1-\rho^{2n-2})\sigma_B^2 = (1-\rho^{2n-2})\sigma^2$$

Therefore, when $n$ goes to infinity,

$$\lim_{n\to\infty} E[\gamma^{*(n)}] = \begin{cases} \frac{1}{2}(L_1+L_2) & \text{if } \rho < 1 \\ \frac{1}{2}\left((B_1^{(0)}+B_2^{(0)})+L_1+L_2\right) & \text{if } \rho = 1 \end{cases} \quad (151)$$

$$\lim_{n\to\infty} Var[\gamma^{*(n)}] = \begin{cases} \sigma^2 & \text{if } \rho < 1 \\ 0 & \text{if } \rho = 1 \end{cases} \quad (152)$$

The case $\rho = 1$ corresponds to the situation where the link gain matrix $\mathbf{G}$ is fixed.

Assume that $\rho < 1$. For an SIR requirement of $\gamma_0$, the outage probability, $\delta_{bal}$, is given by

$$\delta_{bal} = \lim_{n\to\infty} \text{Prob}[\gamma^{*(n)} \leq \gamma_0] = \int_{-\infty}^{\gamma_0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-(L_1+L_2)/2)^2}{2\sigma^2}\right) dx \quad (153)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{L_1+L_2-2\gamma_0}{2\sqrt{2}\sigma}\right) \quad (154)$$

## 10.4 Optimal Power Control

The optimal power control scheme behaves the same as the instantaneous SIR balancing when $\gamma^{*(n)} \geq \gamma_0$. When $\gamma^{*(n)} < \gamma_0$, both users are in outage in the previous scheme. However, under optimal control, the transmission of either of them will be suppressed ($P_i = 0$ or sufficiently small). In other words, one of them will be in outage while the other one will not. If we assume that the user being suppressed transmission is randomly chosen, then the outage probability of the two users will be the same. Otherwise, their outage probability will be different. However, in both cases, the outage probability of the system will be reduced to one half of the instantaneous SIR balancing.

Therefore, we have

$$\delta_{opt} = \frac{1}{2}\delta_{bal} \quad (155)$$

$$= \frac{1}{4}\text{erfc}\left(\frac{L_1+L_2-2\gamma_0}{2\sqrt{2}\sigma}\right) \quad (156)$$

Equation (155) provides a performance bound on all practical power control algorithms for the two-user case.

## 10.5 Cooperative Algorithm

Finally, we consider one form of the Cooperative Algorithm. We assume that in the $(n+1)^{th}$ step, each mobile unit adjusts its transmitted power $P_i^{(n+1)}$ according to the following rule:

$$P_i^{(n+1)} = \alpha_i^{(n)} P_i^{(n)} \quad (157)$$

$$\alpha_i^{(n)} = \sqrt[m]{\frac{\mathrm{GM}_{k\in\mathcal{N}_i\cup\{i\}}(\Gamma_k^{(n)})}{\Gamma_i^{(n)}}} \quad (158)$$

where $\mathrm{GM}(x_i)$ is defined as the geometric mean of $x_i$'s and $m$ is a control parameter. In our analysis, we let $m$ equal to two. The reason for this choice of $m$ is that the algorithm converges in one step if the link gain matrix $\mathbf{G}$ is fixed.

According to the algorithm, the power evolution of the two mobiles, if measured in decibels, can be expressed as

$$P_1^{(n+1)} = P_1^{(n)} + \frac{\Gamma_2^{(n)} - \Gamma_1^{(n)}}{4} \quad (159)$$

$$P_2^{(n+1)} = P_2^{(n)} + \frac{\Gamma_1^{(n)} - \Gamma_2^{(n)}}{4} \quad (160)$$

By equation (140), (141), (159) and (160),

$$\begin{aligned}
\Gamma_1^{(n)} &= P_1^{(n)} - P_2^{(n)} + B_1^{(n)} + L_1 \\
&= P_1^{(n-1)} - P_2^{(n-1)} + \frac{\Gamma_2^{(n-1)} - \Gamma_1^{(n-1)}}{2} + B_1^{(n)} + L_1 \\
&= B_1^{(n)} - \frac{1}{2}B_1^{(n-1)} + \frac{1}{2}B_2^{(n-1)} + \frac{1}{2}(L_1 + L_2) \\
&= (\rho - \frac{1}{2})B_1^{(n-1)} + \sqrt{1-\rho^2}W^{(n-1)} + \frac{1}{2}B_2^{(n-1)} + \frac{1}{2}(L_1 + L_2)
\end{aligned}$$

where $W^{(n)}$ is a Gaussian random variable with mean zero and variance $\sigma_B^2$.

Therefore, $\Gamma_1^{(n)}$ is Guassian distributed and by equation (149) and (150),

$$E[\Gamma_1^{(n)}] = \frac{1}{2}\left[(2\rho-1)\rho^{n-1}B_1^{(0)} + \rho^{n-1}B_2^{(0)} + L_1 + L_2\right] \quad (161)$$

$$Var[\Gamma_1^{(n)}] = 2\left[(\rho^2 - \rho + \frac{1}{2})(1-\rho^{2n-4}) + (1-\rho^2)\right]\sigma^2 \quad (162)$$

When $n$ goes to infinity,

$$\lim_{n\to\infty} E[\Gamma_1^{(n)}] = \begin{cases} \frac{1}{2}(L_1 + L_2) & \text{if } \rho < 1 \\ \frac{1}{2}\left((B_1^{(0)} + B_2^{(0)}) + (L_1 + L_2)\right) & \text{if } \rho = 1 \end{cases} \quad (163)$$

$$\lim_{n\to\infty} Var[\Gamma_1^{(n)}] = \begin{cases} (3-2\rho)\sigma^2 & \text{if } \rho < 1 \\ 0 & \text{if } \rho = 1 \end{cases} \quad (164)$$

By symmetry, the distribution of $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$ are identical.

Now let us compare the performance of instantaneous SIR balancing, path loss compensation and Cooperative Algorithm. By equation (144), (151), and (163), we can see that the SIR achieved by these three schemes have the same mean. The instantaneous SIR balancing has the smallest variance, thus having the smallest outage probability.

Furthermore, it is worth noting that the performance of instantaneous SIR balancing and path loss compensation do not depend on the correlation coefficient, $\rho$. However, the value of $\rho$ affects the performance of Cooperative Algorithm. Notice that $\rho$ reflects the power control rate with respect to the rate of environmental

Table 1: Results for two cochannel users

| | Optimal Control | Instantaneous SIR balancing | Slow Path Loss Compensation | Cooperative Algorithm |
|---|---|---|---|---|
| mean SIR | - | $\frac{L_1+L_2}{2}$ | $\frac{L_1+L_2}{2}$ | $\frac{L_1+L_2}{2}$ |
| variance of SIR | - | $\sigma^2$ | $2\sigma^2$ | $(3-2\rho)\sigma^2$ |
| outage probability | $\frac{1}{4}\text{erfc}\left(\frac{L_1+L_2-2\gamma_0}{2\sqrt{2}\sigma}\right)$ | $\frac{1}{2}\text{erfc}\left(\frac{L_1+L_2-2\gamma_0}{2\sqrt{2}\sigma}\right)$ | $\frac{1}{2}\text{erfc}\left(\frac{L_1+L_2-2\gamma_0}{4\sigma}\right)$ | $\frac{1}{2}\text{erfc}\left(\frac{L_1+L_2-2\gamma_0}{2\sqrt{2(3-2\rho)}\sigma}\right)$ |

change. When $\rho \to 1$, the environment appears to be static and the performance approaches the instantaneous SIR balancing. When $\rho < 0.5$, the environment changes too rapidly that the control mechanism cannot catch up. Therefore, it performs even worse than path loss compensation.

In the special case where $\rho = 0$, i.e. the link gain matrix $\mathbf{G}$ is fixed, the variance of SIR under Cooperative Algorithm is zero. It implies that the algorithm converges under a fixed link gain model.

Assume that $\rho < 1$. For an SIR requirement of $\gamma_0$, the outage probability, $\delta_{coop}$, is given by

$$\delta_{coop} = \lim_{n\to\infty} \text{Prob}[\Gamma_1^{(n)} \leq \gamma_0] = \frac{1}{2}\text{erfc}\left(\frac{L_1+L_2-2\gamma_0}{2\sqrt{2(3-2\rho)}\sigma}\right) \tag{165}$$

Our results are summarized in Table 1.

# 11 Multi-Rate Data Applications

For voice applications, the quality of a connection is considered acceptable if the BER meets certain maximum requirement. This required BER can be translated directly into a required SIR threshold. The basic idea of the QoS tracking is to keep the SIR above this threshold. On the other hand, it is unnecessary to keep the SIR far beyond the threshold because it has little improvement on the utility of a voice user.

The situation is quite different for data applications. A data connection generally has much more stringent BER requirement. Due to this nature, a packet in error has to be retransmitted. Since a higher SIR will result in less retransmissions, the delay experienced by a user can be reduced and the throughput can be increased. This implies that the quality of a data connection is a continuous, increasing function of the SIR.

As there is a world of difference between the requirement of voice and data connections, it is inappropriate to directly apply the QoS tracking paradigm to the power control problem for data traffic. Instead, the problem should be reformulated so as to better exploit the distinct nature of data traffic.

Assume that there are $N$ data users in a single-cell CDMA system using bandwidth $W$. The data rate of each user, in general, can be different. We denote the raw data rate of user $i$ by $R_i$. Thus, the processing gain of user $i$ is $W/R_i$.

We define $f$ as the rate in bits per channel use at which information can be reliably sent through the channel. In general, it is an increasing function of the product of the SIR and the processing gain, while its explicit form depends on the modulation and coding scheme.

Since user $i$ accesses the channel at a rate $R_i$, his information rate is given by

$$R_i f(\frac{\Gamma_i W}{R_i}) \tag{166}$$

We call it the *throughput* of user $i$. The total throughput of the system, $C_T$, is given by

$$C_T = \sum_{i=1}^{N} R_i f(\frac{\Gamma_i W}{R_i}) \tag{167}$$

As we consider a single-cell system, the SIR can be expressed in terms of the received power, $Q_i$, at the base station.

$$\Gamma_i = \frac{Q_i}{\sum_{j \neq i} Q_j + \eta} \tag{168}$$

We would like to find an optimal power allocation, $\mathbf{Q} = [Q_1, Q_2, \cdots, Q_N]^\top$, which maximizes $C_T$. From equation (168), we can see that scaling up any vector $\mathbf{Q}$ will improve the SIR of all users. As a result, no global maximum exists. In practice, however, the received power cannot be infinitely large. So we normalize the power levels by imposing an additional constraint.

$$\sum_{i=1}^{N} Q_i = Q_T \tag{169}$$

where $Q_T$ is a constant.

As it is difficult to deal with the variables $Q_i$'s directly, we employ the following coordinate transformation.

$$1 + \Gamma_i = \frac{\sum_j Q_j + \eta}{\sum_{j \neq i} Q_j + \eta} \tag{170}$$

$$= \frac{Q_T + \eta}{Q_T + \eta - Q_i} \tag{171}$$

$$\frac{1}{1 + \Gamma_i} = 1 - \frac{Q_i}{Q_T + \eta} \tag{172}$$

There is a one-to-one mapping between nonnegative $\Gamma_i$ and $Q_i$. The constant received power constraint becomes

$$\sum_{i=1}^{N} \frac{1}{1 + \Gamma_i} = N - \frac{Q_T}{Q_T + \eta} \tag{173}$$

This problem can be tackled by means of *Lagrange multiplier method.* We define the Lagrangian

$$L = \sum_{i=1}^{N} R_i f(\frac{\Gamma_i W}{R_i}) + \lambda(\sum_{i=1}^{N} \frac{1}{1 + \Gamma_i} - K) \tag{174}$$

where $K = N - \frac{Q_T}{Q_T + \eta}$ is a constant.

In order for $C_T$ to attain an extremum, the following $N + 1$ equations must be satisfied.

$$\frac{\partial L}{\partial \Gamma_i} = W f'(\frac{\Gamma_i W}{R_i}) - \frac{\lambda}{(\Gamma_i + 1)^2} = 0 \text{ for } i = 1, 2, \cdots, N \tag{175}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{N} \frac{1}{1 + \Gamma_i} - K = 0 \tag{176}$$

This system of $N + 1$ equations enables us to solve for the $N + 1$ unknowns $\Gamma_1, \Gamma_2, \cdots \Gamma_N$ and $\lambda$. We call equation (175) the *key equation.* It can also be expressed as follows.

$$W k_i(x_i) = \lambda \tag{177}$$

where $x_i = \Gamma_i W / R_i$, and

$$k_i(x_i) = (1 + \frac{x_i R_i}{W})^2 f'(x_i) \tag{178}$$

If a solution exists, we denote it by $x_i^*$. The corresponding value of $\Gamma_i$ is denoted by $\Gamma_i^*$.

The nature of the stationary point is governed by the second derivative.

$$\frac{\partial^2 L}{\partial \Gamma_i^2} = \frac{W^2}{R_i} f''(\frac{\Gamma_i W}{R_i}) + \frac{2\lambda}{(\Gamma_i + 1)^3} \tag{179}$$

and

$$\frac{\partial^2 L}{\partial \Gamma_i \partial \Gamma_j} = 0, \qquad i \neq j \tag{180}$$

Thus a sufficient condition for the stationary point to be a local maximum is [22]

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{\mathbf{\Gamma}^*,\lambda^*} < 0 \quad \text{for } i = 1, 2, \cdots, N \tag{181}$$

and that for it to be a local minimum is

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{\mathbf{\Gamma}^*,\lambda^*} > 0 \quad \text{for } i = 1, 2, \cdots, N \tag{182}$$

If we substitute equation (175) into (179), we have

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{\mathbf{\Gamma}^*,\lambda^*} = \; = \frac{W^2}{R_i} f''(\frac{\Gamma_i^* W}{R_i}) + \frac{2W}{\Gamma_i^* + 1} f'(\frac{\Gamma_i^* W}{R_i}) \tag{183}$$

$$= W \left[ \frac{W}{R_i} f''(x_i^*) + \frac{2}{\Gamma_i^* + 1} f'(x_i^*) \right] \tag{184}$$

$$= \frac{W^2}{R_i (1 + R_i x_i^* / W)^2} k_i'(x_i^*) \tag{185}$$

where

$$k_i'(x_i) = \frac{R_i}{W}(1 + \frac{x_i R}{W}) \left[ 2f'(x_i) + (\frac{W}{R_i} + x_i) f''(x_i) \right] \tag{186}$$

is obtained by differentiating $k_i(x_i)$.

This equation will be used in the sequel to determine whether a stationary point yields a maximum or a minimum.

Now we come to the point to show the existence of $\mathbf{\Gamma}^*$. If it exists and attains a global maximum, then the optimality can be achieved by allowing all users to transmit simultaneously. We call such a solution a *harmonious schedule*. Otherwise, if the solution precludes some users from transmitting, we call it a *dominated schedule*. To detemine whether an optimal solution is harmonious or dominated, we need to have some information about the function $f$.

In information theoretic sense, $f$ can be regarded as the channel capacity of a discrete-time channel. If we assume a binary-input Gaussian-output (BIGO) channel, $f$ takes the following form [35].

$$f_{BIGO}(x) = -\frac{1}{2} \log_2 2\pi e - \int_{-\infty}^{\infty} P(y) \log_2 P(y) dy \tag{187}$$

where

$$P(y) = \frac{P_0(y) + P_0(-y)}{2} \tag{188}$$

and

$$P_0(y) = \frac{1}{\sqrt{2\pi}} \exp[-(y - \sqrt{2x})^2 / 2] \tag{189}$$

If the Gaussian output is hard quantized into two levels, then the channel becomes a binary symmetric channel (BSC) with crossover probability

$$p(x) = \frac{1}{2}\mathrm{erfc}(\sqrt{x}) \tag{190}$$

The channel capacity is then given by [7]

$$f_{BSC}(x) = 1 + p(x)\log_2 p(x) + (1 - p(x))\log_2(1 - p(x)) \tag{191}$$

Equations (187) and (191) give two examples of $f$. In fact, these two forms belong to a wider class of $f$ which satisfies certain technical conditions [32]. For example, the crossover probability is actually the BER of the Binary Phase Shift Keying (BPSK) modulation. If we replace it by the BER of Differential Phase Shift Keying (DPSK) modulation, the resultant function $f$ still belongs to the class we considered [32].

Before we describe the conditions imposed on $f$, we first introduce the notion $\Phi$.

For a function $g(x)$, we define $\Phi(g(x), c)$ as the property that there exists $c > 0$ such that

$$g(x)\begin{cases} > 0 & \text{for } 0 < x < c \\ = 0 & \text{for } x = c \\ < 0 & \text{for } x > c \end{cases}$$

The conditions are as follows.

1. $f : [0, \infty) \to [0, 1)$ is continuous.
2. $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 1$.
3. For any $x \in (0, \infty)$, the first three derivatives of $f$ exists and the third derivative is continuous.
4. $f' : (0, \infty) \to (0, f'_{\max})$, for some constant $f'_{\max} > 0$.
5. $f'' : (0, \infty) \to (f''_{\min}, f''_{\max})$, for some constants $f''_{\min}$ and $f''_{\max}$ where $f''_{\min} < f''_{\max}$.
6. Either $f''(x) < 0$ or $\Phi(f''(x), \beta_0)$.
7. $f'(x) = o(x^{-2})\ \ (x \to \infty)$
8. Define $h(x) = x^2 f'(x)$. We have $\Phi(h'(x), x_0)$.

Note that conditions 1 to 5 are satisfied for many functions arising in engineering applications. Condition 7 is another way of saying that $\lim_{x\to\infty} h(x) = 0$. Condition 8 is equivalent to $\mathcal{P}(xf''(x) + 2f'(x), x_0)$. Condition 4 and 8 together imply that $f''(x) < 0$ for $x > x_0$.

For this family, we have obtained some results which provide an interesting criterion on the existence of harmonious schedule [32]. We summarize our results in a series of lemmas, theorems and corollaries.

**Lemma 5** *Assuming $W$ is large enough,*

- *If $f''(x) < 0\ \forall x$, then given any $\lambda$ where $0 < \lambda < Wf'(0)$, there exists an unique $x_i^* > 0$ such that*

$$\left.\frac{\partial L}{\partial \Gamma_i}\right|_{x_i^*} = 0$$

*Furthermore, we have*

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{x_i^*} < 0$$

- *If $\Phi(f''(x), \beta_0)$, then given any $\lambda$ where $0 < \lambda < Wf'(x_0)$, there exists an unique $x_i^*$ such that*

$$\left.\frac{\partial L}{\partial \Gamma_i}\right|_{x_i^*} = 0$$

*and*

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{x_i^*} < 0$$

*Furthermore, $x_i^* \geq x_0$.*

*In both cases, $x_i^*$ is a strictly decreasing, continuous function of $\lambda$. In particular, in the first case, when $\lambda \to Wf'(0)$, we have $x_i^* \to 0$.*

**Proof:**

By a lemma in [32], for sufficiently large $W$, if $f''(x) < 0$ for all $x$, we have $k_i'(x_i) < 0$ for all $x_i > 0$. Otherwise, if $\Phi(f''(x), \beta_0)$, then we have $\mathcal{P}(k_i'(x_i), \alpha_i)$, where $\alpha_i \in (\beta_0, x_0)$ for all $i$.

In the first case where $\alpha_i$ does not exist, we define $\alpha_i = 0$ for all $i$, and we let $\tilde{x} = 0$. In the second case, we let $\tilde{x} = x_0$. Then in both cases, $k_i(x_i)$ is a strictly decreasing function of $x_i$ for $x_i \in (\alpha_i, \infty)$.

It is easy to see that

$$Wk_i(\alpha_i) \geq Wf'(\tilde{x}) \tag{192}$$

for all $i$.

Condition 7 implies that

$$W \lim_{x_i \to \infty} k_i(x_i) = 0 \tag{193}$$

Hence, given any $\lambda$ where $0 < \lambda < Wf'(\tilde{x})$, we can find an unique $x_i^* \in (\alpha_i, \infty)$ such that

$$Wk_i(x_i^*) = \lambda \tag{194}$$

Furthermore, $x_i^*$ is a strictly decreasing, continuous function of $\lambda$. In the first case where $\tilde{x} = 0$, if $\lambda \to Wf'(0)$, we have $x_i^* \to 0$.

In the second case where $\alpha_i > 0$, we assume there exists $x_i' < \alpha_i$ such that

$$Wk_i(x_i') = \lambda \tag{195}$$

By equation (185) and the property that $\Phi(k_i'(x_i), \alpha_i)$, we have

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{x_i^*} < 0 \tag{196}$$

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{x_i'} > 0 \tag{197}$$

Thus the claim follows.

□

**Lemma 6** *If $W$ is large enough, $R_T$ has a unique extremum at $\mathbf{x}^*$, where $x_i^* > x_0$ for all $i$. Furthermore, it is a local maximum.*

**Proof:**

We have already shown that $\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{x_i^*} < 0$ for all $i$. However, we need to ensure that the constraint

$$\sum_i \frac{1}{1+\Gamma_i^*} = K \tag{198}$$

is met.

Suppose we fix $\lambda$ such that $\lambda$ equals $Wf'(x_0)$. Then $x_i^*$ is the root of $k_i(x_i) = f'(x_0)$. If we increase $W$, the curve $k_i(x_i)$ will move down uniformly. (Note: the function $k_i$ depends on the value of $W$, but this fact is not shown explicily.) As a result, $x_i^*$ will decrease. Thus, when $W \to \infty$, $\Gamma_i^* = \frac{x_i^* R_i}{W} \to 0$.Hence, if $W$ is sufficiently large, we must have

$$\sum_i \frac{1}{1+\Gamma_i^*} > K \tag{199}$$

for any $K < N$.

To meet the equality constraint, we can decrease $\lambda$. The consequence is that $x_i^*$ will increase and in turn $\Gamma_i^*$ will also increase. If $\lambda^* \to 0$, then $\Gamma_i^* \to \infty$ and $\frac{1}{1+\Gamma_i^*} \to 0 < K$. Hence, there exists a $\lambda^*$ such that the constraint is satisfied.

Note that in $\{\mathbf{x}|x_i > \alpha_i\ \forall i\}$, the stationary point, $\mathbf{x}^*$, is unique. By Lemma 5, it attains a local maximum.

Now we consider the case where $\alpha_i > 0$ for some $i$. If there is another stationary point, $\mathbf{x}'(\neq \mathbf{x}^*)$, in $\{\mathbf{x}|x_i > 0\ \forall i\}$, there must be some $x_i' < \alpha_i$. Therefore, we have

$$\left.\frac{\partial^2 L}{\partial \Gamma_i^2}\right|_{\mathbf{x}_i'^*, \lambda^*} > 0 \tag{200}$$

Due to the equality constraint, it is impossible that $x_i' < x_i^*$ for all $i$. Therefore, $\mathbf{x}'$ does not yield a local minimum. It can only yield a saddle point. Hence, $R_T$ possesses only one extremum and the claim follows.

□

Finally, we show that this local maximum is in fact a global maximum.

**Theorem 13** *For the multirate power control problem with $N$ users, if the bandwidth is large enough, then the capacity optimization problem has a harmonious solution at which a strong* [2] *global maximum can be attained.*

**Proof:**

By Lemma 6, $R_T$ has only one extremum. Hence the global maximum can be attained either at the boundary or at $\mathbf{x}^*$. At the boundary, we have $\Gamma_i = 0$ for some $i$. By condition 2, the total effective rate, $R_T$, is bounded by

$$R_T(\mathbf{\Gamma}) \leq \max_j \sum_{i=1, i\neq j}^{N} R_i \tag{201}$$

given that $\mathbf{\Gamma}$ is a point at the boundary.

If $R_T(\mathbf{\Gamma}^*)$ is greater than the upper bound shown above, it possesses a global maximum at $\mathbf{\Gamma}^*$. If not, we let $\mathbf{\Gamma}' = \mathbf{\Gamma}^*$ and keep $\mathbf{\Gamma}'$ constant. Suppose now we increase $W$. The local maximum changes accordingly, and we still denote it by $\mathbf{\Gamma}^*$. Since $\Gamma_i'$ is fixed and $x_i' = \Gamma_i' W/R_i$, $x_i'$ will increase. When $W \to \infty$, we have $x_i' \to \infty$. Thus, by condition 2, $R_T(\mathbf{\Gamma}') \to \sum_{i=1}^N R_i$. Since $R_T(\mathbf{\Gamma}^*) > R_T(\mathbf{\Gamma}')$, $R_T$ attains a strong global maximum at $\mathbf{\Gamma}^*$.

[2] A function, $f$, attains a *strong global maximum* at $\mathbf{x}_0$ if $f(\mathbf{x}_0) > f(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{x}_0$ [20].

□

A stationary harmonious solution does not necessarily yield a maximum. If $W$ is very small, the harmonious solution may yield a global minimum. Thus we have a dominated schedule, which requires some users to stop their transmission. We state our result in Theorem 14 [32].

**Theorem 14** *For the multirate power control problem with $N$ users, if the bandwidth is small enough, then the capacity optimization problem has a harmonious solution at which a strong global minimum can be attained.*

Now assume that the bandwidth is sufficiently large. Then an approximation to the optimal solution can be obtained easily [32].

**Theorem 15** *(Proportional solution)*
*If the bandwidth is large enough, then the optimal solution is close to the "proportional" solution:*

$$\Gamma_i \propto R_i \ \ \forall i$$

In particular, for the BIGO and BSC model, we have obtained the following asymptotic result.

**Theorem 16** *(Asymptotic analysis for BIGO and BSC)*
*For an interference-limited channel ($\eta = 0$), when either or both the number of terminals, $N$, and the bandwidth, $W$, are large, the globally maximal solution is attained at $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_N^*)$ where $x_i^* = \beta \ \ \forall i$ and*

$$\beta = \frac{W}{\sum_{j=1}^{N} R_j} \tag{202}$$

*The spectral efficiency, $\zeta$, defined as the total throughput per unit bandwidth, is maximized when $\beta \to 0$, and we have*

$$\zeta_{\max} = \begin{cases} \log_2 e & \approx 1.443 \quad \text{for BIGO} \\ \frac{2}{\pi} \log_2 e & \approx 0.918 \quad \text{for BSC} \end{cases} \tag{203}$$

# 12 A Game Theoretic Framework for Power Control

The power control problem can be formulated as a noncooperative $N$-person game[15, 26, 24]. Each mobile user is one player of the game. We let the interval $\mathcal{P}_i = [0, M_i]$ be the strategy space of player $i$. The joint strategy space $\mathcal{S} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_N$ is the Cartesian product of all the individual strategy spaces. Each player chooses a power level $P_i \in \mathcal{P}_i$. The payoff function of player $i$ is $u_i(\mathbf{P})$. Occasionally, we will use an alternative notation $u_i(P_i, \mathbf{P}_{-i})$, where $\mathbf{P}_{-i}$ denotes the power vector of all users except user $i$.

The *power control game* (PCG) can be formally expressed as

$$\max_{P_i} u_i(P_i, \mathbf{P}_{-i}) \quad P_i \in \mathcal{P}_i, \ \forall i = 1, 2, \cdots, N \tag{204}$$

A solution concept which is most widely used in game theoretic problems is the *Nash equilibrium.*

**Definition 5** *A power vector $\mathbf{P}^*$ is a Nash equilibrium if, for every user $i$,*

$$u_i(P_i^*, \mathbf{P}_{-i}^*) \geq u_i(P_i, \mathbf{P}_{-i}^*) \quad \forall P_i \in \mathcal{P}_i \tag{205}$$

Many games have several Nash equilibria. A concept which compares the qualities of two different solutions is called *Pareto dominance.*

**Definition 6** *A power vector* $\mathbf{P}$ *Pareto dominates another vector* $\mathbf{P}'$ *if, for all* $i$,

$$u_i(\mathbf{P}) \geq u_i(\mathbf{P}') \tag{206}$$

*and for some* $j$,

$$u_j(\mathbf{P}) > u_j(\mathbf{P}') \tag{207}$$

*Furthermore, a power vector* $\mathbf{P}^*$ *is Pareto optimal if there exists no vector which Pareto dominates* $\mathbf{P}^*$.

In the following two subsections, we will use two examples to illustrate how the power control problem can be put into the framework of game theory.

## 12.1 Power Balancing

In Section 4, we define power balancing as an optimization problem which aims at maximizing the minimum SIR in the system. The objective function is global in nature as we need to know the minimum SIR among all the users. However, it is possible to construct distributed objective function which arrives at the same solution. To achieve this, we define the payoff function of each player as follows.

$$u_i(\mathbf{P}) = \min(\Gamma_i, \Gamma_{i+1}) \text{ for } i = 1, 2, \cdots, N-1, \quad u_N(\mathbf{P}) = \min(\Gamma_N, \Gamma_1) \tag{208}$$

We consider the following distributed strategy, which is essentially the Cooperative Algorithm described in Section 4.

$$\begin{cases} P_i^{(0)} &= M_i \\ P_i^{(n+1)} &= \alpha_i^{(n)} P_i^{(n)} \\ \alpha_i^{(n)} &= \left[\min(\Gamma_i(\mathbf{P}^{(n)}), \Gamma_{i+1}(\mathbf{P}^{(n)}))/\Gamma_i(\mathbf{P}^{(n)})\right]^{\epsilon} \quad \text{where } 0 \leq \epsilon \leq 1 \end{cases} \tag{209}$$

Under this strategy, we have the following property regarding the minimum SIR in the system.

**Theorem 17** *The minimum SIR among the* $N$ *players is non-decreasing.*

**Proof:**

$$\begin{aligned} \Gamma_i(\mathbf{P}^{(n+1)}) &= \Gamma_i(\alpha_i^{(n)} P_1^{(n)}, \cdots, \alpha_N^{(n)} P_N^{(n)}) && (210) \\ &\geq \alpha_i^{(n)} \Gamma_i(\mathbf{P}^{(n)}) && (211) \\ &= \min\left[\Gamma_i(\mathbf{P}^{(n)}), \Gamma_{i+1}(\mathbf{P}^{(n)})\right]^{\epsilon} \Gamma_i(\mathbf{P}^{(n)})^{1-\epsilon} && (212) \\ &\geq \min\left[\Gamma_i(\mathbf{P}^{(n)}), \Gamma_{i+1}(\mathbf{P}^{(n)})\right] && (213) \end{aligned}$$

Hence, we have

$$\min_k \Gamma_k(\mathbf{P}^{(n+1)}) \geq \min_k \Gamma_k(\mathbf{P}^{(n)}) \tag{214}$$

□

With this property, we can show the following.

**Theorem 18** *The power of user* $i$, $P_i^{(n)}$, *converges to* $P_i^* > 0$ *for all* $i$. *Furthermore, the vector* $\mathbf{P}^* = [P_1^*, P_2^*, \cdots, P_N^*]$ *is a Nash equilibrium, at which the SIR's of all users are equal.*

**Proof:**

By construction, $P_i^{(n)}$, is nonincreasing. By Theorem 17, we have

$$\Gamma_i(\mathbf{P}^{(n)}) \geq \min_k \Gamma_k(\mathbf{P}^{(n)}) \geq \min_k \Gamma_k(\mathbf{P}^{(0)}) > 0 \tag{215}$$

which implies that $P_i^{(n)}$ has a strictly positive lower bound. Thus, $P_i^{(n)}$ converges to a value $P_i^* > 0$ for all $i$.

At the point $\mathbf{P}^*$, we have

$$P_i^* = [\min(\Gamma_i(\mathbf{P}^*), \Gamma_{i+1}(\mathbf{P}^*))/\Gamma_i(\mathbf{P}^*)]^\epsilon P_i^* \tag{216}$$

As a direct consequence, we have

$$\Gamma_i(\mathbf{P}^*) \leq \Gamma_{i+1}(\mathbf{P}^*) \tag{217}$$

for all $i$, and $\Gamma_N(\mathbf{P}^*) \leq \Gamma_1(\mathbf{P}^*)$. Hence, at $\mathbf{P}^*$, all the SIR's are equal.

In particular,

$$\Gamma_i(\mathbf{P}^*) = \Gamma_{i+1}(\mathbf{P}^*) \tag{218}$$

If $P_i$ deviates from $P_i^*$, either $\Gamma_i$ or $\Gamma_{i+1}$ will decrease, thus reducing $u_i$. Hence, $\mathbf{P}^*$ is a Nash equilibrium.

□

## 12.2 Single-Cell CDMA Data Network

Next we consider a wireless CDMA data network. We define the payoff function of player $i$ as the throughput of user $i$.

$$u_i(\mathbf{P}) = R_i f(\frac{\Gamma_i W}{R_i}) \tag{219}$$

where $f$ is assumed to satisfy the conditions stated in the previous section. Specifically, we assume that $f''(x) < 0$ for all $x$. (Note that a different payoff function is used in [24] and [26].)

Since $f$ is a strictly increasing function of $P_i$ given any $\mathbf{P}_{-i}$, it is trivial that this PCG has an unique Nash equilibrium, which can be achieved by setting every power level to its maximum value. However, this solution is not necessarily desirable from a global viewpoint. Therefore, we would like to find other power vectors which improve the payoffs in a global sense. This can be done by means of pricing.

When a user transmits his information through the network, it causes interference to other users. To optimize the system performance in a global sense, we can charge the user some price for creating the harm to the network. This pricing mechanism can implicitly bring cooperation to the users, yet maintaining the noncooperative nature of the game. We let $c_i(\mathbf{P})$ be the pricing function of player $i$. The modified payoff function is defined as

$$v_i(\mathbf{P}) = u_i(\mathbf{P}) - c_i(\mathbf{P}) \tag{220}$$

We are now confronted with a *power control game with pricing* (PCGP):

$$\max_{P_i} v_i(P_i, \mathbf{P}_{-i}) \quad P_i \in \mathcal{P}_i, \ \forall i = 1, 2, \cdots, N \tag{221}$$

Essentially, PCGP is the same as PCG, except with a different payoff function. To distinguish between $u_i$ and $v_i$, from now on, we call $u_i$ the *utility* of player $i$ and $v_i$ the *payoff* of player $i$.

We focus on a single cell in a CDMA system. In such a system, the SIR of user $i$, $\Gamma_i$, can be written as

$$\Gamma_i = \frac{Q_i}{I_i} \tag{222}$$

where $Q_i = G_i P_i$ is the received power of user $i$ and $I_i = \sum_{j \neq i} Q_j + \eta_i$ is the interference plus noise power of user $i$.

Since $Q_i$ is just a linear function of $P_i$, for notational simplicity, we treat $Q_i$'s as our independent variables. The strategy space of user $i$ becomes $\mathcal{Q} = [0, \tilde{M}_i]$ where $\tilde{M}_i = G_i M_i$.

It is easy to show that $Q_i$'s and $\Gamma_i$'s are related by the following equation.

$$Q_i = \frac{\Gamma_i}{\Gamma_i + 1} \times \frac{\eta}{1 - \sum_j \frac{\Gamma_j}{\Gamma_j + 1}} \tag{223}$$

Note that all the $Q_i$'s increase if one or more of the $\Gamma_j$'s increase. Furthermore, if we assume that there is no maximum power constraint (i.e. $\tilde{M}_i \to \infty$), then a necessary and sufficient condition for the vector $[\Gamma_1, \Gamma_2, \cdots, \Gamma_N]^\top$ to be feasible is

$$\sum_i \frac{\Gamma_i}{\Gamma_i + 1} < 1 \tag{224}$$

Concerning Pareto optimality, the following theorem states that a solution should be located at the boundary of the strategy space.

**Theorem 19** *In the original PCG with utility functions $u_i$'s, a power vector $\mathbf{Q}$ is Pareto optimal if and only if $Q_i = \tilde{M}_i$ for some $i$.*

**Proof:**

First of all, note that $u_i$ is a strictly increasing function of $\Gamma_i$.

If $Q_i < \tilde{M}_i$ for all $i$, we can scale up all the $Q_i$'s by a factor $c > 1$. The resultant $\Gamma_i$'s will all increase, thus improving the utilites of all players. Hence, $\mathbf{Q}$ must not be Pareto optimal.

Now consider a vector $\mathbf{Q}$ which is not Pareto optimal, but with $Q_i = \tilde{M}_i$ for some $i$. Since $\mathbf{Q}$ is not Pareto optimal, we can find another vector $\mathbf{Q}'$ which Pareto dominates $\mathbf{Q}$. It implies that $\Gamma_i' \geq \Gamma_i$ for all $i$, and $\Gamma_j' > \Gamma_j$ for some $j$. However, by equation (223), $Q_i' > Q_i$ for all $i$, which leads to a contradiction.

□

As we mentioned before, for the PCG, the vector $[\tilde{M}_1, \tilde{M}_2, \cdots \tilde{M}_N]$ is an unique Nash equlibrium. Furthermore, it is Pareto optimal. However, this solution is not necessarily desirable from a global viewpoint. To distort the players' behaviour, we introduce the following pricing function:

$$c_i(\mathbf{Q}) = \frac{\lambda \Gamma_i}{1 + \Gamma_i} \tag{225}$$

where $\lambda$ is a system parameter.

With this pricing function, the payoff function becomes

$$v_i = R_i f(\frac{\Gamma_i W}{R_i}) - \frac{\lambda Q_i}{Q_i + I_i} \tag{226}$$

If we differentiate equation (226) with respect to $Q_i$, we have

$$\frac{\partial v_i}{\partial Q_i} = f'(\frac{\Gamma_i W}{R_i})\frac{W}{I_i} - \frac{\lambda I_i}{(Q_i + I_i)^2} \tag{227}$$

From this equation, we see that for the pricing mechanism to be effective, $\lambda$ should not be too large or too small. If $\lambda > (\frac{\tilde{M}_i}{I_i} + 1)^2 f'_{\max} W$, then $\frac{\partial v_i}{\partial Q_i} < 0$. Thus $Q_i^* = 0$. On the other extreme, if $\lambda = 0$, it becomes the same as the original PCG.

The following theorem describes some properties of the PCGP for a certain range of values of $\lambda$.

**Theorem 20** *There exists an unique $\lambda^*$ such that for any $\lambda \in [\lambda^*, Wf'(0))$, we can find an unqiue Nash equilibrium, $\mathbf{Q}^*(\lambda)$, for the PCGP, and when $\lambda = \lambda^*$, the solution is Pareto optimal for the original PCG.*

**Proof:**

We rewrite equation (227) as follows.

$$I_i \frac{\partial v_i}{\partial Q_i} = Wf'(\frac{\Gamma_i W}{R_i}) - \frac{\lambda}{(\Gamma_i + 1)^2} \tag{228}$$

Note that this is the same as the key equation shown in the previous section. Assuming that $0 < \lambda < Wf'(0)$ and $W$ is large enough, by Lemma 5, $v_i$ has a unique global maximum at $\Gamma_i^*$.

If there is no maximum power constraint (i.e. $\tilde{M}_i \to \infty$), we require $\sum_i \frac{\Gamma_i^*}{\Gamma_i^*+1} < 1$.

By Lemma 5, when $\lambda \to Wf'(0)$, we have $\Gamma_i^* \to 0$. Since $\Gamma_i^*$ is a strictly decreasing, continuous function of $\lambda$, and $\Gamma_i^* \to \infty$ when $\lambda \to 0$, we can find a unique $\underline{\lambda}$ such that

$$\sum_i \frac{\Gamma_i^*}{\Gamma_i^* + 1} = 1 \tag{229}$$

Thus, the feasibility condition holds if $\lambda$ is within the following range.

$$\underline{\lambda} < \lambda < Wf'(0) \tag{230}$$

By equation (223), the optimal power vector $\mathbf{Q}^*$ is strictly increasing (component-wise) from the zero vector when $\lambda$ decreases from $Wf'(0)$. Thus, there exists an unique $\lambda^*$ such that $Q_i^* \leq \tilde{M}_i$ for all $i$, and $Q_j^* = \tilde{M}_j$ for some $j$. (If $\tilde{M}_i < \infty$ for some $i$, we must have $\lambda^* > \underline{\lambda}$.)

Since $Q_j^* = \tilde{M}_j$ for some $j$, by Theorem 19, the solution $\mathbf{Q}^*$ is Pareto optimal.

□

Given any $\lambda$, we let $Q_T(\lambda) = \sum_i Q_i^*(\lambda)$ be the sum of optimal powers. In addition, we let $U_T(\mathbf{Q}) = \sum_i u_i(\mathbf{Q})$ be the sum of utilities of all players. The following theorem shows that the solution $\mathbf{Q}^*$ has the following global property.

**Theorem 21** *For any $\lambda \in [\lambda^*, Wf'(0))$, the Nash equilibrium of the PCGP, $Q^*(\lambda)$, maximizes $U_T$, subject to the constraint $\sum_i Q_i = Q_T(\lambda)$, where $Q_T(\lambda)$ is a strictly decreasing function of $\lambda$.*

**Proof:**

To solve the global constrained optimization problem, we can make use of the method of Lagrange multiplier. By the result in the previous section, the optimal solution can be obtained by solving

$$Wf'(\frac{\Gamma_i W}{R_i}) - \frac{\tilde{\lambda}}{(\Gamma_i + 1)^2} = 0 \quad \forall i \tag{231}$$

$$\sum_{i=1}^{N} \frac{\Gamma_i}{1 + \Gamma_i} = \frac{Q_T}{Q_T + \eta} \tag{232}$$

where $\tilde{\lambda}$ is the Lagrange multiplier and $Q_T$ here is a given constant.

The solution is exactly the same as our Nash equilibrium for the PCGP with $\tilde{\lambda}$ as the pricing parameter.

□

Although the solution $\mathbf{Q}^*$ maximizes $U_T$ on the hyperplane $\sum_i Q_i = Q_T$, it may not be a global maximum over the whole strategy space. Let the maximal value of $U_T$ be $U_{\max}$. Asymptotically, when the reciever noise is small, we have the following result.

**Theorem 22** *When $\eta \rightarrow 0$, $U_T(\mathbf{Q}^*) \rightarrow U_{\max}$.*

**Proof:**

When $\eta \rightarrow 0$, scaling a power vector has no effect on $U_T$. Thus, there is no loss of generality to restrict the strategy space into a hyperplane. Thus, $\mathbf{Q}^*$ is asymptotically optimal.

□

We have demonstrated how to use pricing technique to implicitly bring cooperation among a group of competitive mobile users. Moreover, the solution is shown to possess nice global property.

# 13 Supermodular Game

In the literature, a well studied class of noncooperative games is the so-called *supermodular game*. Games belonging to this class are particularly well behaved, thus providing an analytical framework to tackle some variations of the power control problem. First of all, let us introduce the definition of a supermodular game. Again we let $u_i$ be the payoff function of player $i$.

**Definition 7** *A power control game is a supermodular game if $u_i(P_i, \mathbf{P}_{-i})$ has increasing differences in $(P_i, \mathbf{P}_{-i})$.*

The notion of increasing differences is defined as follows.

**Definition 8** *$u_i(P_i, \mathbf{P}_{-i})$ has increasing differences in $(P_i, \mathbf{P}_{-i})$ if, for all $P_i \geq P_i'$ and $\mathbf{P}_{-i} \geq \mathbf{P}'_{-i}$,*

$$u_i(P_i, \mathbf{P}_{-i}) - u_i(P_i', \mathbf{P}_{-i}) \geq u_i(P_i, \mathbf{P}'_{-i}) - u_i(P_i', \mathbf{P}'_{-i}) \tag{233}$$

If $u_i$ is twice differentiable, then $u_i(P_i, \mathbf{P}_{-i})$ has increasing differences in $(P_i, \mathbf{P}_{-i})$ if and only if

$$\frac{\partial^2 u_i}{\partial P_i \partial P_j} \geq 0 \quad \forall j \neq i \tag{234}$$

Increasing difference for $u_i$ is equivalent to the property that increases in powers of other users raises the desirability of increasing the power for user $i$.

Supermodular games have the following nice property [33].

**Theorem 23** *For supermodular game, the set of Nash equlibria is nonempty and possesses greatest and least equlibrium points $\overline{\mathbf{P}}$ and $\underline{\mathbf{P}}$.*

Theorem 23 ensures the existence of Nash equilibria for a supermodular game. To reach one of the equilibria, the following two algorithms can be used [33].

**Algorithm 1** *The $n$ players take turns with each player successively maximizing that player's own payoff function while the decisions of the other $n-1$ players are held fixed.*

**Algorithm 2** *Each of the n players concurrently and individually chooses the next decision by maximizing that player's own payoff function under the assumption that the other $n-1$ players will hold their decisions unchanged.*

Both algorithms have the following convergence property.

**Theorem 24** *Starting from the least (or greatest) element of $\mathcal{P}$, the sequences of power vectors generated by both Algorithm 1 and 2 converge to $\underline{\mathbf{P}}$ (or $\overline{\mathbf{P}}$).*

For example, the QoS tracking problem can be formulated as a supermodular game. Let $\mathcal{P}_i = [m_i, M_i]$ be the strategy space of player $i$. We defined the payoff function as follows.

$$u_i = -(\Gamma_i - \gamma_i)^{2n} \tag{235}$$

In other words, each user tries to minimize the "distance" between his received SIR and his specified target. Differentiating $u_i$, we have

$$\frac{\partial^2 u_i}{\partial P_i \partial P_j} = \frac{G_{ii} G_{ij}}{(\sum_{j \neq i} G_{ij} P_j + \eta_i)^2} 2n(\Gamma_i - \gamma_i)^{2n-2}(2n\Gamma_i - \gamma_i) \quad j \neq i \tag{236}$$

If $m_i > 0$, then $\Gamma_i$ has the following lower bound.

$$\Gamma_i \geq \frac{G_{ii} m_i}{\sum_{j \neq i} G_{ij} M_j + \eta_i} = b_i > 0 \tag{237}$$

If we choose $n$ such that $n \geq \gamma_i / 2b_i$, then

$$\frac{\partial^2 u_i}{\partial P_i \partial P_j} \geq 0 \quad \forall j \neq i \tag{238}$$

Thus, the QoS tracking problem can be put within the framework of supermodular game. Another example of applying supermodular game to wireless network can be found in [24]. We omit the details and refer the interested readers to [24].

# 14 Conclusion

We have described different facets of the power control problem in wireless networks, with emphasis on the mathematical issues. We have introduced a number of key concepts, which form an analytical framework for the power-controlled architecture. Based on this framework, research on designing multimedia wireless network is underway. Furthermore, concerning the distributed nature of the power computation, the problem fits well into a game theoretic setting, which is a promising approach to reach good technological solutions to the resource management of third generation wireless networks.

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