Pinning Control of Complex Dynamical Networks

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***Abstract*—This article introduces the notion of pinning control for complex dynamical networks regarding their stabilization, synchronization and control. Specifically, it will first review the concept of network pinning control and then address the fundamental issues of network stabilizability, synchronizability and controllability. Basic ideas will be explained, technical derivations will be outlined, and important theoretical problems will be briefly discussed. It will show that the self-contained theoretical framework of pinning control technology is promising for practical applications in network science and engineering.**

***Index Terms*—Complex network, pinning control, stabilization, synchronization, controllability**

I. Introduction

NETWORK science and engineering, emerged as an interdisciplinary technology based on mathematical graph theory and big-data analysis, finds applications in almost all practical fields, such as IoT, computers, consumer electronics, logistics, statistical physics, structural biology, brain science, social networks and various multi-agent systems [1], [2], [3].

To study network science and engineering, similarly to the investigation of other technical subjects, mathematical modeling is essential. In the real world, natural and man-made networks usually have complicated topologies (i.e., structures), which are commonly referred to as complex networks. From a graph-theoretic perspective, the most typical and representative network models are the random-graph model [4], small-world network model [5] and scale-free network model [6], [7] (see [1], [2], [3] for more detailed descriptions and discussions). A mathematical model of a complex network of dynamical systems is typically represented by a mathematical graph with nodes being dynamical systems, called node-systems, which can be continuous or discrete, time-invariant or varying, lower- or higher-dimensional, and linear or nonlinear in general.

There are many important topics to study about complex networks, including complex dynamics, information spreading, control and synchronization, interaction and evolution, and so on. This article presents only one aspect of the many, specifically pinning control of complex dynamical networks, addressing a few important issues in the studies of stabilization, synchronization and control.

To control a network of dynamical systems for achieving some desirable goals, for example some specific engineering applications, the most important question to ask is whether or not this network is controllable, which is referred to as the network controllability problem. For a given network of many node-systems, the question can be more specific: how many controllers are needed to ensure the network be controllable? The next question is where to input (to "pin") these controllers in order to ensure the task can be done?

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To address the above concerned issues, an effective technique is "pinning control", developed since 2002 [8], [10], [11], to answer the fundamental questions of *how many controllers are needed* and *where to pin these controllers*, so as to ensure the whole network be controllable for achieving some control objectives such as stabilization and synchronization [12], [13], [14], [15], [16].

For simplicity of discussions, hereafter consider a given and fixed network of N identical node-systems in the continuous-time and time-invariant setting, which is connected, unweighted and undirected, with state-feedback couplings, represented by

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j - x_i)\,, \quad i = 1,2,...,N, \tag{1}$$

where the ith node-system state $x_i \in R^n$, $f : R^n \to R^n$ is a nonlinear function satisfying the Lipschitz condition, $c > 0$ is the coupling strength constant, H is a coupling constant matrix, and $A = [a_{ij}]$ is the adjacency matrix defined by $a_{ij} = 1$ if node i is connected with node j but $a_{ij} = 0$ otherwise, with $a_{ii} = 0$, for all $i, j = 1, 2, ..., N$.

In network (1), the summation term may be considered as a linear state-error feedback control input to node i, for all $i = 1, 2, .., N$. This mathematical model, or its enhanced versions, could be used to describe a physical network such as a logistics network interconnecting some storages or containers represented by node-systems, while the components of the state vectors could be prices, costs and features of consumer electronic products.

The network model (1) can be rearranged as

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} \ell_{ij} H x_j\,, \quad i = 1,2,...,N, \tag{2}$$

where $L = [\ell_{ij}]$ is the Laplacian matrix defined by $\ell_{ij} = -a_{ij} = -1$ if node i is connected with node j but $\ell_{ij} = -a_{ij} = 0$ otherwise, with $\ell_{ii} = \sum_{j=1,j\neq i}^{N} \ell_{ij}$, for all $i, j = 1, 2, ..., N$, which describes the diffusive coupling among all node-systems in a way similar to the Kirchhoff law of currents in balance. For the connected undirected network (2), matrix L is symmetrical with all real eigenvalues satisfying

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_N\,, \tag{3}$$

where $\lambda_2 > 0$ is particularly important in many applications such as the network synchronization to be discussed below.

The network model (2) is illustrated by Fig. 1. In the figure, the state-feedback controllers $u_i = Bx_i$ can be understood as

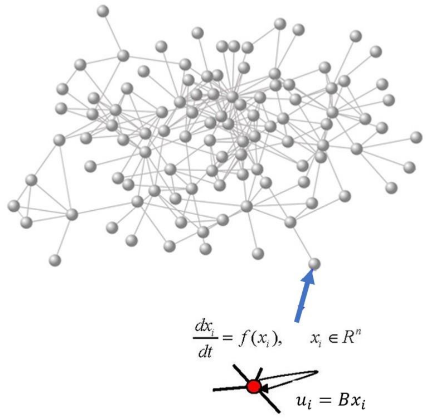

Fig. 1: A network example for illustrating pinning control.

any means of control actions, such as signals, forces, chemicals or policies that can modify the state vectors of the node-systems. The "pinning control" problem is to answer the aforementioned two fundamental questions: (1) how many state-feedback controllers are needed, and (2) at which nodes to pin them, so that a desirable goal (e.g., network controllability) can be achieved?

This article is by no means a comprehensive survey of the subject of pinning control, but only a brief introduction to the notion with detailed discussions on network stabilization, synchronization and controllability. Section II introduces pinning control and network stabilization. Section III discusses network synchronization. Section IV addresses network controllability. Section V concludes the article with a future research outlook.

II. Pinning Control and Network Stabilization

A systematic study of the "pinning control" framework was started in 2002 [8], where the following pinning control stabilization problem of network (2) was investigated:

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} \ell_{ij} H x_j + \delta_i u_i\,, \quad i = 1, 2, ..., N. \tag{4}$$

In this network, the controller $u_i = Bx_i$ is pinned at node i with constant control gain matrix B. By appropriately labeling indices, let $\delta_i = 1$ for $i = 1, 2, ..., k$, and $\delta_i = 0$ for $i = k+1, k+2, ..., N$, with $1 \le k \le N$; in particular, if $k = 1$ then only one controller is used. The pinning control stabilization problem is to determine both k and B such that the whole controlled network is stabilized to its equilibrium state $\bar{x} \in R^n$ satisfying $f(\bar{x}) = 0$, assuming it exists, which may be set as $\bar{x} = 0$ without loss of generality in a nonlinear setting.

Solving the above problem will completely answer the two pinning control questions: k controllers are needed and they should be pinned at node $i = 1, 2, ..., k$.

To solve the problem, the Lyapunov first method is applied, which guarantees that if all the eigenvalues of its Jacobian matrix at an equilibrium state have negative real parts then the system is locally stable about the equilibrium state [9].

First, by linearizing (4) at $\bar{x}$ and using $u_i = cdH(x_i - \bar{x})$ in which $B = cdH$ is chosen with constant $d > 0$, one obtains

$$\dot{e} = e[Df(\bar{x})] - cAeH\,, \tag{5}$$

where $[Df(\bar{x})] \in R^{n\times n}$ is the Jacobian matrix evaluated at $\bar{x}$, $e = [e_1, e_2, \cdots, e_N]^T \in R^{N\times n}$ is the error state with

$$e_i = x_i - \bar{x}\,, \quad i = 1, 2, ..., N,$$

and $A = L - diag\{d_1, d_2, ..., d_N\}$, in which $d_h = d$ for $1 \le h \le k$ and $d_h = 0$ for $k+1 \le h \le N$.

Then, by letting $\mu_1 \le \mu_2 \le \cdots \le \mu_N$ be the (real) eigenvalues of the matrix A and $\Phi = [\phi_1, \phi_2, \cdots, \phi_N] \in R^{N\times N}$ be the corresponding generalized eigenvector basis satisfying $A\phi_h = \mu_h\phi_h$, $h = 1, 2, ..., N$, one can expand each column of e on the basis of Φ in the form of $e = \Phi w$ with a matrix $w \in R^{N\times n}$ satisfying

$$\dot{w} = w[Df(\bar{x})] - cMwH\,,$$

where $M = diag\{\mu_1, \mu_2, \cdots, \mu_N\}$. Finally, denoting the hth row of w as w_h, one can write

$$\dot{w}_h^T = [Df(\bar{x}) - c\mu_h H]w_h^T\,, \quad h = 1, 2, ..., N. \tag{6}$$

To this end, the $(n \times N)$-dimensional network (4) has been decomposed to N of n-dimensional linearized systems (6).

Now, consider $[Df(\bar{x}) - \rho H]$, the so-called master stability matrix [17] for systems (6), with a constant $\rho > 0$. Since the nonlinear function f is assumed to be Lipschitz type, for large enough ρ, the second term (e.g., with $H = I$) will dominate the first term in this master stability matrix, so that the matrix is Hurwitz stable in the sense that the real parts of its eigenvalues are all negative. Clearly, this ρ depends on parameters c, k, d, and matrix M, which in turn depends on the matrix A (hence, its eigenvalues) in system (5).

To this end, it was shown in [8] that if

$$\rho < c\mu_1\,, \tag{7}$$

namely, if all (real) eigenvalues of matrix A are large enough, namely larger than constant $\rho/c > 0$, then the error state of system (5) is locally exponentially stable about zero, hence the equilibrium state $\bar{x}$ of network (4) is stable, implying that the network can be stabilized to its equilibrium state by using k controllers pinning at node i for $i = 1, 2, ..., k$.

III. Network Synchronization

The synchronization phenomenon between two coupled pendulum clocks was first observed by the Dutch scientist Christian Huygens in 1665, who started a systematic study leading to an active research subject for its broad range of applications in almost all fields of science and engineering.

The notion of network synchronization can be classified as state synchronization and phase synchronization, but only the state synchronization is discussed here. In most cases, synchrony is desirable, for instance regarding the coordination of multi-agents, while in some cases it is undesirable such as data traffic congestion. A comprehensive survey of some earlier works on network synchronization is given in [18] and a recent one in [19], mostly investigating the intrinsic relationship between the topology and the synchronizability of a general complex network. This is now discussed from a pinning control perspective.

Consider network (2) again. The network is said to achieve (complete) state synchronization if and only if

$$\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0 \quad \text{for all} \quad i, j = 1, 2, .., N, \qquad (8)$$

where $||\cdot||$ is the Euclidian norm. For notational convenience, define $s(t) = x_1(t)$ (or any x_i, $i = 1, 2, ..., N$), which satisfies an individual node-system $\dot{s} = f(s)$. The synchronization problem (8) can thus be equivalently reformulated as

$$\lim_{t\to\infty} ||x_i(t) - s(t)|| = 0 \quad \text{for all} \quad i = 1, 2, .., N. \qquad (9)$$

Note that, since $s(t) = x_1(t)$ is chosen here, the case of $i = 1$ in (9) is automatically satisfied.

A. Network synchronization criteria

To proceed, set $y_i(t) = x_i(t) - s(t)$ for all $i = 1, 2, ..., N$, where $y_1(t) \equiv 0$. Consider the variation of network (2), which yields a locally linearized system:

$$\dot{y}_i = [Df(s)]\, y_i - c\sum_{j=1}^{N} \ell_{ij} H\, y_j\,, \quad i = 1, 2, ..., N, \qquad (10)$$

where $[Df(s)]$ is the Jacobian matrix of f evaluated at $s(t)$. Let $Y = [y_1, y_2, \cdots, y_N]$ and rewrite (10) in a compact form

$$\dot{Y} = [\nabla f]\, Y - cHYL^T\,. \qquad (11)$$

Then, diagonalize the matrix $L = [\ell_{ij}]$ as $L = P\Lambda P^{-1}$ by a nonsingular matrix P, with $\Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_N\}$, where $\{\lambda_i\}$ are the eigenvalues of L satisfying (3). Further, let $\eta = [\eta_1, \eta_2, \cdots, \eta_N] = YP$. Since $\lambda_1 = 0$, one has

$$\dot{\eta}_i = [Df(s)]\, \eta_i - c\lambda_i H\eta_i\,, \quad i = 2, 3, ..., N.$$

Now, denote $\sigma = \inf\{c\lambda_i : i = 2, 3, ..., N\}$ and consider the so-called master stability equation [17]

$$\dot{\xi} = \big[[Df(s)] - \sigma H\big]\xi\,. \qquad (12)$$

According to the master stability principle [17], if the maximum conditional Lyapunov exponent L_{max} of the system (12) is negative then the system is stable about zero, thus $\xi \to 0$, so $Y \to 0$, namely all $y_i \to 0$, as $t \to \infty$. Consequently, the network achieves synchronization (9) or (8).

The master stability principle [17] can be understood similarly to the above discussion on system (6): if the term σH in equation (12) is large enough then it will dominate the term $[\nabla f]$, which is uniformly bounded since f is Lipschitz, so that the system (12) is stable about zero, i.e., $\xi \to 0$ as $t \to \infty$.

To derive some precise criteria for synchronization of the network (2), spectral analysis involving the network Laplacian eigenvalues (3) is useful. In retrospect, the first synchronization criterion was established in [20], [21], in terms of the Laplacian eigenvalue λ_2 in (3), as follows:

$$0 < \sigma_0 \le \lambda_2 < \infty\,, \qquad (13)$$

namely, condition (13) guarantees network (2) to synchronize. Here, the constant σ_0 is implicitly determined by the given network, which exists but is not needed to know in the following analysis.

The criterion (13) is consistent with (7). The main idea lies in that, if all the positive eigenvalues of the Laplacian matrix in (3) are large enough, then the second term in the above master stability equation (12) will dominate the node-system function in the first term, as explained above. When the stabilization criterion (7) is applied to synchronization, it can also be similarly understood as follows. Consider any two system states, x_p and x_q, both satisfying the same network model (2), and denote the error $e_{p,q} = x_p - x_q$. Then, by subtracting two linearized systems of the same form (10) with different indices p and q, one obtains the error dynamical systems in the form of $\dot{e}_{p,q} = F(x_p, x_q) - c\sum\{e_{p,q}\}$. In this implicit formulation, $F(x_i, x_j) = f(x_i) - f(x_j)$ is Lipschitz, therefore its Jacobian is uniformly bounded, so that the second summation term, which contains some combinations of the errors $e_{p,q}$, dominates the first bounded term. Consequently, applying criterion (13) guarantees that the errors $e_{p,q} = x_p - x_q \to 0$ as $t \to \infty$, namely the network synchronization (8) is achieved.

For convenience, denote $S_{\max} = [\sigma_0, \infty)$, called the synchronization region of network (2). Then, criterion (13) can be written as $\lambda_2 \in S_{\max}$, as visualized by Fig. 2 (b).

The second criterion was derived slightly later in [22], also based on the master stability principle [17]. This criterion is presented in terms of the ratio of the smallest and the largest eigenvalues in (3), as follows:

$$0 < \sigma_1 \le \lambda_2/\lambda_N \le \sigma_2 \le 1\,, \qquad (14)$$

where the constants σ_1, σ_2 are implicitly determined by the given network parameters, which likewise are not needed in the following discussions. Similarly, denote the synchronization region by $S_{\max} = [\sigma_1, \sigma_2]$. Then, this criterion can be written as $\lambda_2/\lambda_N \in S_{\max}$, as visualized by Fig. 2 (c).

In Fig. 2, the curve in each figure is the maximum conditional Lyapunov exponent L_{max} of the network, mentioned above, which bounds the Laplacian eigenvalue set (3) on the real σ-axis. In Fig. 2 (a), the curve is never negative ($S_{\max}$ is empty), therefore the network will not synchronize in general. In Fig. 2 (b), the curve is negative over an unbounded internal $[\sigma_0, \infty)$ on the σ-axis ($S_{\max}$ is unbounded). In Fig. 2 (c), the

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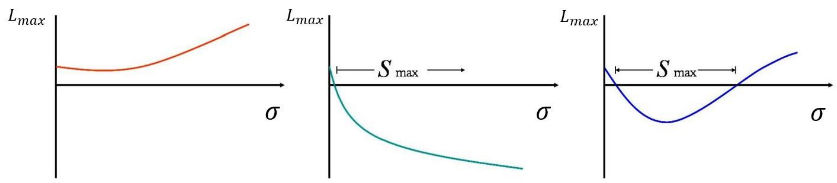


(a) $S_{\max} = \emptyset$; (b) $S_{\max} = [\sigma_0, \infty)$; (c) $S_{\max} = [\sigma_1, \sigma_2]$

Fig. 2: Network synchronization regions.

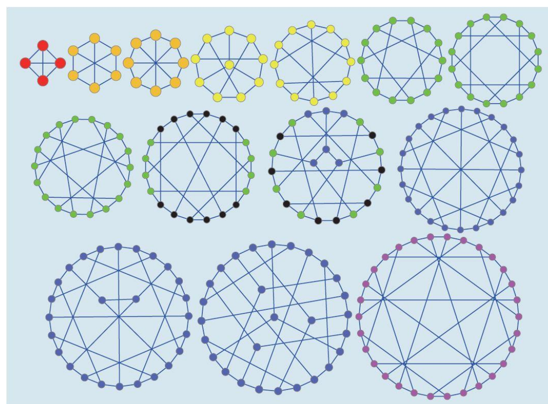


Fig. 3: Optimal homogeneous network examples [26].

curve is negative over a bounded internal $[\sigma_1, \sigma_2]$ on the σ-axis ($S_{\max}$ is bounded).

Lately, through numerical simulation [23] and graph-theoretic analysis [24], it was found that the curve in Fig. 2 (c) may bend down and then bend up again alternatively around the σ-axis, where the number of bending times depends on the order of the Laplacian matrix L. In this case, the network synchronization region $S_{\max}$ is a union of several subintervals. It was observed that the eigen-ratio criterion (14) may not work properly if the eigen-ratio falls into somewhere between two of such subintervals [25]. By comparison, the first criterion (13) works well because the region is unbounded, which has no such non-synchronization region gaps.

B. Optimal network synchronizability

To compare the synchronization performances of two networks, a measure is the network synchronizability, which represents the network self-synchronizing ability, without additional pinning control involved. The network synchronizability is measured by the eigenvalue λ_2 in criterion (13) or the eigen-ratio λ_2/λ_N in criterion (14), such that they can stay inside the corresponding synchronization region and, moreover, they can be as robust as possible against perturbations in the sense that they will not move out from the synchronization regions due to the perturbations. It is clear that, for the first criterion, the larger the λ_2, the better the synchronizability; for the second criterion, likewise, the larger the ratio λ_2/λ_N, the better the synchronizability.

In searching for the best possible synchronizability, it was found in [26] that, the totally homogeneous networks are optimal in any group of comparable networks with same number of nodes and same number of edges. A totally homogeneous network is characterized by the degrees, girths and path-sums of all its nodes [26]:

(i) Degree k_i of a node i: the number of its adjacent edges.

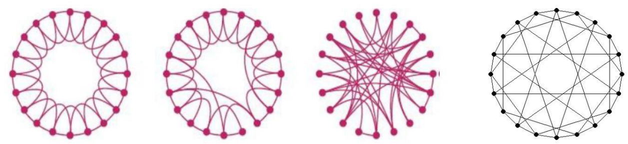
(ii) Girth g_i of a node i: the number of edges within a shortest cycle passing this node i; the minimum value of all node girths is the girth of the network.

(iii) Path-sum l_i of a node i: the total number of edges from all other nodes to this node i through their shortest distances; the average path-sum of the network is $\langle l \rangle = \frac{1}{N(N-1)} \sum_{i=1}^{N} l_i$.

If a network satisfies $k_1 = k_2 = \cdots = k_N$, $g_1 = g_2 = \cdots = g_N$ and $l_1 = l_2 = \cdots = l_N$ simultaneously, then it is called a totally homogeneous network. Furthermore, if such a network has a maximum girth g, a minimum average path-sum $\langle l \rangle$, and a maximum eigenvalue λ_2 or maximum eigen-ratio λ_2/λ_N, then it is called an optimal homogeneous network.

For illustration, consider the examples of totally homogeneous networks shown in Fig. 3, which are optimal respectively in their own groups of comparable networks with same number of nodes and same number of edges, with maximum g, minimum $\langle l \rangle$, as well as maximum λ_2 and λ_2/λ_N [26].

As can be seen from Fig. 3, all optimal homogeneous networks are homogeneous and symmetrical geometrically, having many cycles, verified by extensive simulations [26], [27] and also by higher-order network topologies [28].



regular network; small-world network; random-graph network; totally homogeneous network

Fig. 4: Four typical types of complex networks [28].

TABLE I. Characteristic numbers of higher-order topologies [28].

Define
$m_0 =$ number of nodes,
$m_1 =$ number of edges,
$m_2 =$ number of triangles,
$m_3 =$ number of tetrahedrons,
and so on. The Euler characteristic number is:
$\chi = m_0 - m_1 + m_2 - m_3 + \cdots$.
Define
$r_0 = 0$ by convention,
$r_1 =$ rank of node-edge adjacency matrix,
$r_2 =$ rank of edge-face adjacency matrix,
$r_3 =$ rank of face-polyhedron adjacency matrix,
and so on. The Betti numbers are:
$\beta_k = m_k - r_k - r_{k+1}\,, \quad k = 0, 1, 2, \ldots\,,$
where
$\beta_0 =$ number of 0th-order cavities,
$\beta_1 =$ number of 1st-order cavities,
$\beta_2 =$ number of 2nd-order cavities,
and so on. The Euler-Poincaré formula is:
$\chi = m_0 - m_1 + m_2 - m_3 + \cdots = \beta_0 - \beta_1 + \beta_2 - \beta_3 + \cdots$

TABLE II. Simulation and calculation results of four types of networks [28].

Network type	Euler number	Betti numbers	λ_2	λ_2/λ_N
Regular network	$\chi = 0$	$\beta_0 = 1, \beta_1 = 1, \beta_2 = 0$	0.4799	0.0769
Small-world	$\chi = -3$	$\beta_0 = 1, \beta_1 = 4, \beta_2 = 0$	0.5035	0.0714
Random-graph	$\chi = -5$	$\beta_0 = 1, \beta_1 = 6, \beta_2 = 0$	0.7947	0.0812
Totally homogeneous	$\chi = -20$	$\beta_0 = 1, \beta_1 = 21, \beta_2 = 0$	2.0000	0.2982

C. *Higher-order network topologies*

As observed above, cycles are beneficial for optimal network synchronizability. Higher-order network topologies involve many cycles of different orders, for example cliques (fully-connected subgraphs) of different orders such as triangles and tetrahedrons, and cavities of different orders [28].

For a given network, first recall the Euler characteristic number, Petti numbers and the Euler-Poincaré formula, as summarized in Table I.

To show how these higher-order topological characteristics may be applied to investigate the network synchronizability, consider as an example four typical types of complex networks: regular network, small-world network, random-graph network and totally homogeneous network, as shown in Fig. 4, where each has 20 nodes and 40 edges in the simulations reported below [28].

Table II shows the simulation and calculation results. As can be seen, the synchronizabilities of the four types of networks are in the following ordering: regular network $<$ small-world network $<$ random-graph network $<$ totally homogeneous network, where $<$ means "worse than", which is consistent with other reports in the literature. This ordering is obtained and can be verified by the first criterion based on λ_2 (13) and also by the Euler characteristic number (the bigger in magnitude, the better) as well as the Betti numbers (the bigger, the better).

One may find that the eigen-ratio criterion has a small inconsistency in judging between regular network and small-world network, likely due to the imprecise definition of a small-world network, which actually is not much different from the regular network in this example, and possibly due to the multiple synchronization region problem mentioned above.

Based on extensive simulations, the Euler characteristic numbers or the Betti numbers as those in Table II appear to be very promising for measuring the network synchronizability because they are all integers and are clearly distinguished by integers. However, eigenvalues differ from each other only by decimals, which might contain numerical errors.

IV. Network Controllability

The interdisciplinary field of network science with control systems theory has rapidly developed in recent years [8], [10], [12], [11], [13], as comprehensively surveyed in [14].

The question here is: if a network is not controllable, how to make it controllable by pinning control? This is the issue of network controllability, which measures the ability of a network of dynamical node-systems that can be controlled by some inputs to some nodes, called driver nodes. Here, the controllability is the ability of the networked system to move from any initial state to any target state in finite time [29], [30], [31]. This network controllability can be quantified by counting how many driver nodes are needed and where they are located [12], which is a typical pinning control problem.

A general linear time-invariant (LTI) dynamical system is described by $\dot{x} = A\,x + B\,v$, where $x \in R^n$ is the state vector, $v \in R^m$ is the control input, and $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are constant matrices, with $1 \le m \le n$. This LTI system is often abbreviated as (A, B). This system is (completely) state controllable if and only if there exists a control input, v, that can drive the state x from any initial state to any target state in finite time. A basic criterion for the LTI system to be state controllable is that the controllability matrix

$$Q = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] \tag{15}$$

has full row-rank [29], [30], [31]. To make an uncontrollable network become controllable, the number of the needed driver nodes actually fills the rank deficit of the network controllability matrix Q, so as to make it be of full row-rank [32].

An equivalent necessary and sufficient condition is the so-called PBH criterion [31]: Rank$[(sI - A)\;\; B] = n$ for any complex number s, which is equivalent to that there is a constant vector ξ satisfying both $\xi^T A = \lambda \xi^T$ and $\xi^T B = 0$.

There is a slight generalization of the state controllability notion, known as the structural controllability [33]. This concept concerns two parameterized matrices $A(p)$ and $B(p)$, which contains some parameters p characterizing the structure of the underlying system: if those parameters have some values that can make the system be state controllable, then the parameterized system $(A(p), B(p))$ is structurally controllable. Hereafter, by "controllable" it refers to either state or structural controllability, which would be clear within the context.

There is a dual concept to the controllability, that is, the observability, using the measurement data $y(t) = C\,x(t) \in R^p$ with a constant matrix C. The LTI system (A, B, C) is said to be observable if there is a finite time interval $[t_1, t_2] \subset [0, \infty)$ over which the available data $\{y(t)\}$ can uniquely determine the initial state $x(0)$ [29], [30], [31]. This initial state can then be used to determines the whole state trajectory $\{x(t) :\, t \in [0, \infty)\}$ because for an LTI system (A, B, C) its solution is given by $x(t) = x(0)e^{tA} + \int_0^t e^{(t-\tau)A} Bu(\tau)d\tau$.

A. Pinning controllability

A complex dynamical network typically has many node-systems and many connecting edges, for which to achieve a certain objective such as stabilization or synchronization, it is impossible to control the network through a large number of driver nodes. Therefore, pinning control with a small number of deriver nodes is more practical and hence preferable.

In the current literature, there are many studies of the theory and methodologies of pinning control on various topics, such as network controllability [34], network synchronization [35], optimizing control [36], and nonlinear control [37]. This article discusses only the topic of network controllability, for directed networks of LTI node-systems. Here, the single-input single-output (SISO) setting will be discussed, where all node-systems and all edge connections are one-dimensional.

Again, consider network (2), where the edges are now directed, which will only change the adjacency matrix and the Laplacian matrix to be non-symmetrical thereby having complex eigenvalues. Nevertheless, spectral analysis will not be performed here, so the direction issue is not an obstacle.

Now, consider a directed network with N node-systems, which is assumed uncontrollable. Then, by pinning certain type of controllers to some nodes, it is possible to make the network controllable. As discussed above, a linear state-feedback controller $u_i = \delta_i B_i x_i$ is used at node i, where $\delta_i = 1$ or 0 depending on if the controller is pinned to node i, for $i = 1, 2, ..., N$.

As a simple example for illustration, consider the 3-node network shown in Fig. 5 (a), where all constants are nonzero to maintain the network connectivity. Suppose that this network already has one controller pinned at node 1, $u_1 = B_1 x_1$, but the controlled network is still not controllable since its controllability matrix (15) is not of full row-rank:

$$Q = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_1\, a_{12} & 0 \\ 0 & B_1\, a_{13} & 0 \end{bmatrix}$$

By adding one controller $u_3 = B_3 x_3$ to pin node 3, as shown in Fig. 5 (b), the controllability matrix (15) becomes

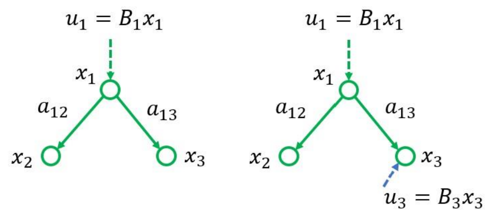
$$Q = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_1 a_{12} & 0 \\ 0 & B_1 a_{13} & B_1 B_3 a_{13} \end{bmatrix}$$

which has full row-rank. This example shows that the network (a) with one controller pinned at node 1 is not controllable, therefore it needs one more controller, u_3, to pin to node 3 (or, by symmetry, node 2), as shown in (b), to become controllable. This answers the two pinning control questions: how many controllers are needed and where to pin them.

Next, consider a general setting of a multiple-input multiple-output (MIMO) network with N LTI node-systems:

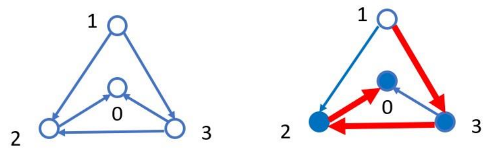
$$\dot{x}_i = A\, x_i + c \sum_{j=1}^{N} \alpha_{ij} H\, y_j + \delta_i\, u_i\,, \quad y_i = C\, x_i\,, \tag{16}$$

where $i = 1, 2, ..., N$. Similarly, $x_i \in R^n$ is the system state, $u_i = Bx_i \in R^m$ is the control input, with $1 \le m \le n \le N$, $c > 0$ is the coupling strength, and H and C are constant



(a) One controller (b) Two controllers

Fig. 5: Example for illustrating the concept of pinning controllability.



(a) A directed network (b) A maximum matching

Fig. 6: Example for illustrating the concept of matching.

matrices. Here, only linear node-systems are considered, with A being a constant matrix, which could be the Jacobian matrix of a nonlinear system f linearized at some equilibrium state. Moreover, $\Lambda = [\alpha_{ij}]$ is the adjacency matrix defined by $\alpha_{ij} = 1$ if node j points to node i, otherwise $\alpha_{ij} = 0$. Let $\Delta = \text{diag}\{\delta_1, \delta_2, \cdots, \delta_N\}$ be the pinning control matrix defined as follows: if node i is pinned by a control input u_i then $\delta_i = 1$, otherwise $\delta_i = 0$, for $i = 1, 2, ..., N$. When $n = m = 1$, this model reduces to the SISO setting discussed above.

For network (16), the pinning control problem becomes: How many δ_i should be 1 and which δ_i should be 1?

B. Determining driver nodes

The number of driver nodes needed to ensure the network to be controllable, denoted by n_D, is commonly used to measure the controllability of network (16): the smaller the n_D, the better the network controllability in the sense that less controllers are needed for maintaining the network to be controllable.

1) SISO networks: First, the problem is addressed in the SISO network setting.

To calculate n_D, the concept of matching from graph theory is useful. In a connected directed network, a matching is a set of edges that do not have common head and common tail. For illustration, Fig. 6 (a) shows a directed network, while (b) shows a matching, which is marked by red wide arrows. For example, the edge $(1, 2)$ alone is a matching; edge $(1, 2)$ and edge $(2, 0)$ together constitutes another matching. It is easy to see that there are other matchings. The one marked by red wide arrows are maximal in the sense that it cannot be further extended to include more edges, which may not be unique in a large network but they all have the same number of edges. Among all possible maximal matchings, the one with the largest number of edges is a maximum matching, which likewise may not be unique but they all have the same largest number of edges. Regarding the nodes, in any matching, the node at the head of a directed edge is a matched node, while the one at the tail is an unmatched node. In Fig. 6 (b), those shaded nodes are matched nodes while the empty node is unmatched. Furthermore, if all nodes in a maximum matching are matched nodes, then the maximum matching is called a perfect matching. For instance, a one-directional directed cycle, such as a triangle or a rectangle, is a perfect matching; however, the one shown in Fig. 6 (b) is maximum but not perfect.

Now, return to the number n_D of driver nodes needed for achieving or maintaining the network controllability. By either the minimum inputs theorem (MIT) [12] (for directed networks) or the exact controllability theorem (ECT) [32] (for both directed and undirected networks), the number n_D can be calculated as follows:

$$n_D = \begin{cases} \max\{1, N - |E|\}, & \text{using MIT,} \\ \max\{1, N - \text{rank}(A)\}, & \text{using ECT,} \end{cases} \tag{17}$$

where $|E|$ is the number of edges in a maximum matching E. It was shown in [12] that, for a directed network, if a maximum matching is perfect, then only one controller is needed and the controller can be pinned at any node; if the maximum matching is not perfect, then the number of controllers needed is equal to the number of unmatched nodes, and the controllers have to be pinned at all unmatched nodes. As a result, formulas (17) completely answers the two pinning control questions as how many controllers are needed and where to pin them, for any SISO directed network of LTI node-systems.

For example, in the network shown in Fig. 6 (a), with a maximum matching for instance the one shown in (b), to make the network controllable only one controller is needed and it should be pinned at the unmatched node, i.e., node 1.

It should be noted that formulas (17) gives an algorithm to find driver controllers, but it does not characterize a controllable network. A necessary and sufficient condition for the network (16) to be controllable, in the SISO setting, is given in [16] as follows:

$$\begin{array}{l}(A,H) \text{ is controllable; } (A,C) \text{ is observable;}\\ \text{For any eigenvalue } \lambda \text{ of } A,\ (\mathrm{Re}\{\lambda\})\Lambda \neq 0, \text{ and}\\ \mathrm{Rank}\,[I-\Lambda\Gamma_1 \quad \Delta\Gamma_2] = N, \text{ where}\\ \quad \Gamma_1 = C[\lambda I - A]^{-1}H, \quad \Gamma_2 = C[\lambda I - A]^{-1}B\,.\end{array}$$

2) MIMO networks: For a higher-dimensional network (16) in the MIMO setting, with $1 < m \leq n \leq N$, a necessary and sufficient condition for the network to be controllable is characterized by two algebraic matrix equations [15]:

$$\Delta^T XB = 0 \text{ and } \Lambda^T XHC = X(\lambda I - A)^{-1}\,, \tag{18}$$

which together, for any $\lambda \in \mathcal{C}$, have a unique matrix solution $X = 0$.

In fact, this characterization also provides an algorithm to find a solution to the pinning control problem, by tuning the diagonal matrix Δ to determine how many δ_i should be 1 and which δ_i should be 1, such that the two algebraic matrix equations (18) have a unique zero-matrix solution.

This provides a complete answer to the two pinning control questions as how many controllers are needed and at which nodes to pin them.

V. Conclusions

This article introduces the notion of pinning control for complex dynamical networks, regarding their stabilizability, synchronizability and controllability. The concept of network pinning control is first reviewed, and the network stabilization is discussed with a sufficient condition derived. Then, the network synchronization is discussed, regarding both optimal synchronizability and some general network topologies that can have the best possible synchronizability. Finally, the issue of network pinning controllability is addressed, with necessary and sufficient conditions as well as implementable algorithms derived.

It should be remarked that the complete pinning control solutions for SISO networks (17) and for MIMO networks (18) summarized in this article are established only for directed networks, based on the classical structural controllability criterion and powerful graph matching theory. For undirected networks, however, there exists no similar algorithms and characterization, which are yet to be developed. Moreover, although this article has shown that higher-order topologies contribute to optimal network synchronization, how the higher-order structures is related to the spectral criteria needs more exploration and analysis. Last but not least, how the higher-order topologies are related to the network controllability needs further investigation.

The self-contained theoretical framework of pinning control technology is promising, which will find more practical applications in network engineering, especially for IoT, logistics networks, consumer electronics marketing and transportation networks, as well as their associated sensing and communication networks, among many others.

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