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# Network synchronizability analysis: A graph-theoretic approach

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This paper addresses the fundamental problem of complex network synchronizability from a graph-theoretic approach. First, the existing results are briefly reviewed. Then, the relationships between the network synchronizability and network structural parameters (e.g., average distance, degree distribution, and node betweenness centrality) are discussed. The effects of the complementary graph of a given network and some graph operations on the network synchronizability are discussed. A basic theory based on subgraphs and complementary graphs for estimating the network synchronizability is established. Several examples are given to show that adding new edges to a network can either increase or decrease the network synchronizability. To that end, some new results on the estimations of the synchronizability of coalescences are reported. Moreover, a necessary and sufficient condition for a network and its complementary network to have the same synchronizability is derived. Finally, some examples on Chua circuit networks are presented for illustration. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2965530]

**First, two simple examples of regular symmetrical graphs are presented, to show that they have the same structural parameters (average distance, degree distribution, and node betweenness centrality) but have very different synchronizabilities. This demonstrates the intrinsic complexity of the network synchronizability problem. Several examples are then provided to show that adding new edges to a network can either increase or decrease the network synchronizability. However, it is found that for networks with disconnected complementary graphs, adding edges never decreases their synchronizability. On the other hand, adding one edge to a cycle of size $N \geq 5$ definitely decreases the network sychronizability. Then, since sometimes the synchronizability can be enhanced by changing the network structures, the question of whether the networks with more edges are easier to synchronize is addressed. It is shown by examples that the answer is no. This reveals that generally there are redundant edges in a network, which not only make no contributions to synchronization but actually may reduce the synchronizability. Moreover, three types of graph operations are discussed, which may change the network synchronizability. Further, subgraphs and complementary graphs are used to analyze the network synchronizability. Some sharp and attainable bounds are provided for the eigenratio of the network structural matrix, which characterizes the network synchronizability especially when the network's corresponding graph has cycles, bipartite graphs or product graphs as its subgraphs. Finally, by analyzing the effects of a newly added node, some results on estimating the synchronizability of coalescences are presented. It is found that a network and its complementary network have the same synchronizability when the sum of the smallest nonzero and largest eigenvalues of the corresponding graph is equal to the size of the network, i.e., the total number of its nodes. Some equilibrium synchronization examples are finally simulated for illustration.**

## I. INTRODUCTION

Systems composed of dynamical units are ubiquitous in nature, ranging from physical to technological, and to biological fields. These systems can be naturally described by networks with nodes representing the dynamical units and links representing interactions among them. The topology of such networks has been extensively studied and some common architectures such as small-world and scale-free networks have been discovered.[1,2] It has been known that these topological characteristics have strong influences on the dynamics of the structured systems, such as, epidemic spreading, traffic congestion, collective synchronization, and so on. From this viewpoint, systematically studying the network structural effects on their dynamical processes has both theoretical and practical importance.

In the study of collective behaviors of complex networks, the synchronous behavior in particular as a widely observed phenomenon in networked systems has received a great deal of attention in the past decades,[3–23] e.g., there are some recently published review articles on network synchronization in the literature.[24,25] Oscillator network models have been commonly used to characterize synchronous behaviors. In this setting, a synchonizability theorem provided by Pecora and Carroll[3] indicates that the collective synchronous behavior of a network is completely determined by the network structure. In fact, the network synchronizability is completely determined by two factors: one is the synchronized

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region related to the node dynamics and the inner linking function; the other is related to the eigenvalues of the network structural matrix. Based on this basic understanding, the synchronized region problems were studied and disconnected synchronized regions were found in Refs. 4, 22, and 23. On the other hand, the relationships between the network synchronizability (the eigenratio of the network topological matrix) and the network structural parameters were studied in detail in Refs. 6, 11, 14, 16, 20, and 21. Since the synchronizability is correlated with many topological properties, it is hard to give a direct relationship between the synchronizability and those topological properties. Donetti *et al.*[15] pointed out that a network with optimized synchronizability should have an extremely homogeneous structure, i.e., the distributions of topological properties should be very narrow.

Without considering the node and inner-linking dynamics, a network is completely determined by its outer-linking structure, i.e., the corresponding graph. Algebraic graph theory has been well studied (see Refs. 26–34, and references therein). In recent years, there are some research works[35–43] which combine the graph theory and complex networks to study network synchronization; for example, network synchronizability was analyzed by a graph-theoretic method in Ref. 35, and the effects of complementary graphs and graph operations on network synchronizability were studied in Refs. 37 and 38. It was shown[39] that network synchronizability has no relations with some statistical properties, and a theory of subgraphs and complementary graphs was established for studying network synchronization in Refs. 40 and 41. In addition, algebraic graph theory was used to study consensus problems in vehicle systems.[44,45] All these research works show that better understanding and careful manipulation of graphs can be very helpful for network synchronization.

Motivated by the above-mentioned works, this paper focuses on the graph-theoretic approach to network synchronization. Clearly, complex networks are closely related to graphs. Consider a dynamical network consisting of $N$ coupled identical nodes, with each node being an $n$-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}H(x_j), \quad i = 1,2,\ldots,N, \tag{1}$$

where $x_i=(x_{i1},x_{i2},\ldots,x_{in}) \in \mathbb{R}^n$ is the state vector of node $i$, $f(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-valued function, constant $c>0$ represents the coupling strength, $H(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^n$ is called the inner-linking function, and $A=(a_{ij})_{N\times N}$ is called the outer-coupling matrix or topological matrix, which represents the coupling configuration of the entire network. This paper only considers the case that the network is diffusively connected, i.e., the entries of $A$ satisfy

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, \quad i = 1,2,\ldots,N.$$

Further, suppose that if there is an edge between node $i$ and node $j$, then $a_{ij}=a_{ji}=-1$, i.e., $A$ is a Laplacian matrix. In this setting, if the graph corresponding to $A$ is connected, then 0 is an eigenvalue of $A$ with multiplicity 1 and all the other eigenvalues of $A$ are strictly positive, which are denoted by

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N. \tag{2}$$

The dynamical network (1) is said to achieve (asymptotical) synchronization if $x_1(t) \to x_2(t) \to \cdots \to x_N(t) \to s(t)$, as $t \to \infty$. Because of the diffusive coupling configuration, the synchronous state $s(t) \in \mathbb{R}^n$ is a solution of an individual node, i.e., $\dot{s}(t)=f(s(t))$. As shown in Ref. 3, the local stability of the synchronized solution $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$ can be determined by analyzing the so-called master stability equation,

$$\dot{\omega} = \{Df[s(t)] + \alpha DH[s(t)]\}\omega, \tag{3}$$

where $\alpha \in \mathbb{R}$, and $Df[s(t)]$ and $DH[s(t)]$ are the Jacobian matrices of functions $f$ and $H$ at $s(t)$, respectively. The largest Lyapunov exponent $L_{\max}$ of network (1), which can be calculated from system (3) and is a function of $\alpha$, is referred to as the master stability function. In addition, the region $S$ of negative real $\alpha$, where $L_{\max}$ is also negative is called the synchronized region. Based on the results of Refs. 3 and 19, the synchronized solution of dynamical network (1) is locally asymptotically stable if

$$-c\lambda_k \in S, \quad k = 2,3,\ldots,N. \tag{4}$$

It is well known that if the synchronized region $S$ is unbounded, in the form of $(-\infty, \alpha]$, then the eigenvalue $\lambda_2$ of $A$ characterizes the network synchronizability. On the other hand, if the synchronized region $S$ is bounded, in the form of $[\alpha_1, \alpha_2]$, then the eigenratio $r(A)=\lambda_2/\lambda_N$ of the network structural matrix $A$ characterizes the synchronizability. By condition (4), the larger the $|\lambda_2|$ or the $r(A)$ is, the better the synchronizability will be, depending on the types of the synchronized region. This paper focuses on the analysis of the network synchronizability index $r(A)$ for the case of bounded regions from a graph-theoretic approach.

Throughout this paper, for any given undirected graph $G$, eigenvalues of $G$ mean eigenvalues of its corresponding Laplacian matrix. For convenience, notations for graphs and their corresponding Laplacian matrices are not differentiated, and networks and their corresponding graphs are not distinguished, unless otherwise indicated.

## II. RELATIONSHIPS BETWEEN THE NETWORK SYNCHRONIZABILITY AND NETWORK STRUCTURAL PARAMETERS

The relationships between network synchronizability index $r(A)$ and network structural characteristics such as average distance, node betweenness, degree distribution, clustering coefficient, etc. have been well studied.[6,11,14–16] However, there are also references [36,37,39] showing that for some special networks the synchronizability has no direct relations with the network structural parameters, at least some network characteristics alone cannot determine the synchronizability. This shows the complexity of the problem.

For a given degree sequence, a construction method for finding two types of graphs was given in Ref. 36, where one resultant graph has large $\lambda_2$ and $r=\lambda_2/\lambda_N$, i.e., good synchronizability, and the other has small $\lambda_2$ and $r=\lambda_2/\lambda_N$, i.e., bad

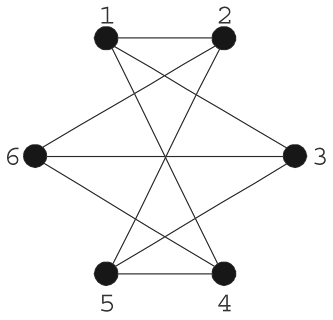

FIG. 1. Graph $G_1$.

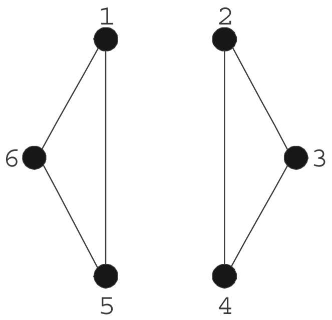

FIG. 3. Graph $G_1^c$.

synchronizability. This shows that the degree sequence by itself is not sufficient to determine the synchronizability.

Further, two simple graphs $G_1$ and $G_2$ on six nodes, shown in Figs. 1 and 2, were presented in Ref. 37, where $G_1$ is a typical bipartite graph with many interesting properties.[26,27] These two graphs have the same degree sequence, where the degree of every node is 3; the same average distance $\frac{7}{5}$; and the same node betweenness centrality 2,[16] but the synchronizability of network $G_1$ is better than that of network $G_2$: $\lambda_2(G_1)=3, r(G_1)=0.5$; $\lambda_2(G_2)=2, r(G_2)=0.4$.

*Remark 1:* Although only two six-node graphs are shown in Figs. 1 and 2, they can be easily generalized to graphs of size $N=2n$ with the same conclusion. Suppose that graph $G_1$ is bipartite, which means that it contains two sets of nodes, each set containing $n$ isolated nodes, and each node in one set connects to all the nodes in the other set. Graph $G_2$ is composed of two fully connected subgraphs, each has size $n$, where each node in one subgraph to connected by one edge corresponding node in the other subgraph. In this case, the smallest nonzero eigenvalue and the largest eigenvalue of $G_1$ are $N/2$ and $N$, respectively, so $r(G_1)=\frac{1}{2}$. On the other hand, the smallest nonzero eigenvalue and the largest eigenvalue of $G_2$ are 2 and $(N/2)+2$, respectively,[27,34] with $r(G_2)=4/(N+4)\rightarrow 0$ as $N\rightarrow +\infty$. Therefore, these two graphs have the same structural parameters but have very different synchronizabilities.

*Remark 2:* It was pointed out in Ref. 39 that small local changes in the network structure can sensitively affect the graph eigenvalues relevant to the synchronizability, while some basic statistical network properties such as degree distribution, average distance, degree correlation, and clustering coefficient remain essentially unchanged. Together with the results discussed above, it is clear that more work is needed in order to unveil the effects of network structural parameters on the network synchronizability. According to the existing results, the statistical properties can distinguish some types of networks with good or bad synchronizabilities, but there are also special networks for which the statistical properties fail to tell any difference between their synchronizabilities. Therefore, more work is needed in order to truly understand the essence of the complex network synchronizability.

## III. EFFECTS OF EDGE ADDITION AND COMPLEMENTARY GRAPHS

Consider a task of enhancing $\lambda_2$ and $r$ by adding some edges to a graph. For this purpose, the following result is useful.[34]

*Lemma 1:* For any given connected undirected graph $G$ of size $N$, its nonzero eigenvalues indexed as in Eq. (2) grow monotonically with the number of added edges, that is, for any added edge $e$, $\lambda_i(G+e)\geq\lambda_i(G)$, $i=1,\ldots,N$.

Therefore, if only the change of the eigenvalue $\lambda_2$ is concerned, adding edges never decreases the synchronizability. However, for the eigenratio $r=\lambda_2/\lambda_N$, this is not necessarily true. For example, adding an edge between node 1 and node 3 in graph $G_2$ (Fig. 2), denoted by $e\{1,3\}$, leads to a new graph $G_2+e\{1,3\}$, whose eigenvalues are 0, 2.2679, 3, 4, 5 and 5.7321. Thus, $r(G_2+e\{1,3\})=0.3956$ is even smaller than the original $r(G_2)=0.4$. This means that the synchronizability of network $G_2+e\{1,3\}$ is worse than that of network $G_2$. Adding a new edge between node 1 and node 4 instead, one gets $r(G_2+e\{1,3\})<r(G_2+e\{1,3\}+e\{1,4\})=0.3970<r(G_2)$. This means that, the synchronizability of network $G_2+e\{1,3\}+e\{1,4\}$ is better than $G_2+e\{1,3\}$, but

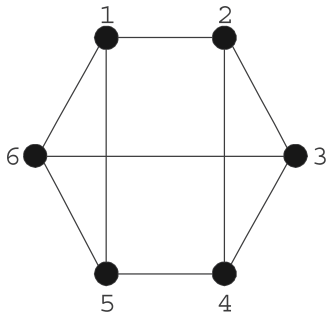

FIG. 2. Graph $G_2$.

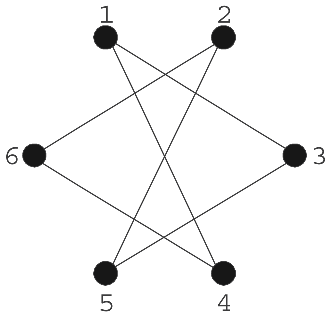

FIG. 4. Graph $G_2^c$.

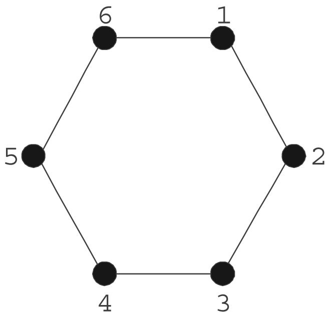

FIG. 5. Graph $C_6$.

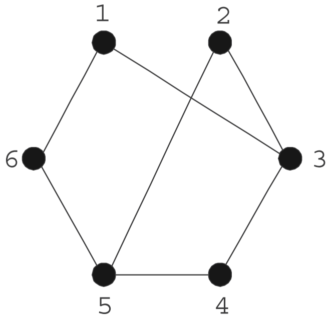

FIG. 7. Graph $C_{6o}$.

still worse than $G_2$. Therefore, by adding edges, the network synchronizability may increase or decrease. In searching for a condition under which adding edges may enhance the synchronizability, it was found[37] that for networks with disconnected complementary graphs, adding edges never decreases their synchronizability. For a given graph $G$, the complement of $G$, denoted by $G^c$, is the graph containing all the nodes of $G$ and all the edges that are not in $G$. For example, the complementary graphs of $G_1$ and $G_2$ in Figs. 1 and 2 are shown in Figs. 3 and 4, respectively. The following results are needed for discussing complementary graphs.[27,34]

*Lemma 2:* For any given graph $G$, the following statements hold:

(i) $\lambda_N(G)$, the largest eigenvalue of $G$, satisfies $\lambda_N(G) \leq N$.
(ii) $\lambda_N(G)=N$ if and only if $G^c$ is disconnected.
(iii) If $G^c$ is disconnected and has exactly $q$ connected components, then the multiplicity of $\lambda_N(G)=N$ is $q-1$, $1 \leq q \leq N$.
(iv) $\lambda_i(G^c)+\lambda_{N-i+2}(G)=N$, $2 \leq i \leq N$.

The complementary graph of $G_1$, shown in Fig. 3, is disconnected. The largest eigenvalue of $G_1$ is 6, which remains the same when the graph receives additional edges. Hence, combining with Lemmas 1 and 2, one concludes that the synchronizability of all the networks built on graph $G_1$ never decreases with adding edges, as detailed in Ref. 37.

On the other hand, under what conditions will adding one edge decrease the synchronizability? It was found[41] that adding one edge to a given cycle with $N$ ($N \geq 5$) nodes definitely decreases the synchronizability. To show this, the following lemma for eigenvalues of cycles is needed.[34,36]

*Lemma 3:* For any cycle $C_N$ with $N$ ($\geq 4$) nodes, its eigenvalues are given by $\mu_1, \ldots, \mu_N$ [not necessarily ordered as in Eq. (2)] with $\mu_1=0$ and

$$\mu_{k+1} = 3 - \frac{\sin\left(\frac{3k\pi}{N}\right)}{\sin\left(\frac{k\pi}{N}\right)}, \quad k=1,\ldots,N-1.$$

By the above lemma, one can get the following result for cycles.[41]

**Theorem 1:** For any cycle $C_N$ with $N \geq 4$ nodes, adding one edge will never enhance but possibly decrease its synchronizability $r(C_N)$; specifically, $r(C_4+e)=r(C_4)$ and $r(C_N+e) < r(C_N)$.

For example, for $N \geq 5$ adding one edge to cycle $C_6$ with 6 nodes definitely decreases the synchronizability, as shown in Figs. 5 and 6, where $r(C_6)=\frac{1}{4}=0.25$, $r(C_6+e\{1,3\})=\frac{1}{4.4142}=0.2265 < r(C_6)$.

However, the synchronizability may be enhanced by changing the network structure after edge addition. For example, one can change $C_6+e\{1,3\}$ to $C_{6o}$ as shown in Fig. 7, giving $r(C_{6o})=1.2679/4.7231=0.2684$, which is better than $r(C_6)$ (see Ref. 41 for details).

Following above discussions, in optimizing network structures another interesting question is whether networks with more edges are easier to synchronize. It was found[41] that the answer is negative.

*Lemma 4:* For any graph $G$ with 16 edges on 10 nodes, its eigenratio is bounded by $r(G) < \frac{2}{5}$.

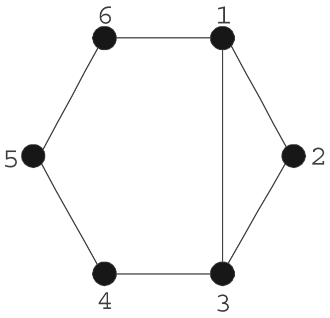

FIG. 6. Graph $C_6+e\{1,3\}$.

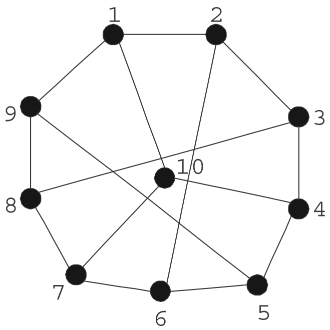

FIG. 8. Graph $\Gamma_1$, $r(\Gamma_1)=\frac{2}{5}$.

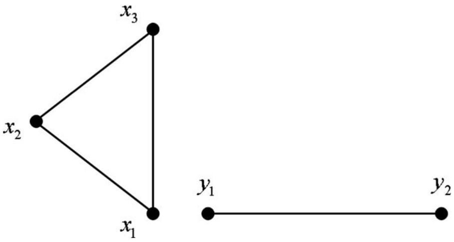

FIG. 9. Two graphs $G=C_3$ and $H=P_2$.

Lemma 4 shows that there is not a graph $G$ with 16 edges on 10 nodes whose synchronizability is $r(G)\geq\frac{2}{5}$. However, there does exist a graph $\Gamma_1$ with 15 edges on 10 nodes whose synchronizability is $r(\Gamma_1)=\frac{2}{5}$ (see Fig. 8), consistent with the result of Ref. 15. This clearly shows that networks with more edges are not necessarily easier to synchronize. In fact, by the optimal result of Ref. 15, $r=\frac{2}{5}$ is the optimal synchronizability for graphs with 15 edges on 10 nodes. For any graph $G$ with 16 edges on 10 nodes, if both $G$ and $G^c$ have even cycles (i.e., cycles with even-degree nodes), then by the result of Ref. 40, $r(G)\leq\frac{2}{6}=\frac{1}{3}$. Therefore, adding one more edge definitely decreases the synchronizability in this case.

*Remark 3:* For simplicity, this section only discusses some six-node graphs and a ten-node graph. Clearly, for the edge-addition and structure-changing effects on the network synchronizability, one can generalize the above discussions to cycles with $N$ nodes, as shown in Figs. 6 and 7. It is still an interesting topic for further research to find more examples, as discussed in Lemma 4, to show the effects of optimizing the network structure by adding or removing edges.

## IV. EFFECTS OF GRAPH OPERATIONS

The effects of graph operations, such as product, join, coalescence, etc., on network synchronization were studied in Ref. 38.

First, consider the product operation. Consider two nonempty graphs $G(V_1,E_1)$ and $H(V_2,E_2)$. Their Cartesian product graph $G\times H$ is defined, as in Refs. 27 and 38, to be a graph obtained as follows:

(i) the set of nodes of $G\times H$ is the Cartesian product $V_1\times V_2$; and
(ii) any two nodes $(u,u')$ and $(v,v')$ are adjacent in $G\times H$ if and only if

(ii.1) $u=v$ and $u'$ is adjacent with $v'$ in $H$; or
(ii.2) $u'=v'$ and $u$ is adjacent with $v$ in $G$.

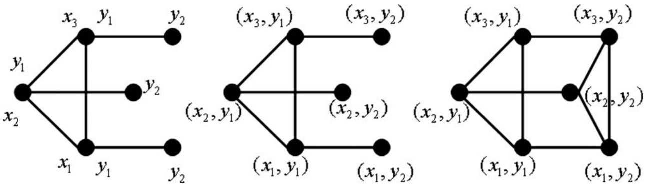

FIG. 10. Generation of a product graph.

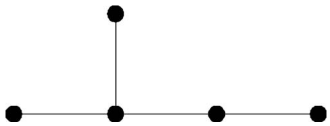

FIG. 11. Graph $\Gamma_2$.

Take the two graphs $G=C_3$ and $H=P_2$ in Fig. 9 as an example. First, establish the Cartesian product space of their nodes, as shown in Fig. 10(a). Then, label the new nodes in the product space, as shown in Fig. 10(b). Finally, connect some of its node pairs in the product space by following (ii) above, as shown in Fig. 10(c), which is isomorphic to graph $G_2$ shown in Fig. 2, as can be verified by folding the node $(x_2,y_2)$ to the outside of the square, namely, $G_2=C_3\times P_2$. For two nonempty graphs $G$ and $H$, it is known that $\lambda_2(G\times H)=\min\{\lambda_2(G),\lambda_2(H)\}$ and $\lambda_{\max}(G\times H)=\lambda_{\max}(G)+\lambda_{\max}(H)$, so $r(G\times H)<\min\{r(G),r(H)\}$.

Then, consider the join operation. Let $G(\mathcal{V}_1,\mathcal{E}_1)$ and $H(\mathcal{V}_2,\mathcal{E}_2)$ be two graphs on disjoint sets of $n$ and $m$ nodes, respectively. Their disjoint union $G+H$ is the graph $G+H=(\mathcal{V}_1\cup\mathcal{V}_2,\mathcal{E}_1\cup\mathcal{E}_2)$, and their joint $G*H$ is the graph on $n+m$ nodes obtained from $G+H$ by inserting new edges from each node of $G$ to each node of $H$, as can be easily imagined.[27,28,38] It is well known that the largest eigenvalue of the graph $G*H$ is $\lambda_{\max}(G*H)=n+m$, and the smallest nonzero eigenvalue is $\lambda_2(G*H)=\min\{\lambda_2(G)+m,\lambda_2(H)+n\}\geq\lambda_2(G)+\lambda_2(H)$. Therefore, $r(G*H)\geq 0.5$. For example, graph $G_1$ in Fig. 1 can be viewed as $O_3*O_3$, where $O_3$ is the graph containing three isolated nodes, which has $r(G_1)=0.5$.

Finally, consider the coalescence operation. A coalescence of two graphs $G$ and $H$, denoted by $G\bigcirc H$, is a graph obtained from the disjoint union $G+H$ by identifying a node of $G$ with a node of $H$, as can be easily imagined. The coalescence generally does not yield a unique graph.[38] It was shown in Ref. 38 that $\lambda_2(G\bigcirc H)\leq\min\{\lambda_2(G),\lambda_2(H)\}$ and $\lambda_{\max}(G\bigcirc H)\geq\max\{\lambda_{\max}(G),\lambda_{\max}(H)\}$, so $r(G\bigcirc H)\leq\min\{r(G),r(H)\}$. For example, graph $\Gamma_2$ in Fig. 11 can be viewed as the coalescence of chain $P_4$ in Fig. 12 and chain $P_2$ in Fig. 9, and indeed $r(\Gamma_2)\leq r(P_4)$. Obviously, chain $P_5$ can also be obtained as the coalescence of $P_4$ and $P_2$.

## V. THEORY OF SUBGRAPHS AND COMPLEMENTARY GRAPHS FOR ESTIMATING THE NETWORK SYNCHRONIZABILITY

The theory of subgraphs and complementary graphs are used to estimate the network synchronizability in Ref. 40. This section briefly discusses such estimation problems.

For a given graph $G(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ denote the set of nodes and the set of edges of $G$, respectively. A graph $G_1$ is called an induced subgraph of $G$, if the node set $\mathcal{V}_1$ of $G_1$ is a subset of $\mathcal{V}$ and the edges of $G_1$ are all edges among

FIG. 12. Chain $P_4$.

nodes $\mathcal{V}_1$ in $\mathcal{E}$. The complementary graph of $G$, denoted by $G^c$, is the graph containing all the nodes of $G$ and all the edges that are not in $G$.

The following lemma is needed for estimating the largest Laplacian eigenvalue.[26]

*Lemma 5:* For any given connected graph $G$ of size $N$, its largest eigenvalue $\lambda_N$ satisfies $\lambda_N \geq d_{\max}+1$, with equality if and only if $d_{\max}=N-1$, where $d_{\max}$ is the maximum degree of $G$.

Combining Lemmas 2 and 5, one can obtain the following corollaries.[40]

*Corollary 1:* For any given graph $G$ of size $N$, if its second smallest eigenvalue equals its smallest node degree, i.e., $\lambda_2(G)=d_{\min}(G)$, then either $G$ or $G^c$ is disconnected; if $\lambda_2(G) > d_{\min}(G)$, then $G$ is a complete graph; if both $G$ and $G^c$ are connected, then $\lambda_2(G) < d_{\min}(G)$.

*Corollary 2:* For any given connected graph $G$ and even its induced subgraph $G_1$, one has $\lambda_{\max}(G) \geq \lambda_{\max}(G_1)$, so the synchronizability index of $G$ satisfies

$$r(G) \leq \frac{d_{\min}(G)}{\lambda_{\max}(G_1)};$$

if both $G$ and $G^c$ are connected, then

$$r(G) < \frac{d_{\min}(G)}{d_{\max}(G)+1}.$$

Since subgraphs have less nodes, this corollary is useful when a graph $G$ contains some subgraphs whose largest eigenvalues can be easily obtained.

*Corollary 3:* For a given graph $G$, if the largest eigenvalue of $G^c$ is $\lambda_{\max}=d_{\max}(G^c)+\alpha$, then $\lambda_2(G) =d_{\min}(G)+1-\alpha$. Consequently, the synchronizability index of $G$ satisfies

$$r(G) = \frac{d_{\min}(G)+1-\alpha}{\lambda_{\max}(G)} \leq \frac{d_{\min}(G)+1-\alpha}{d_{\max}(G)+1}.$$

By Lemma 5, generally $\alpha \geq 1$, so the bound in Corollary 3 is better than the one in Corollary 2.

Subgraphs are further discussed below. First, consider graphs having cycles as subgraphs.

**Theorem 2:** For any given graph $G$, suppose $H$ is its induced subgraph composing of all nodes of $G$ with the maximum degree $d_{\max}(G)$, and $H^c$ is the induced subgraph of $G^c$ composed of all nodes of $G^c$ with the maximum degree $d_{\max}(G^c)$. Then, if both $H$ and $H^c$ have even cycles (i.e., cycles with even number of nodes) as induced subgraphs, then $\lambda_{\max}(G) \geq d_{\max}(G)+2$ and $\lambda_{\max}(G^c) \geq d_{\max}(G^c)+2$. Consequently, the synchronizability index of $G$ satisfies

$$r(G) \leq \frac{d_{\min}(G)-1}{d_{\max}(G)+2}.$$

The smallest even cycle is cycle $C_4$, and its complementary graph $C_4^c$ has two separated edges. $C_4$ and $C_4^c$ are very important in graph theory.[27] A graph has a $C_4$ as an induced subgraph if and only if $G^c$ has $C_4^c$ as an induced subgraph. For example, consider graph $G_2$ in Fig. 2. Its complementary graph is $G_2^c$ in Fig. 4, which is equivalent to $C_6$ in Fig. 5. Testing the eigenvalues of $G_2$ and its synchronizability, one finds that they attain the exact bounds given in Theorem 2.

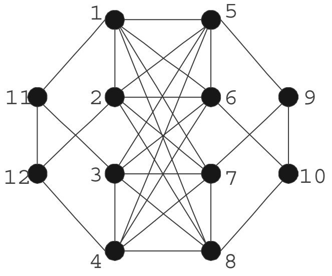

FIG. 13. Graph $\Gamma_3$.

For the case of odd cycles as subgraphs, see Ref. 40.

Next, consider graphs having bipartite graphs as subgraphs. A bipartite graph generated by graphs $G$ and $H$ is the joint $G * H$ of $G$ and $H$, as discussed in the previous section (see Refs. 27 and 28 for more details about bipartite graphs).

**Theorem 3:** Let $H$ be a subgraph of a given graph $G$ containing all nodes of $G$ with the same degree $d$. Suppose $H$ contains a bipartite subgraph $H_1 * H_2$, and the numbers of nodes of $H_1$ and $H_2$ are $n_1$ and $n_2$, respectively. Then, the largest eigenvalue of $G$ satisfies $\lambda_{\max}(G) \geq d+n_1+n_2 -d_{\max}(H_1 * H_2)$.

For example, consider graph $\Gamma_3$ shown in Fig. 13. It can be easily verified that $\Gamma_3$ has a bipartite graph $H$ as its subgraph, which is composed of all nodes with degree 6 from $\Gamma_3$. The largest eigenvalue of this bipartite graph is 8. So, by Theorem 3, $\lambda_{\max}(\Gamma_3) \geq 9$. On the other hand, the maximum degree of the complementary graph $\Gamma_3^c$ of $\Gamma_3$ is 8. And the nodes with degree 8 in $\Gamma_3^c$ form a cycle $C_4$. By Theorem 2, the smallest nonzero eigenvalue of $\Gamma_3$ satisfies $\lambda_2(\Gamma_3) \leq d_{\min}(\Gamma_3)-1=2$. So, $r(\Gamma_3) \leq \frac{2}{9}$. Simply computing the Laplacian eigenvalues of $\Gamma_3$, one obtains $\lambda_2=1.7251$ and $\lambda_{\max} =9.2749$. Consequently, $r(\Gamma_3)=1.7251/9.2749 \approx 0.176$. Therefore, the theorems presented in this section successfully give the upper integer of the largest eigenvalue and the lower integer of the smallest nonzero eigenvalue of $\Gamma_3$.

Then, consider graphs having product graphs as subgraphs, where the concept of product graphs was introduced in the previous section.

**Theorem 4:** For a given graph $G$, let $H$ be a subgraph of $G$ containing all nodes of $G$ with the same degree $d$. Suppose $H$ contains a product graph $H_1 \times H_2$ as its subgraph. Then, the largest eigenvalue of $G$ satisfies $\lambda_{\max}(G) \geq d +\lambda_{\max}(H_1)+\lambda_{\max}(H_2)-d_{\max}(H_1 \times H_2)$.

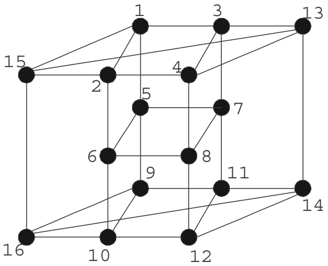

FIG. 14. Graph $\Gamma_4$.

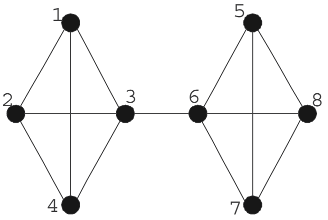

FIG. 15. Graph $\Gamma_5$.

For example, consider graph $\Gamma_4$ in Fig. 14. Obviously, all nodes of $\Gamma_4$ have degree 4. And $\Gamma_4$ has a product graph $C_4 \times P_3$ (nodes 1–12) as its subgraph, where $P_3$ denotes a chain with three nodes. The largest eigenvalue of this product subgraph is 7. So, by Theorem 4, $\lambda_{\max}(\Gamma_4) \geq 7$. On the other hand, the complementary graph $\Gamma_4^c$ of $\Gamma_4$ has a bipartite graph as its subgraph, which is composed of nodes 1–4 and nodes 9–12. By Theorem 3, the largest eigenvalue of $\Gamma_4^c$ satisfies $\lambda(\Gamma_4^c) \geq d_{\max}(\Gamma_4^c)+3$. Thus, by Corollary 3, the smallest nonzero eigenvalue of $\Gamma_4$ satisfies $\lambda_2(\Gamma_4) \leq d_{\min}(\Gamma_4)-2=2$. So, $r(\Gamma_4) \leq \frac{2}{7}$. Simply computing the Laplacian eigenvalues of $\Gamma_4$, one obtains $\lambda_2=1.2679$ and $\lambda_{\max}=7.4142$. Consequently, $r(\Gamma_4)=1.2679/7.4142 \approx 0.171$. Similar to the above example, the corresponding theorems proved in this section successfully give the upper integer of the largest eigenvalue and the lower integer of the smallest nonzero eigenvalue of $\Gamma_4$.

Finally, consider the maximum disconnected subgraph. Given a graph $G$ of size $N$, suppose $H$ is an induced subgraph of $G$, has size $n_1$, and is disconnected. $H$ is called a maximum disconnected subgraph, if the node number of any other disconnected subgraph of $G$ is less than or equal to $n_1$.

**Theorem 5:** For a given connected graph $G$ of size $N$, if the node number of its maximum disconnected subgraph is $n_1$, then the smallest nonzero eigenvalue of $G$ satisfies $\lambda_2 \leq N-n_1$. Consequently,

$$r(G) \leq \frac{N-n_1}{d_{\max}(G)+1}.$$

For example, consider graph $\Gamma_5$ shown in Fig. 15. By deleting node 3 or 6, one can verify that the node number of its maximum disconnected subgraph is 7. So, $\lambda_2(\Gamma_5) \leq 1$. Combining Lemma 5 and Theorem 5, one obtains $r(\Gamma_5) < \frac{1}{5}$ (see Ref. 40 for details).

In fact, for the graphs shown in Fig. 15, one can give a more precise estimation for their smallest nonzero eigenvalues by using the eigenvector method.[27] For example, the graph $(\Gamma_5-e\{3,6\})^c$, i.e., the complementary graph of $\Gamma_5-e\{3,6\}$, is a bipartite graph, so the eigenvector corresponding to its largest eigenvalue is $v=\left(\frac{1}{\sqrt{8}},\frac{1}{\sqrt{8}},\frac{1}{\sqrt{8}},\frac{1}{\sqrt{8}},\frac{-1}{\sqrt{8}},\frac{-1}{\sqrt{8}},\frac{-1}{\sqrt{8}},\frac{-1}{\sqrt{8}}\right)^T$.[27] Suppose the Laplacian matrix of $\Gamma_5$ is $L_5$ (the order of the nodes is as shown in Fig. 15). Then, $v^T L_5 v=0.5$. By the Releigh-quotient theory of algebraic graph theory,[34] one has $\lambda_2(\Gamma_5) \leq 0.5$. By simple computation, one obtains $\lambda_2(\Gamma_5) \approx 0.3542$. Obviously, this new estimation for $\lambda_2$ is sharper than that given by Theorem 5.

It is well known that graphs shown in Fig. 15 have large node and edge betweenness centralities, therefore have bad synchronizabilities in general. Based on the theory of subgraphs, complementary graphs and eigenvectors, this section has given an explanation as why such graphs indeed have bad synchronizabilities in general.

## VI. MORE RESULTS ON THE SYNCHRONIZABILITY OF COALESCENCES

Given a connected graph $G$, adding a new node $g$ and a new edge to connect $G$ and $g$ will form a new graph $G+g$, which can be viewed as the coalescence of $G$ and $P_2$ (Fig. 9) as discussed in Sec. IV. Now, consider the synchronizability of $G+g$. By the result of Ref. 38, $r(G+g) \leq r(G)$. In fact, one can get more results for this synchronizability estimation, as shown by the following theorem.

**Theorem 6:** Given a connected graph $G$, one has

$$\lambda_{\max}(G) \leq \lambda_{\max}(G \bigcirc P_2) < \lambda_{\max}(G) + 1 + \frac{1}{\lambda_{\max}(G)}$$

and $r(G \bigcirc P_2) \leq 1/\lambda_{\max}(G)$. If the complementary graph of $G \bigcirc P_2$ is connected, then $r(G \bigcirc P_2) < 1/\lambda_{\max}(G)$.

*Proof:* From the results of Sec. IV, $\lambda_{\max}(G) \leq \lambda_{\max}(G \bigcirc P_2)$ holds obviously. On the other hand, without loss of generality, suppose that the Laplacian matrix of $G \bigcirc P_2$ is

$$L=\begin{pmatrix} 1 & -1 & 0 \\ -1 & d+1 & L_{12} \\ 0 & L_{12}^T & L_{22} \end{pmatrix},$$

where $L_1=\begin{pmatrix} d & L_{12} \\ L_{12}^T & L_{22} \end{pmatrix}$ is the Laplacian matrix of $G$. By the Schur complement, one has

$$\left[\lambda_{\max}(G) + 1 + \frac{1}{\lambda_{\max}(G)}\right] I - L > 0,$$

if and only if

$$\begin{pmatrix} \Lambda(G) - d - \dfrac{1}{\Lambda(G)} & L_{12} \\ L_{12}^T & (\Lambda(G)+1)I - L_{22} \end{pmatrix} > 0,$$

where $\Lambda(G)=\lambda_{\max}(G)+1/\lambda_{\max}(G)$. Since $\lambda_{\max}(G)$ is the largest eigenvalue of $L_1$, the above inequality holds obviously. Hence, $\lambda_{\max}(G \bigcirc P_2) < \lambda_{\max}(G)+1+1/\lambda_{\max}(G)$. Further, by Corollary 2 in Sec. V, the last part of the theorem can be verified. ■

In addition, in Theorems 5 and 6, one can give an estimation of the smallest nonzero eigenvalue of $G \bigcirc H$.

**Theorem 7:** Given two connected graphs $G$ and $H$, one has $\lambda_2(G \bigcirc H) \leq 1$.

*Proof:* Let the node numbers of $G$ and $H$ be $n$ and $m$, respectively, and $g$ be the identifying node of $G$ in the coalescence of $G$ and $H$. By deleting node $g$, the maximum disconnected subgraph $H_1$ of $G \bigcirc P_2$ will have $n+m-1$ nodes. Here, by Theorem 6, $\lambda_2(G \bigcirc H) \leq 1$. ■

By Theorems 6 and 7, one knows that generally the synchronizability of the coalescences of graphs is weak. Let $K_N$ denote a complete graph of size $N$. Consider the coalescence of $K_N$ and $P_2$ (shown in Fig. 9). Obviously, the complemen-

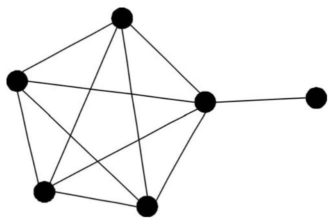

FIG. 16. Graph $\Gamma_6$.

tary graph of $K_N \bigcirc P_2$ is disconnected, so $\lambda_{\max}(K_N \bigcirc P_2)=N+1$. Further, by Lemma 2, one knows that $\lambda_2(K_N \bigcirc P_2)=1$, so the synchronizability index is $r(K_N \bigcirc P_2)=1/N+1$. But, on the other hand, $r(K_N)=1$. Therefore, in this case, a newly added node severely decreases the synchronizability. For example, $r(\Gamma_6)=\frac{1}{6}$, where $\Gamma_6$ is shown in Fig. 16. In addition, graph $\Gamma_5$ in Fig. 15 can be viewed as a coalescence $K_4 \bigcirc P_2 \bigcirc K_4$, which also has weak synchronizability. In order to improve the synchronizability of such graphs, their graph structures must be modified.[24]

In what follows, consider the coalescences of star-shaped graphs and $P_2$ (shown in Fig. 9). Let $S_N$ denote a star-shaped graph of size $N$. For $S_6$, shown in Fig. 17, it can be viewed as a coalescence of $S_5$ and $P_2$ by identifying the central node of $S_5$ and one node of $P_2$. In this case, $r(S_6)=\frac{1}{6}<r(S_5)=\frac{1}{5}$. Compared with graph $K_N \bigcirc P_2$, one added node does not result in severe decrease of the synchronizability of $S_6$. As another example, $\Gamma_8$ in Fig. 18 can also be viewed as a coalescence of $S_5$ and $P_2$. By Lemma 5 and Corollary 1, $\lambda_{\max}(\Gamma_8)>5$ and $\lambda_2(\Gamma_8)<1$. On the other hand, $\Gamma_8$ can also be viewed as a coalescence of $P_4$ and $S_3$, so $\lambda_2(\Gamma_8)\leq\lambda_2(P_4)=0.5858$. Therefore, $r(\Gamma_8)<0.5858/5$. By simple computation, one obtains $r(\Gamma_8)=0.4859/5.0861$, which is worse than $r(S_6)$. From these examples, one can see that the effects of adding one node and one edge (i.e., coalescing $P_2$ to a given graph) on the synchronizability are very different for different graphs. How to reconstruct a new graph by adding a new node to surely improve the synchronizability is still an interesting open problem.

## VII. CONDITIONS FOR A NETWORK AND ITS COMPLEMENTARY NETWORK TO HAVE THE SAME SYNCHRONIZABILITY

From the above discussions, one can see that complementary graph is very important for the study of the network

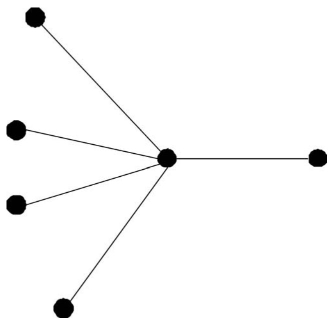

FIG. 17. Graph $\Gamma_7=S_6$.

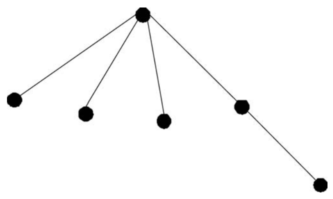

FIG. 18. Graph $\Gamma_8$.

synchronization. Therefore, it is interesting to consider the synchronizabilities of a network and its complementary network simultaneously.

**Theorem 8:** For a given connected graph $G$ of size $N$, if both $G$ and $G^c$ are connected, then $G$ and $G^c$ have the same synchronizability if and only if $\lambda_2(G)+\lambda_N(G)=N$, i.e., $\lambda_2(G)=\lambda_2(G^c)$ and $\lambda_N(G)=\lambda_N(G^c)$.

*Proof:* In Lemma 2, $G$ and $G^c$ have the same synchronizability, i.e.,

$$\frac{\lambda_2(G)}{\lambda_N(G)}=\frac{N-\lambda_N(G)}{N-\lambda_2(G)}.$$

So, $N\lambda_2(G)-\lambda_2^2(G)=N\lambda_N(G)-\lambda_N^2(G)$, i.e., $N[\lambda_N(G)-\lambda_2(G)]=[\lambda_N(G)+\lambda_2(G)][\lambda_N(G)-\lambda_2(G)]$. Since $G^c$ is connected, $\lambda_N(G)-\lambda_2(G)\neq 0$. Therefore, $\lambda_N(G)+\lambda_2(G)=N$. Thus, Lemma 2 leads to the conclusion.

It is well known that the complementary graphs of chain $P_4$ (shown in Fig. 12) and cycle $C_5$ are the same as $P_4$ and $C_5$, respectively. Hence, $P_4$ and $C_5$ satisfy the condition given in Theorem 8. In addition, graph $C_{6o}$ shown in Fig. 7 satisfies $\lambda_2(G)+\lambda_6(G)=6$, so $C_{6o}$ and its complementary graph $C_{6o}^c$ have the same synchronizability. Further, delete the edge $e\{3,6\}$ from graph $G_2$ in Fig. 2, and denote the resultant graph by $G_2-e\{3,6\}$. Then, by simple computation, one can verify that $G_2-e\{3,6\}$ and its complementary graph have the same synchronizability, where $r(G_2-e\{3,6\})=0.2$. From these examples, one knows that there do exist many graphs which have the same synchronizability as their complementary graphs.

Next, consider the effects of edge-adding on the synchronizabilities of a graph and its complementary graph. First, consider graph $C_{6o}+e\{3,5\}$, where $C_{6o}$ is given as in Fig. 7. By simple computation, one obtains that

$$r(C_{6o}+e\{3,5\})=\frac{1.2679}{5.4142}<r(C_{6o})=\frac{1.2679}{4.7321},$$

and obviously

$$r((C_{6o}+e\{3,5\})^c)=\frac{0.5858}{4.7321}<r(C_{6o}^c)=\frac{1.2679}{4.7321}.$$

This means that adding an edge to $C_{6o}$ decreases the synchronizabilities of $C_{6o}$ and $C_{6o}^c$ simultaneously. On the other hand, by adding the edge $e\{3,6\}$ to $G_2-e\{3,6\}$, one can find that this added edge increases the synchronizabilities of $G_2-e\{3,6\}$ and $(G_2-e\{3,6\})^c$ simultaneously. Of course, there are other examples in which adding one edge increases the synchronizability of either the original graph or the complementary graph. This shows the complexity in this kind of edge-adding problems.

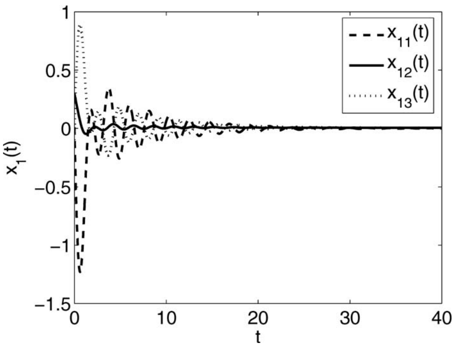

FIG. 19. Network on $G_1$.

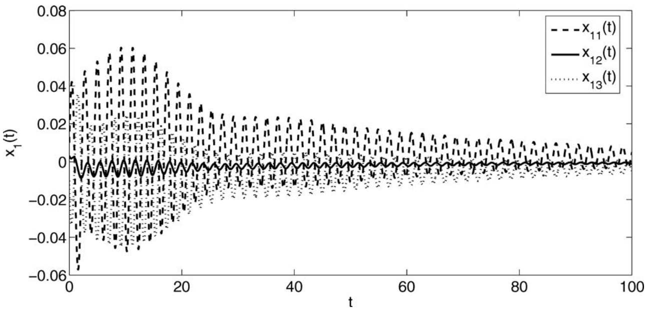

FIG. 20. Network on $C_6+e\{1,3\}$.

It should be pointed out that although Theorem 8 gives a condition for a graph and its complementary graph to have the same synchronizability, the eigenvalue conditions therein are generally not easy to test. It is more interesting to give a new condition for this problem which is solely based on graph characteristics. This leaves an open problem for future research.

## VIII. EQUILIBRIUM SYNCHRONIZATION OF A CHUA CIRCUIT NETWORK

Consider network (1) consisting of the third-order smooth Chua's circuits,[46] in which each node is described by

$$\begin{aligned} \dot{x}_{i1} &= -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha(a x_{i1}^3 + b x_{i1}), \\ \dot{x}_{i2} &= k x_{i1} - k x_{i2} + k x_{i3}, \\ \dot{x}_{i3} &= -k\beta x_{i2} - k\gamma x_{i3}. \end{aligned} \tag{5}$$

Linearizing Eq. (5) about its zero equilibrium gives

$$\dot{x}_i = F x_i, \quad F = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{pmatrix}, \tag{6}$$

where $x_i=(x_{i1},x_{i2},x_{i3})^T$.

Take $k=1, \alpha=-0.1, \beta=-1, \gamma=1, a=1, b=-25$. Then, $F$ is stable, i.e., the node system (5) is locally stable about zero. Further, take the inner linking matrix

$$H = \begin{pmatrix} 0.8348 & 9.6619 & 2.6591 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -7.0837 & -0.8042 \end{pmatrix}.$$

Then, by simple computation, one can verify that $F+\alpha H$ has two disconnected stable regions: $S_1=[-0.01,0]$ and $S_2=[-3.3,-0.74]$, so the entire synchronized region is $S=S_1\cup S_2$. Moreover, let $N=6$ and the outer coupling matrix $A$ be equal to the Laplacian matrix $G_1$ shown in Fig. 1. According to the eigenvalues of $G_1$ given in Sec. II, one may take the coupling strength $c=\frac{1}{2.1}$. Then, for all eigenvalues of $G_1$, one has $c\lambda_i \in S_2$, $i=1,\ldots,6$. By condition (4), network (1) specified with the above data achieves synchronization. Figure 19 shows the state of node 1 in this network. The other nodes behave similarly.

With the above node dynamics and inner linking function, similar synchronization behaviors can be observed on other graphs with suitable coupling strengths $c$, for example, on $G_2$ $(c=\frac{1}{2.1})$, $G_2^c$ $(c=\frac{1}{1.3})$, $C_{6o}$ $(c=\frac{1}{1.5})$, and $C_6+e\{1,3\}$ $(c=\frac{1}{1.345})$. Figure 20 shows a similar synchronization result on graph $C_6+e\{1,3\}$ (shown in Fig. 6). Obviously, the network on $G_1$ synchronizes much faster than that on $C_6+e\{1,3\}$. This also demonstrates that the synchronizability of $G_1$ is better than that of $C_6+e\{1,3\}$. In fact, the region of the coupling strength $c$ is very narrow for achieving synchronization of $C_6+e\{1,3\}$.

However, for the outer coupling matrix $G_2-e\{3,6\}$ or $(G_2-e\{3,6\})^c$, for any coupling strength $c\in[0.003,+\infty)$, condition (4) does not hold. Therefore, for the above case of node equation, inner coupling matrix and coupling strength, the network built on $G_2-e\{3,6\}$ does not synchronize at all. Figure 21 shows the state of node 1 in this network with $c=\frac{1}{1.8}$; the other nodes behave similarly.

For simplicity, this section only discusses some synchronization and nonsynchronization behaviors of Chua's circuit networks on six-node graphs. However, similar synchronization problems can be discussed on some graphs with $N$ nodes. For example, with the node equation as given in Eq. (5) and with the above data, for any natural number $n$, a network of $N=2n$ nodes in type $G_1$ (Fig. 1), as discussed in Remark 1, achieves synchronization with the coupling strength $c=1/n$. However, a network of $N=2n$ $(n\geq 5)$ nodes in type $G_2$ (Fig. 2), as discussed in Remark 1, cannot achieve synchronization with any coupling strength $c>0.01/n$. This also verifies the statement given in Remark 1 that the synchronizability index of networks of type $G_2$ (Fig. 2) tends to 0 as $n\to+\infty$.

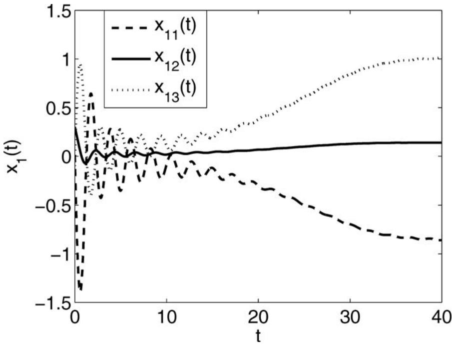

FIG. 21. Network on $G_2-e\{3,6\}$.

## IX. CONCLUSION

From both geometric and algebraic points of view, the study on the synchronizability of complex networks can be separated into two parts: one is on the geometric synchronized region,[3,4,22,23,37] for which the larger the synchronized region the better the synchronizability; the other is the algebraic eigenratio $r=\lambda_2/\lambda_N$ of the corresponding Laplacian matrix, for which the larger the $r$ the better the synchronizability.[3,7,11,15,16,21,24] This paper has taken both viewpoints from the graph-theoretic approach to discuss the performance of the network synchronizability, showing an in-depth application of the graph theory to network synchronization studies. Sections II–V introduce the existing results to show the effects of network statistical properties, subgraphs, complementary graphs, graph operations, adding edges, and adding nodes, etc.[15,36–41] In Secs. VI and VII, some new results on the synchronizability of coalescence have been introduced, and a condition for a network and its complementary network to have the same synchronizability has been illustrated. In Sec. VIII, a Chua's circuit network was used to show its synchronization and nonsynchronization behaviors on different graphs and complementary graphs. The study of this paper has demonstrated that better understanding and careful manipulation of the underlying graphs are indeed very important and helpful for investigating complex network synchronization.

## ACKNOWLEDGMENTS

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