

On differences and similarities in the analysis of Lorenz, Chen, and Lu systems



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ABSTRACT

Currently it is being actively discussed the question of the equivalence of various Lorenz-like systems and the possibility of universal consideration of their behavior (Algaba et al., 2013a,b, 2014b,c; Chen, 2013; Chen and Yang, 2013; Leonov, 2013a), in view of the possibility of reduction of such systems to the same form with the help of various transformations. In the present paper the differences and similarities in the analysis of the Lorenz, the Chen and the Lu systems are discussed. It is shown that the Chen and the Lu systems stimulate the development of new methods for the analysis of chaotic systems. Open problems are discussed.

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1. Introduction

Currently it is being actively discussed the question of the equivalence of various Lorenz-like systems and the possibility of universal consideration of their behavior [2–5,11,14,35] in view of the possibility of reduction of such systems to the same form with the help of various transformations.

In the present paper the differences and similarities in the analysis of these systems are discussed and it is shown that the Chen and the Lu systems stimulate for the development of new methods for the analysis of chaotic systems.

2. Lorenz-like systems: Lorenz, Chen, Lu, and Tigan systems

Consider the famous Lorenz system [60]

$$\begin{aligned}\dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\beta z + xy,\end{aligned}\tag{1}$$

where σ , ρ , β are positive parameters.

Consider the Chen system [12]

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$$\begin{aligned}\dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x + cy - xz, \\ \dot{z} &= -bz + xy\end{aligned}\quad (2)$$

and the Lu system [61]

$$\begin{aligned}\dot{x} &= a(y - x), \\ \dot{y} &= cy - xz, \\ \dot{z} &= -bz + xy,\end{aligned}\quad (3)$$

where a , b , c are real parameters. Systems (2) and (3) are Lorenz-like systems, which have been intensively studied in recent years.

In 2012 G.A. Leonov suggested to consider the following substitutions [35]

$$x \rightarrow hx, \quad y \rightarrow hy, \quad z \rightarrow hz, \quad t \rightarrow h^{-1}t \quad (4)$$

with $h = a$. By this transformation for $a \neq 0$ one has in (2) and (3)

$$a \rightarrow 1, \quad c \rightarrow \frac{c}{a}, \quad b \rightarrow \frac{b}{a}.$$

For $a = 0$ the Chen and the Lu systems become linear and their dynamics have minor interest. Thus, without loss of generality, one can assume that $a = 1$. Remark that the transformation (4) with $h = a$ does not change the direction of time for the positive chaotic parameters considered in the works [12,61].

Later, in 2013, the transformation (4) was independently considered in the works [2,3]¹ with $h = -c$ for the reduction of the Chen system (2):

$$\begin{aligned}\dot{x} &= -\frac{a}{c}(y - x), \\ \dot{y} &= \left(\frac{a}{c} - 1\right)x - y - xz, \quad \sigma = -\frac{a}{c}, \quad \rho = \frac{a}{c} - 1, \quad \beta = -\frac{b}{c} \quad (\sigma + \rho = -1), \\ \dot{z} &= \frac{b}{c}z + xy,\end{aligned}\quad (5)$$

and the Lu system (3):

$$\begin{aligned}\dot{x} &= -\frac{a}{c}(y - x), \\ \dot{y} &= -y - xz, \quad \sigma = -\frac{a}{c}, \quad \rho = 0, \quad \beta = -\frac{b}{c} \quad (\rho = 0), \\ \dot{z} &= \frac{b}{c}z + xy,\end{aligned}\quad (6)$$

to the form of the Lorenz system (1).

Note that here in contrast to the previous transformation: 1) the transformation (4) with $h = -c$ change the direction of time for $c > 0$ considered in the works [12,61], 2) for $c = 0$ the Chen and the Lu systems do not become linear and their dynamics may be of interest. For $c < 0$ the transformation (4) with $h = -c$ does not change the direction of time.

For $c = 0$ in [2,3] it is suggested to apply the previous transformation (4) with $h = a$ and is claimed that the Chen and the Lu systems with $c = 0$ are “a particular case of the T-system” [26,77] (which was published later)

$$\begin{aligned}\dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x - axz, \\ \dot{z} &= -bz + xy.\end{aligned}\quad (7)$$

To fill the formal gap in the notation of the parameters in this case it is required the additional transformation $x \rightarrow x/\sqrt{a}$, $y \rightarrow y/\sqrt{a}$, $z \rightarrow z/a$. Finally, to transform the Chen and the Lu systems with $c = 0$ to the T-system one has to apply the following transformation

$$x \rightarrow \sqrt{ax}, \quad y \rightarrow \sqrt{ay}, \quad z \rightarrow z, \quad t \rightarrow a^{-1}t. \quad (8)$$

For $\sigma = 10$, $\beta = 8/3$ and $0 < \rho < 1$, the Lorenz system is stable. For $1 < r < 24.74 \dots$ the zero fixed point loses its stability and two additional stable fixed points appear. For $\rho > 24.74 \dots$ all three fixed points become unstable and trajectories, depending on the initial data, may be repelled by them in a very complex way. For the parameter set

¹ Submission dates: [35] – December 27, 2012; [2] – 22 January 2013; [3] – 22 January 2013.

$\{\sigma, \rho, \beta\} = \{10, 28, 8/3\}$ it was found numerically a chaotic strange attractor in the Lorenz system [60]. Various rigorous approaches to the justification of its existence are based, for example, on the investigation of instability (hyperbolicity) of trajectories with the help of computing Lyapunov exponents, or the computation of attractor dimensions (see, e.g., [27,19] and many others). See also analytical–numerical approach in [78].

The Chen system with the parameter set $\{a, c, b\} = \{35, 28, 3\}$ is chaotic [12], but it may not be chaotic for some other parameter. The Lu system with the parameter set $\{a, c, b\} = \{36, 20, 3\}$ is also chaotic [61] and, likewise, it may not be chaotic for any other parameter.

It is easy to see that a generalized system [36,14,37]

$$\begin{aligned}\dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - dy - xz, \\ \dot{z} &= -bz + xy,\end{aligned}\tag{9}$$

contains, as special cases, systems (1)–(3). Here again $\sigma > 0$, $b > 0$, but r and d are certain real parameters. Note that for the Lorenz system: $d = 1$, the Chen system: $d = -c$, $c > 0$, $r = c - a$, and the Lu system: $d = -c$, $c > 0$, $r = 0$.

As it was noted by one of the reviewers of this material “If we have to give a different name of each planar slice we take in the three-dimensional parameter space of the Lorenz system, we might have a serious problem.” On the other hand, we would like to recall the classical 16th Hilbert problem (second part, [23]) on the number and mutual disposition of limit cycles in two-dimensional polynomial systems, where one of the tasks is to find the simplest system, from a certain class, with the maximum possible number of limit cycles. We can consider its essential “chaotic” analog: *on the number and mutual disposition of chaotic compact invariant connected sets* (e.g. local attractors and repellers) in three- or multi-dimensional polynomial systems. Many chaotic polynomial systems have been discovered (e.g., such particular cases of three-dimensional quadratic systems as the Lorenz, the Rossler, the Sprott, the Chen, the Lu and other systems) and studied over the years, “but it is not known whether the algebraically simplest chaotic flow has been identified” [73,72]. Thus, one of the attractive feature of the Chen and the Lu systems is that the scenarios of transition to chaos in them are similar to those in the Lorenz system, but, in contrast to the latter, nonlinear Chen and Lu systems involve only two parameters hence are simpler.

3. Recent discussion on equivalence of the Lorenz, Chen, and Lu systems

Recently a very interesting discussion on the equivalence of the Lorenz, the Chen and the Lu systems was initiated in [2,3,11].

Below a few remarks, concerning the discussion and being important, are given.

1) The Lorenz system is the system considered in the original work by Edward Lorenz [60]: Lorenz, E. N. (1963). *Deterministic nonperiodic flow*. *J. Atmos. Sci.*, 20(2):130–141. E. Lorenz obtained his system as a truncated model of thermal convection in a fluid layer and the parameters σ , ρ , and β of his system are positive because of their physical meaning (e.g., $\beta = 4(1 + a^2)^{-1}$ is positive and bounded). Thus, from a physical point of view, systems (5) and (6) are not particular cases of the Lorenz system since $\beta = -\frac{b}{c}$ is negative in (5) and (6) for the case $bc > 0$ considered in [12,61] and $\sigma + \rho < 0$ in (5) and $\rho = 0$ in (6).

Formally one may compare the Lorenz system with not necessarily positive parameters in backward time for $c > 0$ or in forward time for $c < 0$ with systems (2) and (3). But non-positive parameters and backward time do not have a clear physical sense for the Lorenz model as well as for many other physical problems.

2) From a mathematical point of view, one may consider non-positive parameters or backward time.

In [2] one can read: “Chen’s attractor exists if Lorenz repulsor exist” and “most of the literature on the Chen system is redundant because the results obtained can be directly derived from the corresponding results on the Lorenz system”.

But the question of importance is what was known about repulsor (or repeller) in the Lorenz system and the dynamics of time-reversal Lorenz system before the works [12,61] were published? Note that in [2,3] there is no discussion of the following important questions for the consideration of the Lorenz system in the backward time or with non positive parameters: the existence of the extension of solutions, the existence of attractors, and the possibility of consideration of invariant sets in the backward time. Some necessary results can be found in [16], [15, p. 35]. See also [68,81,82]. However they differ from similar consideration for the classical Lorenz system with positive parameters in the forward time.

In fact, a consideration of system in the backward time (see the case $c > 0$) seems to be needless if all the objects of interest and their properties can be obtained from the study of this system in the forward time. As rightly noted in the works [1–3], the existence of periodic or homoclinic trajectories can be studied in one of the time directions. However for the study of non-closed trajectories and the sets of such trajectories being invariant in the forward time, it may be not the case.

For example, the definition of a dynamical system and the consideration of limit behavior of trajectories require a proof of trajectory existence. Generally speaking, for quadratic systems the existence of a trajectory on $t \in [t_0, +\infty)$ does not imply its existence on $t \in (-\infty, t_0]$ (see, e.g., examples from the paper [21] on the completeness of quadratic polynomial systems or the classical example $\dot{x} = x^2$).

Remark also that even in the case of the existence of the corresponding objects in forward and backward time their characteristics can be substantially different. For example, in general, absolute values of Lyapunov exponents of a bounded

trajectory in forward and backward time can be quite different. Widely used Kaplan–Yorke dimension (or Lyapunov dimension) of an invariant set can be defined only for one direction of time.

Also, to the best of our knowledge, even visualization and localization of the Lorenz repulsor and attractors in the Lorenz system with non-positive parameters had been unknown. Thus it would be appropriate to note that *some of the literature on the Chen and the Lu systems are new and of interest because the results obtained cannot be directly derived from the corresponding results on the Lorenz system*, since corresponding results on the Lorenz system have been unknown.

Recently interesting examples, that were previously unexplored, were reported in the Lorenz system and the Chen system [74,75].

Recall that in the case $c = 0$ in [2,3,11] there is remarked that the Chen and the Lu systems are only a particular case of the T-system, which was published in 2004 [77] (i.e. later than the Chen and the Lu systems were published). Also an invariant algebraic surface in the Chen system with $c = 0$ (see, e.g., [18,59]) has not correspondence with the Lorenz system [2].

3) Besides the fact of simultaneous existence of the corresponding objects in forward and backward time (being equivalent: closed orbits, homoclinic orbits, invariant sets and others; or dual: attractor – repellor), it is of importance the possibility to find the object and to analyse its properties in forward or backward time. In this case the questions of importance arise: (a) whether the methods, developed for the study of a system in forward time, can be applied in a similar way to the study of a system in backward time; and (b) whether a universal consideration of a system in both forward and backward time is possible.

Next we consider some differences and similarities in the analysis of the above mentioned systems and demonstrate that for the study of some properties of time-reversal Lorenz, Chen, and Lu systems new methods are needed.

4. Differences and similarities in the analysis of the considered systems

4.1. Homoclinic orbits

Consider the Lorenz system with fixed parameters $\sigma = 10$ and $\beta = 8/3$ and a varying parameter ρ , following the works of E. Lorenz. For $0 < \rho < 1$ the origin is a globally stable fixed point. Then, for $\rho > 1$ the origin becomes unstable and two new stable fixed points arise, the basins of attraction of which are separated by the stable manifold of the unstable origin. For $\rho = 13.9\dots$ this stable manifold contains a homoclinic orbit [27]. This result was generalized by G.A. Leonov.

Theorem 4.1 ([45,46,38]). *Let the numbers β and σ be given. For the existence of $\rho > 1$ such that system (1) has a homoclinic trajectory it is necessary and sufficiently that*

$$2\beta + 1 < 3\sigma. \quad (10)$$

The sufficiency of condition (10) was first obtained in [40,41]. The hypothesis that inequality (10) is a necessary condition was accepted in [40,41] and was first proved in [13].

Recently in the papers [37,46] it is proposed a new effective analytical–numerical procedure for localization of homoclinic trajectories (Fishing principle). For applying this method to three-dimensional systems it is of very importance the existence of the two-dimensional stable manifold of a saddle point, on which the trajectories are attracted to the saddle point from which a homoclinic trajectory is outgoing (see Fig. 1).

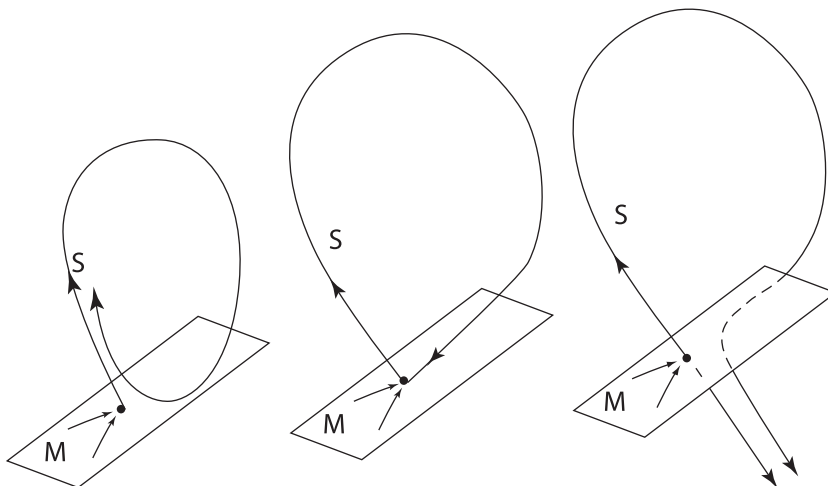


Fig. 1. Bifurcation of the birth of a homoclinic orbit.

Fishing principle is based on the construction of a special two-dimensional manifold such that the separatrix of the saddle intersects and does not intersect the manifold for two different values of a parameter. The continuity implies the existence of some intermediate value of the parameter, at which the separatrix touches the manifold. Since by the construction the only possibility is touching the saddle, one can numerically localize the birth of the homoclinic orbit changing the parameter value.

For the computation of a homoclinic trajectory the initial data in numerical integration are chosen closely to a saddle point and its unstable one-dimensional manifold, for example, on the eigenvector corresponding to the positive eigenvalue of the saddle. The purpose of numerical procedures is to reveal when the outgoing trajectory returns to the stable two-dimensional manifold (see Fig. 1).

Using Fishing principle, one can obtain [35,37,46] the following approximations for the Lorenz system with the parameters $\sigma = 10$, $\beta = 8/3$, and

$$\rho \in [13.92, 13.93]$$

for the Lu system with the parameters $r = 0$, $\sigma = 35$, $d = -28$, and

$$b \in [44.963, 44.974]$$

and for the Chen system with the parameters $r = -7$, $d = 35$, $d = -28$, and

$$b \in [40.914, 40.935].$$

Though the inversion of time does not affect on the existence of the homoclinic trajectory, it makes impossible effective application of modern analytical and numerical methods to the proof of the existence of homoclinic trajectories.

Here (see Fig. 2) because of the existence of the two-dimensional unstable manifold of a saddle: 1) in numerical procedure for each parameters set it is necessary to consider a set of initial data in a neighborhood of the saddle point close to two-dimensional unstable manifold (for example, on a plane, spanned on two eigenvectors corresponding to positive eigenvalues of the saddle), 2) the only one trajectory, corresponding to the homoclinic orbit, returns back to the considered unstable manifold, while the rest of trajectories are repelled by this manifold. Therefore in the numerical analysis of trajectories with the above-mentioned initial data it may be considered a substantial nonuniformity in their behavior, what makes more difficult a numerical analysis (for example, for the Lorenz system in a neighborhood of zero saddle point, for $x = y = 0$ the trajectories of system are exponentially repelled by this point: $z(t) \approx e^{\beta t}$). The above shows the significant multiple increasing of computational complexity and the impossibility of effective numerical analysis.

The existence, in a system, of homoclinic trajectories is a valuable tool in studying chaos. So-called Shilnikov chaos exists in three-dimensional dynamical systems with a homoclinic trajectory of a saddle point of equilibrium if this equilibrium is a saddle-focus with a positive saddle value [70]. While for the Lorenz, the Chen, and the Lu systems there are parameters, corresponding to a homoclinic trajectory of zero equilibrium, but this zero state is not a saddle-focus. Nevertheless in [37] it is shown that a small change of all these systems in a neighborhood of a saddle can lead to the satisfaction of all conditions of the Shilnikov theorem and, consequently, to Shilnikov chaos in the Lorenz, the Chen, and the Lu systems. Such a construction requires also to make the use of an analog of Fishing principle and the existence of a two-dimensional stable manifold.

4.2. Divergence and attractor dimension

Consider a dynamical system

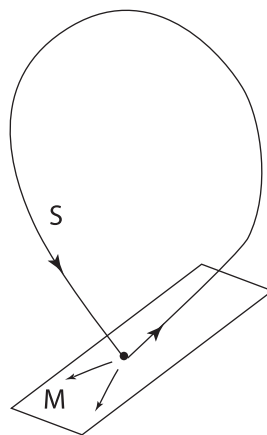


Fig. 2. Invertation of time and homoclinic orbit.

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad f \in \mathbb{C}^1 \quad (11)$$

and its linearization along the solution $x(t) = x(t, x_0)$ for $t \in [0, +\infty)$,

$$\frac{dy}{dt} = J(t)y, \quad J(t) = J(x(t, x_0)) = \left\{ \frac{\partial f_i(x)}{\partial x_j} \Big|_{x=x(t, x_0)} \right\}. \quad (12)$$

Denote by $\lambda_1(x(t, x_0)) \geq \dots \geq \lambda_n(x(t, x_0))$ the eigenvalues of the symmetric Jacobi matrix $(J(x(t, x_0)) + J(x(t, x_0))^*)$.

Important property of dynamical system (11) is its divergence

$$\operatorname{div}f(x(t, x_0)) = \sum_1^n \frac{\partial f_i(x)}{\partial x_i} \Big|_{x=x(t, x_0)} = \frac{1}{2} \sum_1^n \lambda_i(x(t, x_0)), \quad (13)$$

which characterizes the change of volume along the trajectory $x(t, x_0)$ in the phase space.

Suppose $X(t) = \{x_1(t), \dots, x_n(t)\}$ is a fundamental matrix of system (12) and $\alpha_1(X(t)) \geq \dots \geq \alpha_n(X(t)) \geq 0$ are its singular values (the square roots of eigenvalues of the matrix $X(t)^*X(t)$ are renumbered for each t). Geometrically, the values $\alpha_j(X(t))$ coincide with the principal axes of the ellipsoid $X(t)B$, where B is a unit ball. The Lyapunov exponent μ_j at the point x_0 is a number (or the symbol $-\infty$ or $+\infty$)²:

$$\mu_j(x_0) = \lim_{t \rightarrow +\infty} \sup \frac{1}{t} \ln \alpha_j(X(t)).$$

By definition, $\mu_j(x_0)$ is the exact Lyapunov exponent if there exists a finite limit $\lim_{t \rightarrow +\infty} \frac{1}{t} \ln \alpha_j(X(t))$. If the Lyapunov exponents of system exist and are finite, then

$$\lim_{t \rightarrow +\infty} \sup \frac{1}{t} \int_0^t \operatorname{div}f(x(\tau, x_0)) d\tau = \sum_1^n \mu_j(x_0). \quad (14)$$

Proposition 5 [56]. *If $\operatorname{div}f(x(t, x_0)) > 0$, then a stationary solution $x(t, x_0) \equiv x_0$ is Lyapunov unstable and a periodic trajectory $x(t, x_0) = x(t + T, x_0)$ is orbitally unstable.*³

For the Chen systems (2) and (5) and for the Lu systems (3) and (6) the divergence is constant and under the condition $a + b > c > 0$ one has

$$\operatorname{div}_{(2),(3)} = -a + c - b < 0 \quad (15)$$

and

$$\operatorname{div}_{(5),(6)} = \frac{a}{c} - 1 + \frac{b}{c} > 0, \quad (16)$$

respectively. Therefore systems (5) and (6), unlike (2) and (3), are not dissipative in the sense $\operatorname{div} < 0$ and there occurs only volumes increasing. Consequently the bounded set cannot be positively invariant and it is not dissipative in the sense of Levinson [49].

The idea of volume contracting is a base of the dimension theory of attractors⁴ (see, e.g., [8,20,47,76]). For finite-dimensional dynamical systems it was possible to try to get nontrivial results: in this case the estimates of dimension might not be greater than the dimension of the phase space. Then it was essential to try to extend the well-known Liouville theorem on the volume $V(K)$ contracting for the invariant compact set $K \subset U$ of differential equation.

Theorem 5.1. *If*

$$\operatorname{div}f(x) < 0, \quad \forall x \in U \subset \mathbb{R}^n, \quad (17)$$

then $V(K) = 0$.

The mathematical tools, developed independently by Douady Oesterle [20] and Ilyashenko [25], permitted one to obtain the following extension of the Liouville theorem and to estimate the Hausdorff dimension of K .

² There are two widely used definitions of Lyapunov exponents: the upper bounds of the exponential growth rate of the norms of linearized system solutions (LCEs) and the upper bounds of the exponential growth rate of the singular values of linearized system fundamental matrix (LEs). While often these two definitions gave the same values, for a given system they may be different and there are examples in which Benettin algorithm [6] (see, e.g., its MATLAB implementation in [71]) fails to compute the correct values (see the discussion in [29]). It is possible to show that the ordered LCEs majorizes the ordered LEs. The existence of different definitions, computational methods and related assumptions led to the appeal "Whatever you call your exponents, please state clearly how are they being computed" [17]. See, e.g., a more detailed discussion in [29,50].

³ For an arbitrary solution it, generally speaking, is not valid (see, e.g., a gap in the proof of the Chetaev theorem on instability by the first approximation [44,50]).

⁴ Following [9,44], it can be found that an attractor is a bounded, closed, invariant attracting subset of the phase space of a dynamical system. The different types of attraction and rigorous definitions of attractors can be found in [8,63].

Theorem 5.2. *Suppose that the inequality*

$$\lambda_1(x) + \dots + \lambda_k(x) + s\lambda_{k+1}(x) < 0, \quad \forall x \in U, s \in [0, 1] \tag{18}$$

is satisfied. Then the Hausdorff dimension of an invariant compact set $K \subset U$ has the following estimate

$$\dim_H K < k + s. \tag{19}$$

Remark that condition (18) can be satisfied only if the divergence is negative.

In the work [43] it is introduced Lyapunov functions in the estimates of the form (18) and proved the following result.

Theorem 5.3 ([43,47]). *If there exists a differentiable function $v(x)$ such that the inequality*

$$\lambda_1(x) + \dots + \lambda_k(x) + s\lambda_{k+1}(x) + \dot{v}(x) < 0, \quad \forall x \in U$$

is satisfied, then estimate (19) is valid.

Nowadays the various characteristics of attractors of dynamical systems (information dimension, metric entropy, etc) are studied based on Lyapunov exponents computation.⁵

Local Lyapunov dimension of a point x_0 is defined by

$$\dim_L x_0 = j(x_0) + \frac{\mu_1(x_0) + \dots + \mu_j(x_0)}{|\mu_{j+1}(x_0)|}, \tag{20}$$

where $\mu_1(x_0) \geq \dots \geq \mu_n(x_0)$ are Lyapunov exponents; $j(x_0) \in [1, n]$ is the smallest natural number m such that

$$\mu_1(x_0) + \dots + \mu_{m+1}(x_0) < 0, \quad \mu_{m+1}(x_0) < 0, \quad \frac{\mu_1(x_0) + \dots + \mu_m(x_0)}{|\mu_{m+1}(x_0)|} < 1.$$

The Lyapunov dimension of an invariant set $K \subset U$ of a dynamical system is defined as

$$\dim_L K = \sup_{x_0 \in K} \dim_L x_0. \tag{21}$$

Lyapunov dimension is an estimate from above of topological, Hausdorff, and fractal dimensions [7,8,24]

$$\dim_T K \leq \dim_H K \leq \dim_F K \leq \dim_L K.$$

The estimate from above of the Lyapunov dimension by Lyapunov functions [36] and its comparison with the local Lyapunov dimension in zero stationary point permit one to obtain the exact formula of dimension for a generalized Lorenz system (9) with a certain parameter d . For example, for the Lorenz system (where $d = 1$) the following result is known:

Theorem 5.4 ([36,39,55]). *If all the equilibria of the Lorenz system are hyperbolic, then*

$$\dim_L K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4r\sigma}}.$$

However the case $d < 0$ is turned to be more complicated and a similar consideration does not permit one to obtain similar expressions for the attractors of the Chen and the Lu systems for classical parameters (where $d = -1$). Here the assertions on a coincidence of the Lyapunov dimension of attractors with its value at zero stationary point are conjectures, the proof of which requires the construction of new Lyapunov functions and a careful consideration.^{6,7}

Note also that the time inversion (e.g. in the change (4) with time reversal) may lead to quite different values of Lyapunov values and positive divergence (in this case it is impossible to introduce a nontrivial Lyapunov dimension and to estimate it). In general,

$$\limsup_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln |x(\tau)| = -\liminf_{t \rightarrow -\infty} \frac{1}{t} \ln |x(-t)|.$$

⁵ While positive largest Lyapunov exponent, computed along a trajectory, is widely used as indication of chaos, the rigorous mathematical consideration requires the verification of additional properties (e.g., such as regularity, ergodicity) since there are known the Perron effects of the largest Lyapunov exponent sign-reversal for non-regular linearization [30,31,50]. The regularity of almost all linearizations of a dynamical system and the existence of exact limits of Lyapunov exponents (for almost all x_0) with respect to an invariant measure result from the Oseledets theorem [64]. However in the general case there are no effective methods for the construction of an invariant measure in the phase space of a system, the support of which is sufficiently dense. More essential justification of the existence of exact values of Lyapunov exponents in computer experiments may be the following: in calculations with finite precision any bounded pseudo-trajectory $\tilde{x}(t, x_0)$ has a point of self-intersection: $\exists t_1, t_2 : \tilde{x}(t_1, x_0) = \tilde{x}(t_1 + t_2, x_0)$. Then for sufficiently large $t \geq t_1$ the trajectory $\tilde{x}(t, x_0)$ may be regarded as periodic. From a theoretical point of view this fact is relies on the shadowing theory, the closing lemma, and its various generalizations (see, e.g., the surveys [22,62,67,69]).

⁶ For example, in [14] there are given the results on the analysis of the generalized Lorenz system (9). In the case $d > 0$ these results are special cases of more general results, published earlier in [36]. In the case $d < 0$ the authors' students Korzhemanova and Kusakin revealed a gap in the reasoning given in [14]. They found parameters, under which the validity of condition (55) does not imply the validity of condition (49): $a = 1, b = 2, c = 6, d = 1,$ and $\gamma_1 = 5, \gamma_2 = -59/12.$

⁷ In [33] Leonov's conjecture on the Lyapunov dimension of the Rossler attractor was verified numerically, while an analytical proof is still an open problem.

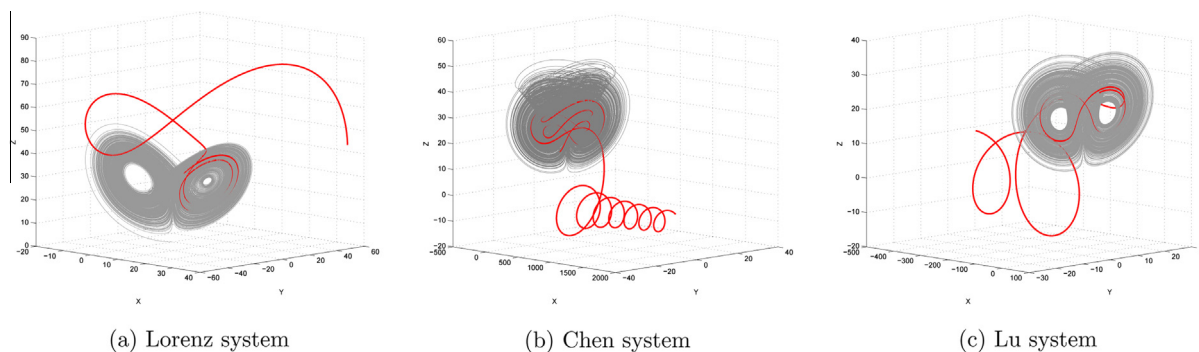


Fig. 3. Numerical computation of invariant sets in forward (gray) and backward (red) time. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5.1. Application of dimension estimates to the problem on stability of stationary sets. Analog of Bendixson criterion

Consider a certain set $D \subset \mathbb{R}^n$, diffeomorphic to closed ball. We assume D is positively invariant with respect to the solutions of system (11).

Theorem 5.5 ([42,48]). Suppose that there exists a continuously differentiable function $v(x)$ such that

$$\lambda_1(x) + \lambda_2(x) + \dot{v}(x) < 0, \quad \forall x \in D. \quad (22)$$

Then any solution $x(t)$ of system (11) with the initial data $x(0) \in D$ tends for $t \rightarrow +\infty$ to a stationary set.

From Theorem 5.5 it follows at once the following.

Theorem 5.6. Suppose that there exists a continuously differentiable function $v(x)$ such that

$$\lambda_1(x) + \lambda_2(x) + \dot{v}(x) < 0, \quad \forall x \in \mathbb{R}^n.$$

Then any bounded for $t \geq 0$ solution of system (9) tends for $t \rightarrow +\infty$ to a stationary set.

For the Lorenz system (1) with hyperbolic equilibria, condition (22) is satisfied for

$$r < (b+1) \left(\frac{b}{\sigma} + 1 \right). \quad (23)$$

5.2. Numerical simulation and visualization of attractors

An oscillation in a dynamical system can be easily localized numerically if the initial conditions from its open neighborhood lead to long-time behavior that approaches the oscillation. Thus, from a computational point of view it is natural to suggest the following classification of attractors, based on the simplicity of finding the basin of attraction in the phase space.

Definition ([32,51,53,54]). An attractor is called a *hidden attractor* if its basin of attraction does not intersect with small neighborhoods of equilibria, otherwise it is called a *self-excited attractor*.

For a *self-excited attractor* its basin of attraction is connected with an unstable equilibrium and, therefore, self-excited attractors can be localized numerically by the *standard computational procedure*, in which after a transient process a trajectory, started from a point of an unstable manifold in a neighborhood of an unstable equilibrium, is attracted to the state of oscillation and traces it. Thus self-excited attractors can be easily visualized.

In contrast, for a hidden attractor its basin of attraction is not connected with unstable equilibria. For example, hidden attractors are attractors in the systems with no equilibria or with only one stable equilibrium (a special case of multistable systems and coexistence of attractors).⁸ Recent examples of hidden attractors can be found in [10,28,34,52,57,58,65,66,79,80,83,84]. Multistability is often an undesired situation in many applications, however coexisting self-excited attractors can be found by the standard computational procedure. In contrast, there is no regular way to predict the existence or coexistence of hidden attractors in a system. Note that one cannot guarantee the localization of an attractor by the integration of trajectories with random initial conditions (especially for multidimensional systems) since its basin of attraction may be very small.

Classical Lorenz, Chen, and Lu attractors are self-excited attractors, and consequently they can be easily found numerically. If E. Lorenz, a pioneer of chaos theory, studied his system with inverted time by a reason of instability, he would

⁸ For example the nested limit cycles in the papers on the 16th Hilbert problem, counterexamples to the Aizerman and Kalman conjectures on the absolute stability of nonlinear control systems, and others examples.

not find by numerical experiments his famous attractor, which became repeller in the case of inverted time, and the theory of chaos would come into being much later. In Fig. 3 it is shown the corresponding computation: first we compute attracting invariant set in the Lorenz, the Chen, and the Lu systems in the forward time (gray), then from the last gray points we compute trajectories in the backward time (red).

While the invariant sets are same for both forward and backward time, we actually have very different pictures.

6. Conclusion

In 2012 G. Leonov in his work [35] demonstrated by the transformation $x \rightarrow hx$, $y \rightarrow hy$, $z \rightarrow hz$, $t \rightarrow h^{-1}t$ with $h = a$ that without loss of generality nonlinear Chen and the Lu systems can be considered as two-parametric systems (for $a = 0$ the Chen and the Lu systems become linear and their dynamics have minor interest). Later, in 2013, in the works [2,3] the transformation with $h = -c$ was applied to transform the Chen and Lu systems for $c > 0$ to the form of the Lorenz system (here for $c = 0$ the Chen and the Lu systems do not become linear and can be transformed by the transformation with $h = a$ to the T-system which was published later).

Here it is demonstrated that the change of variables with time inversion, reducing the Chen and the Lu systems with $c > 0$ to the form of the Lorenz system, make difficult the application of numerical procedures for attractor visualization and localization of homoclinic orbit. Also the transformed Chen (5) and Lu (6) systems, unlike the original Chen (2) and Lu (3), are not dissipative in the sense $\text{div} < 0$ and there volume increasing take place. Consequently a bounded set cannot be positively invariant and the transformed system are not dissipative in the sense of Levinson. Thus time inversion, used for the reduction of the Chen and the Lu systems to the Lorenz system, makes it impossible to introduce a Lyapunov dimension and to estimate it. While Lyapunov dimension of the Lorenz attractor can be obtained analytically [47], for the Chen and Lu attractors it is still an open problem. While the titles “.. *The Chen system is a special case of the Lorenz system*” and “*The Lu system is a particular case of the Lorenz system*” of [2,3] may lead to the conclusion that the study of the Chen and the Lu systems is useless, the Chen and the Lu systems stimulate the development of new methods for the analysis of chaotic systems.

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