广义 Lorenz 系统族
理论及应用简介

Introducing the Generalized Lorenz Systems Family:
Theory and Applications

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9-9-09

To the Memory of

Father of Chaos

Edward N. Lorenz

(23 May 1917 – 16 April 2008)

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……

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- Chen System
- Generalized Lorenz System
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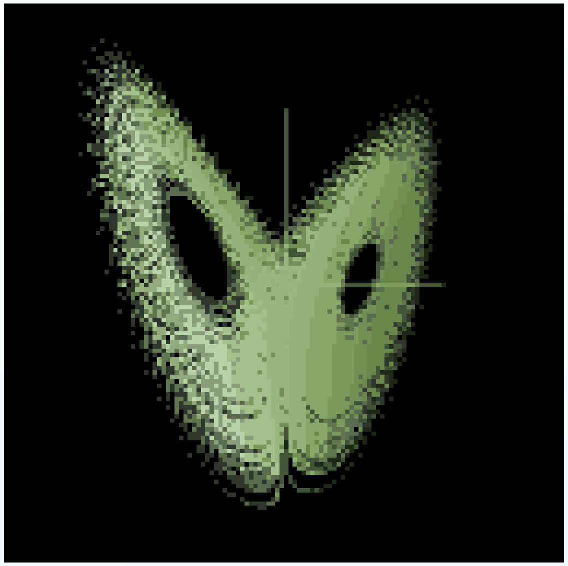
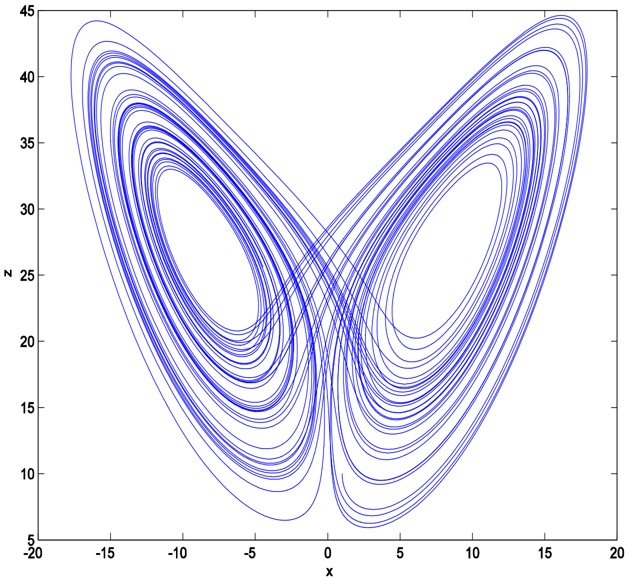
Lorenz System

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz, \end{cases}$$

$$a = 10, b = 8/3, c = 28$$



E. N. Lorenz, "Deterministic non-periodic flow," J. Atmos. Sci., 20, 130-141, 1963.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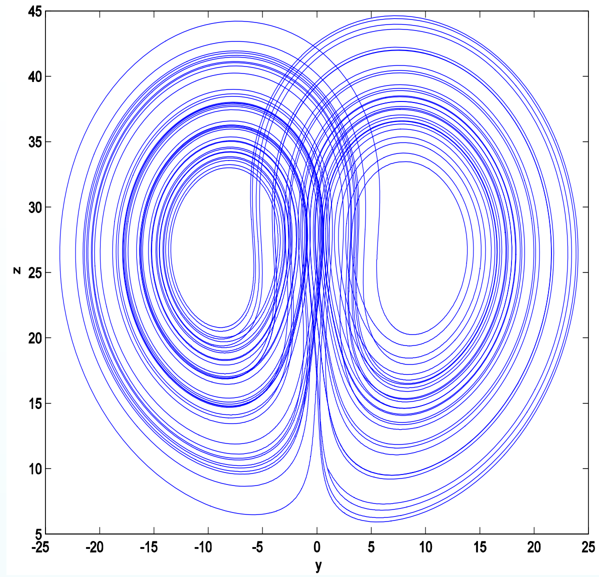
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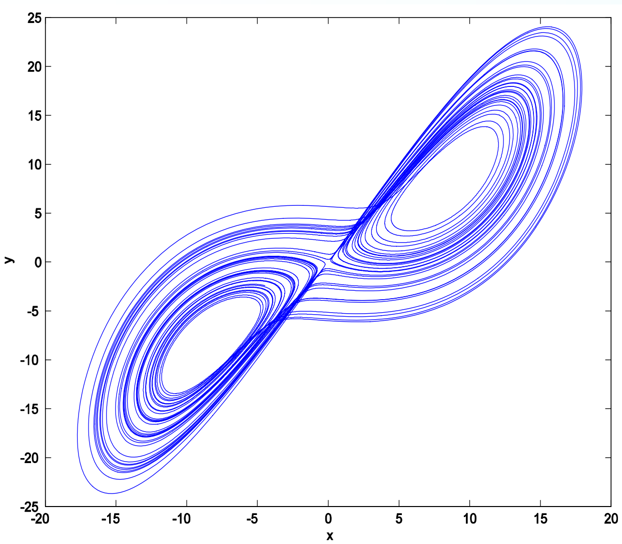
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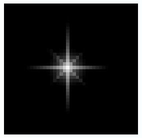
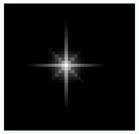
Any Extension or Connection?

Lorenz
(1963)



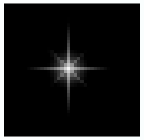


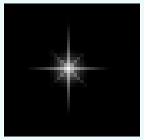
Rössler
(1976)

3-D Autonomous with

1 or 2 Quadratic Terms

Sprott
(1997)





(others)

Lorenz → Chen System

Lorenz System

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz - y + u \\ \dot{z} = xy - bz, \end{cases}$$

→

Chen System

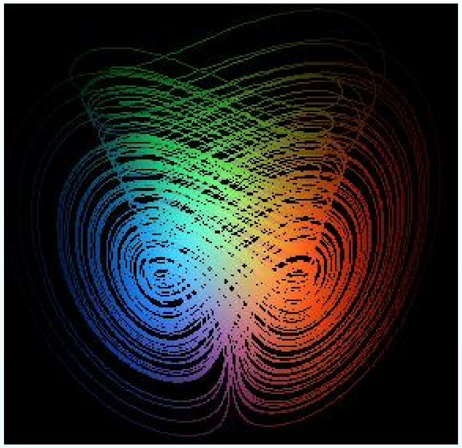
$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = (c-a)x - xz + cy \\ \dot{z} = xy - bz, \end{cases}$$

$$u = -ax + (1-c)y + 0z$$

G. Chen: Anti-Control of Chaos (1996, 1998, 2000, …)

❑ Chen System

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = xy - bz, \end{cases}$$

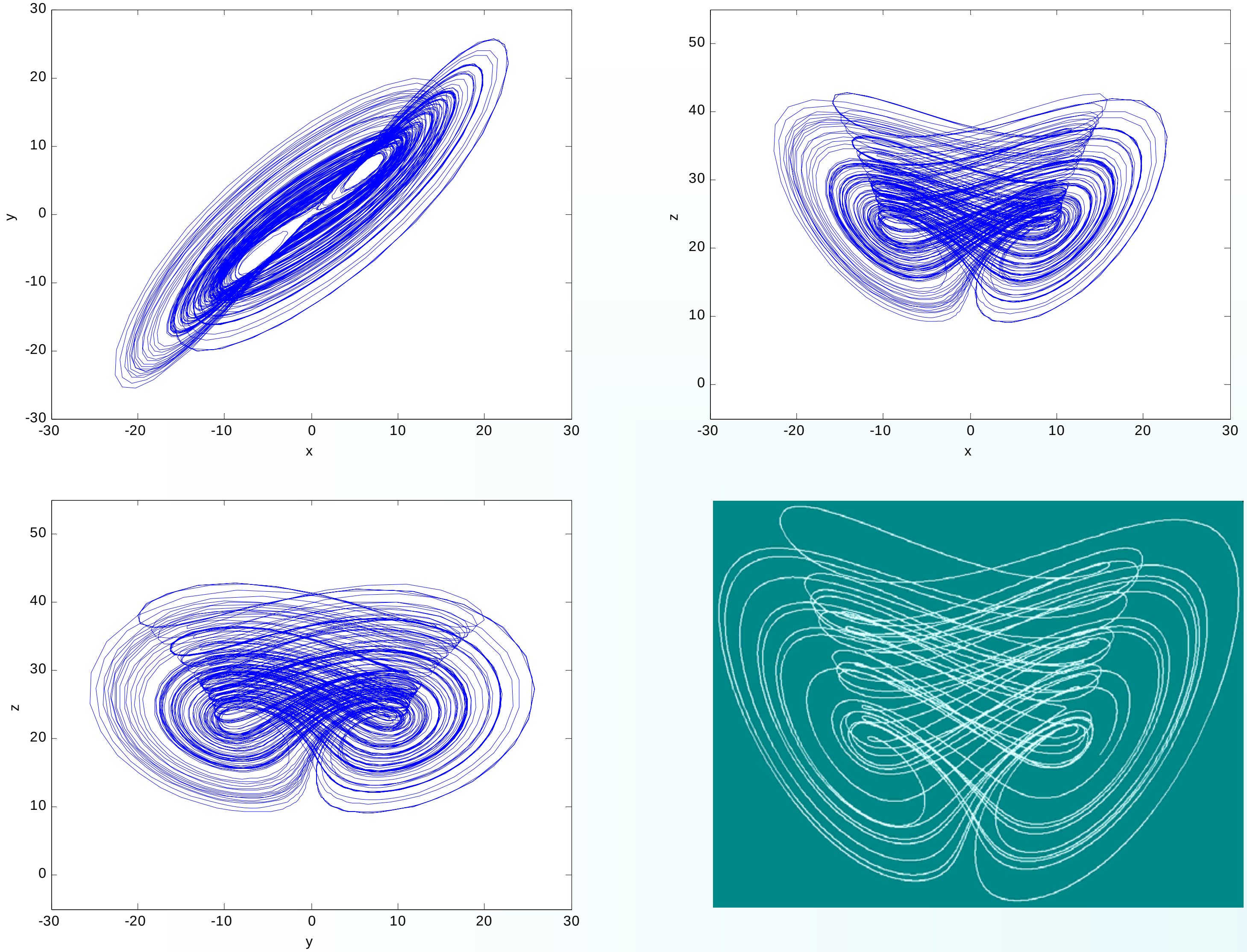


$$a = 35; \quad b = 3; \quad c = 28$$

G. Chen and T. Ueta, "Yet another chaotic attractor," Int J. of Bifurcation and Chaos, 9(7), 1465-1466, 1999.

T. Ueta and G. Chen, "Bifurcation analysis of Chen's equation," Int J. of Bifurcation and Chaos, 10(8), 1917-1931, 2000.

T. S. Zhou, G. Chen and Y. Tang, "Chen's attractor exists," Int. J. of Bifurcation and Chaos, 14, 3167-3178, 2004.



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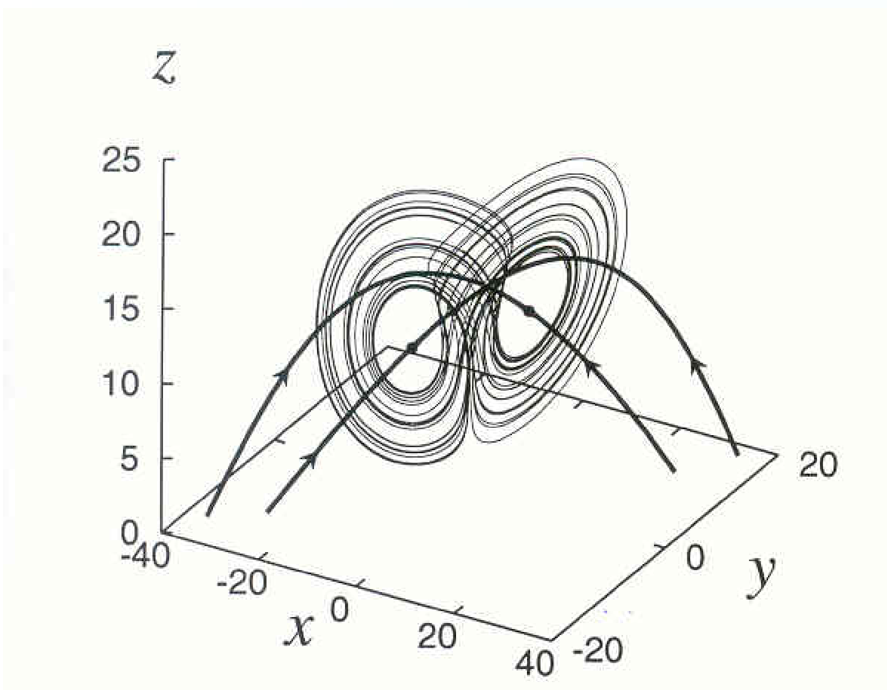
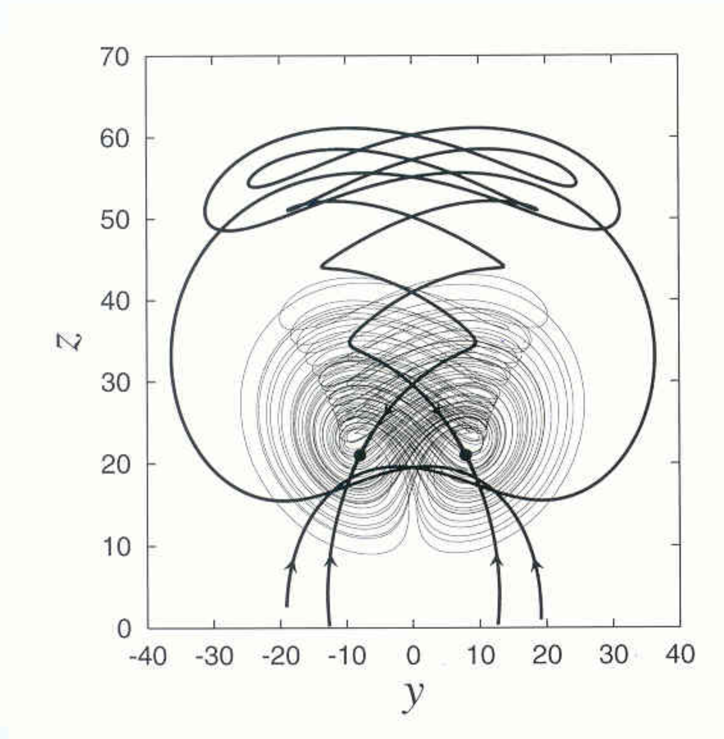
Some Comparisons:

Stable Manifolds





Lorenz Attractor **Chen Attractor**

Remark 1:

Equivalence Lorenz ⟺ Chen ?

- **Early Attempt:** G. R. Chen, Z. R. Liu, Z. Y. Yan, P. Yu, …

- **Progress:** Z. T. Hou, N. Kang, X. X. Kong, G. R. Chen, G. Yan (2009):

  Lorenz system and Chen system are not smoothly equivalent

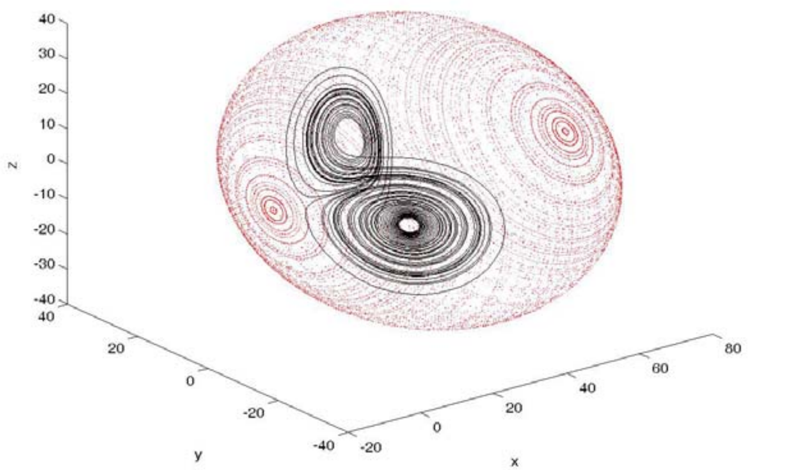
  (i.e., no diffeomorphism between them)

Q: **Are Lorenz system and Chen system topologically equivalent**

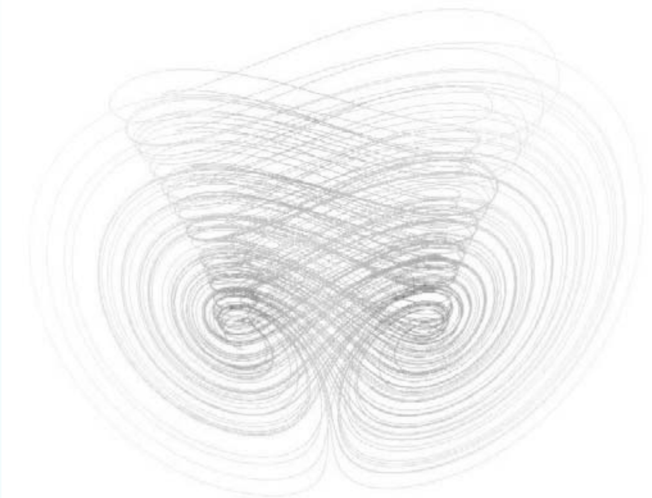
(i.e., any homeomorphism between them) **?**

Remark 2:

Global Boundedness



Lorenz Attractor



Chen Attractor **?**

Early Attempt: G. R. Chen, W. X. Qin, J. A. Lu, D. M. Li, …, R. Barboza

Proof

Existence of a Chaotic Attractor

Shilnikov Theorem (1967):

If a 3D autonomous system has two distinct saddle fixed points and there exists a heteroclinic orbit connecting them, and if the eigenvalues of the Jacobin of the system at these fixed points are

$\alpha_k, \beta_k \pm j\omega_k (k=1,2)$ **satisfying** $|\alpha_k| > |\beta_k| > 0 \quad (k=1,2)$

and $\beta_1\beta_2 > 0 \quad or \quad \omega_1\omega_2 > 0$ **then the system has infinitely many Smale horseshoes and hence has horseshoe chaos.**

Proof

Show the existence of a heteroclinic orbit

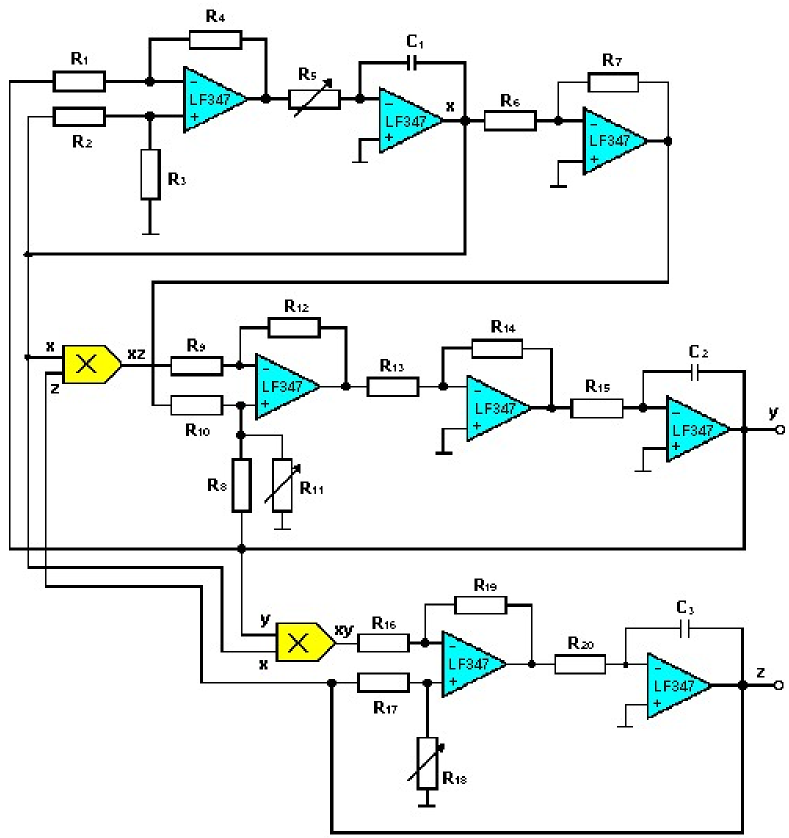
between two saddle-focus fixed points

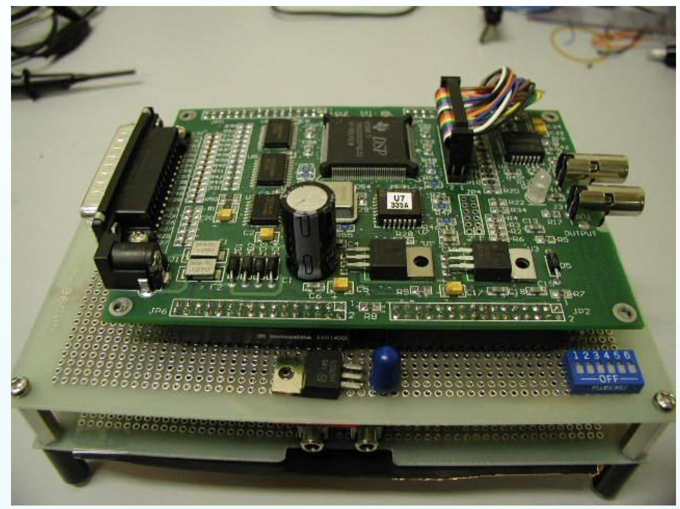
(a constructive approach)

- **Start from a series expansion of the heteroclinic orbit**
- **Substituting it into the characteristic equation of the system**
- **Force it to satisfy the basic properties as a heteroclinic orbit**
- **Force it to satisfy the Shilnikov conditions**
- **Guarantee the uniform convergence of the series expansion**

T. S. Zhou, G. Chen and Y. Tang, “Chen's attractor exists,” Int. J. of Bifurcation and Chaos, 14, 3167-3178, 2004.

Circuit Implementation

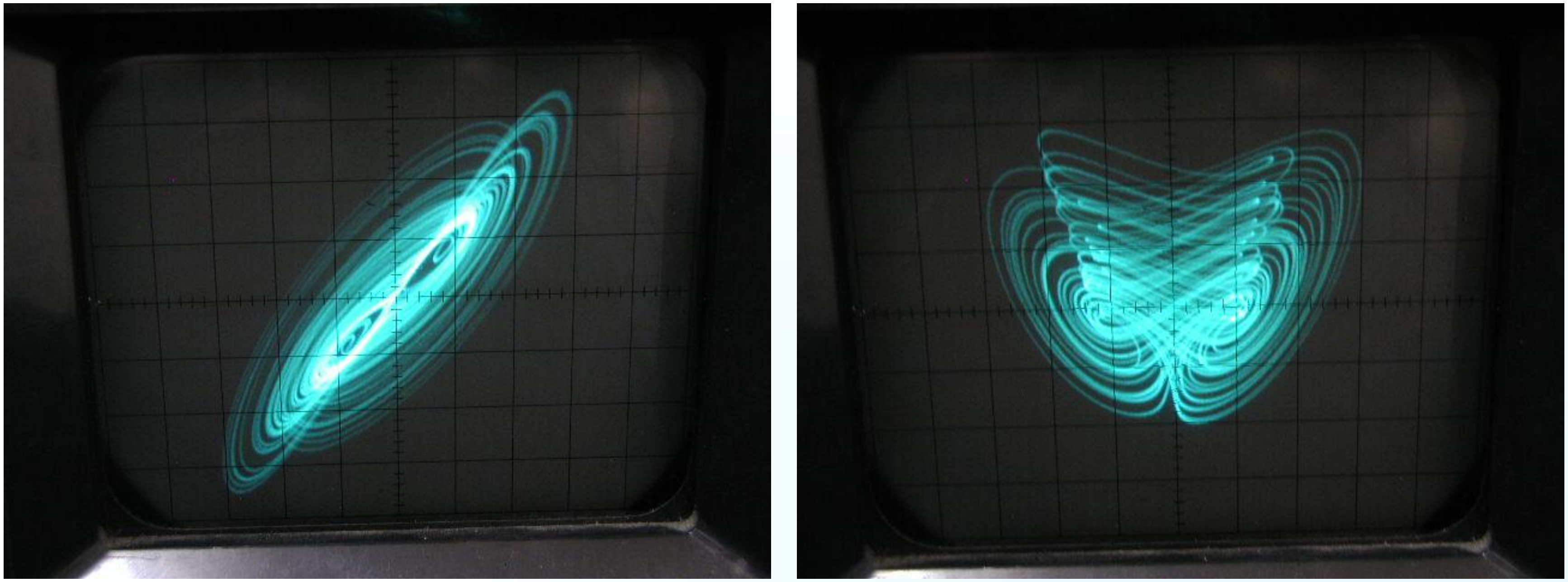




(T. Ueta)

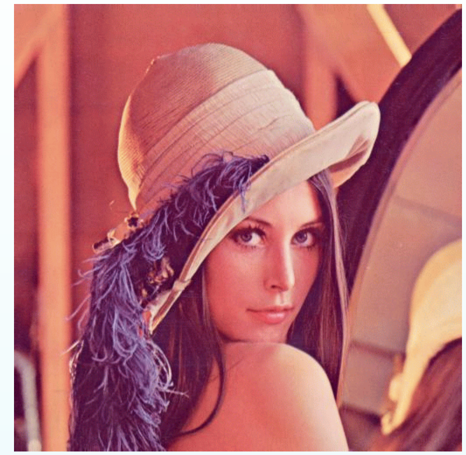
G.-Q. Zhong and K. S. Tang, "Circuitry implementaion and synchronization of Chen's attractor," Int. J. of Bifur. Chaos, 12(6), 1423-1427, 2002.

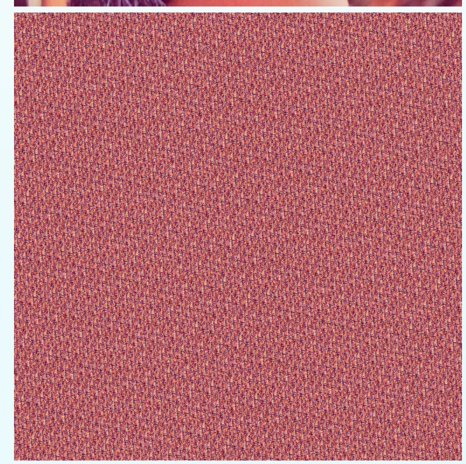
Electronic Attractor

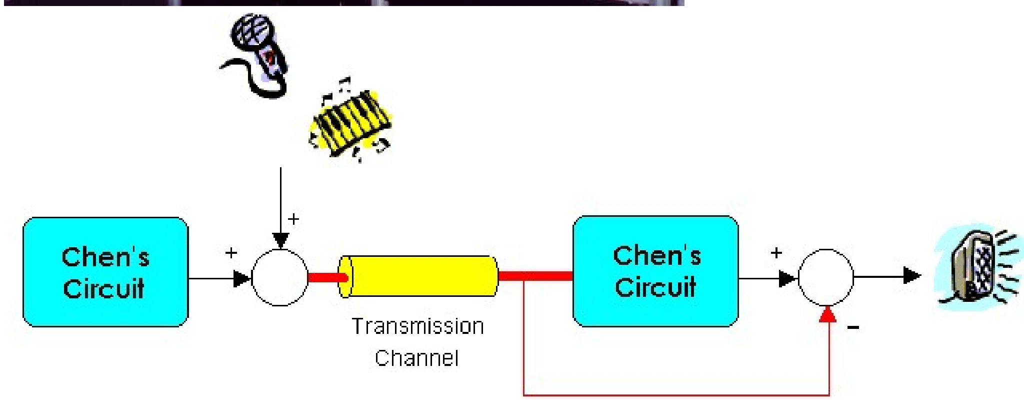


Chen Attractor

Some Applications







Generalized Lorenz System

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

According to the C-V Canonical Form –

Lorenz System satisfies: $a_{12}a_{21} > 0$
Chen System satisfies: $a_{12}a_{21} < 0$

Q: What system satisfies $a_{12}a_{21} = 0$ **?**

A. Vaněček and S. Čelikovský, Control Systems: From Linear Analysis to Synthesis of Chaos, London: Prentice-Hall, 1996 (S. Čelikovský and A. Vaněček, 1994)

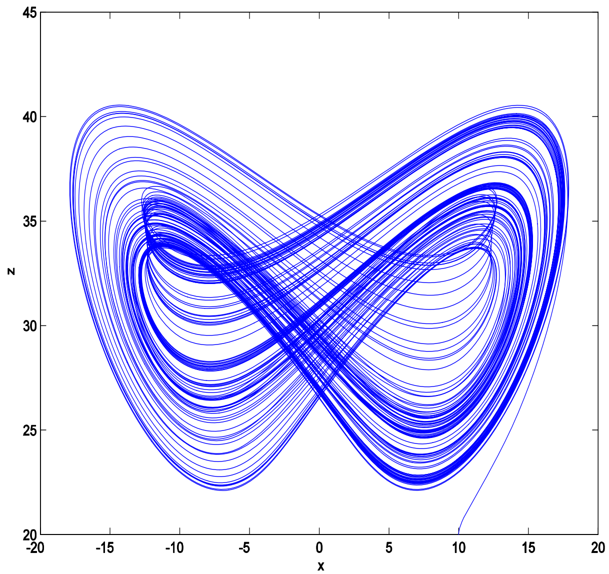
Lü system

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz, \end{cases}$$

$$a = 36; \quad b = 3; \quad c = 20$$

$$\longrightarrow \quad a_{12}a_{21} = 0$$

J. Lü and G. Chen, "A new chaotic attractor coined," Int. J. of Bifurcation and Chaos, 12(3), 659-661, 2002.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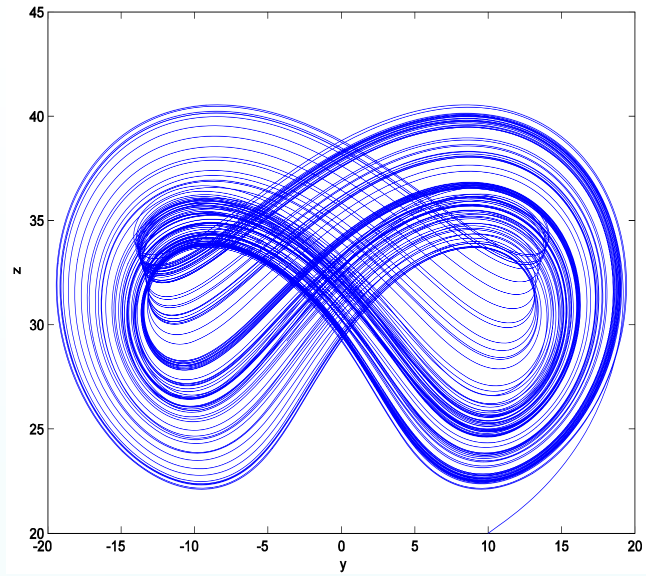
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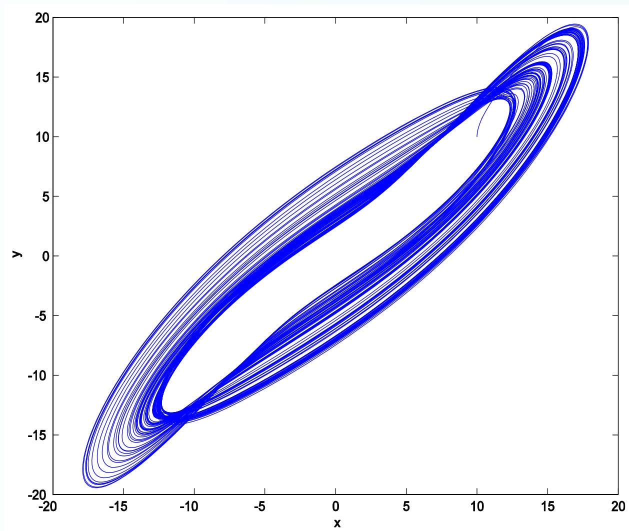
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A Unified Chaotic System

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x) \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y \\ \dot{z} = xy - \frac{1}{3}(\alpha + 8)z, \end{cases}$$

where $a \in [0,1]$

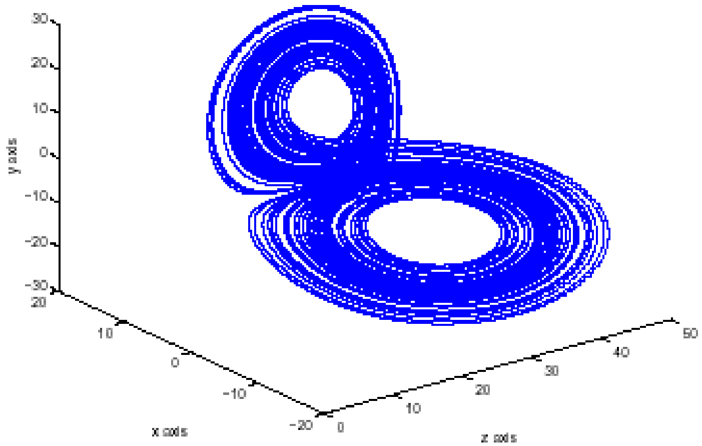
When

$$\alpha = 0, \quad \alpha = 1, \quad \alpha = 0.8$$

it becomes the Lorenz, Chen, or Lü system, respectively.

J. Lü, G. Chen, D. Cheng and S. Čelikovský, "Bridge the gap between the Lorenz system and the Chen system," Int. J. Bifurcation and Chaos,12(12), 2917-2926, 2002.

(a)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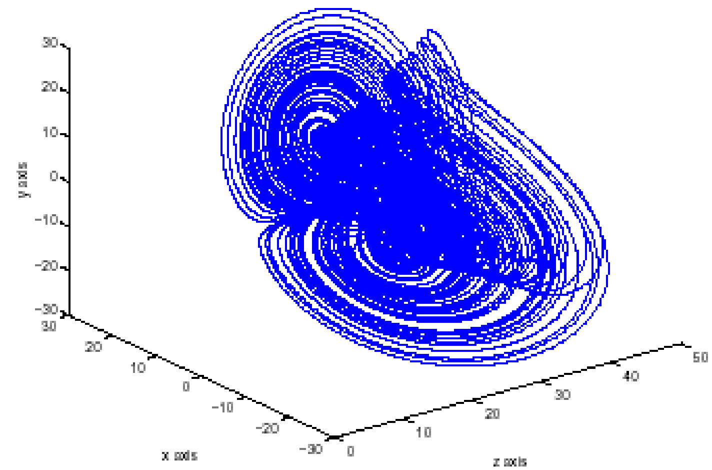
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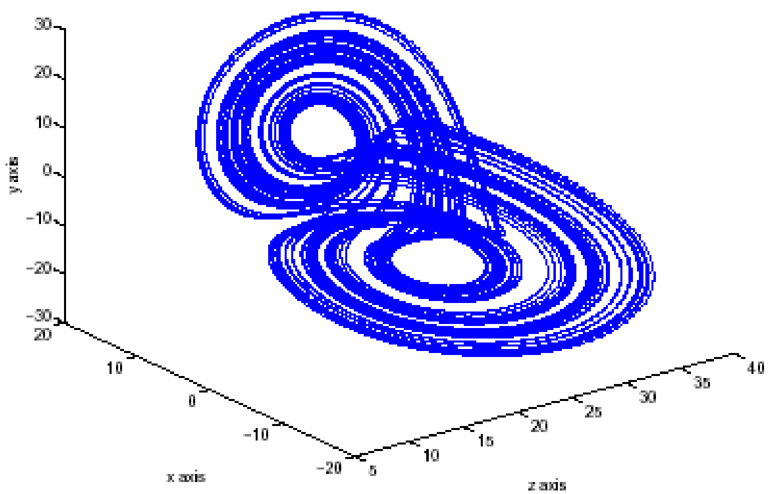
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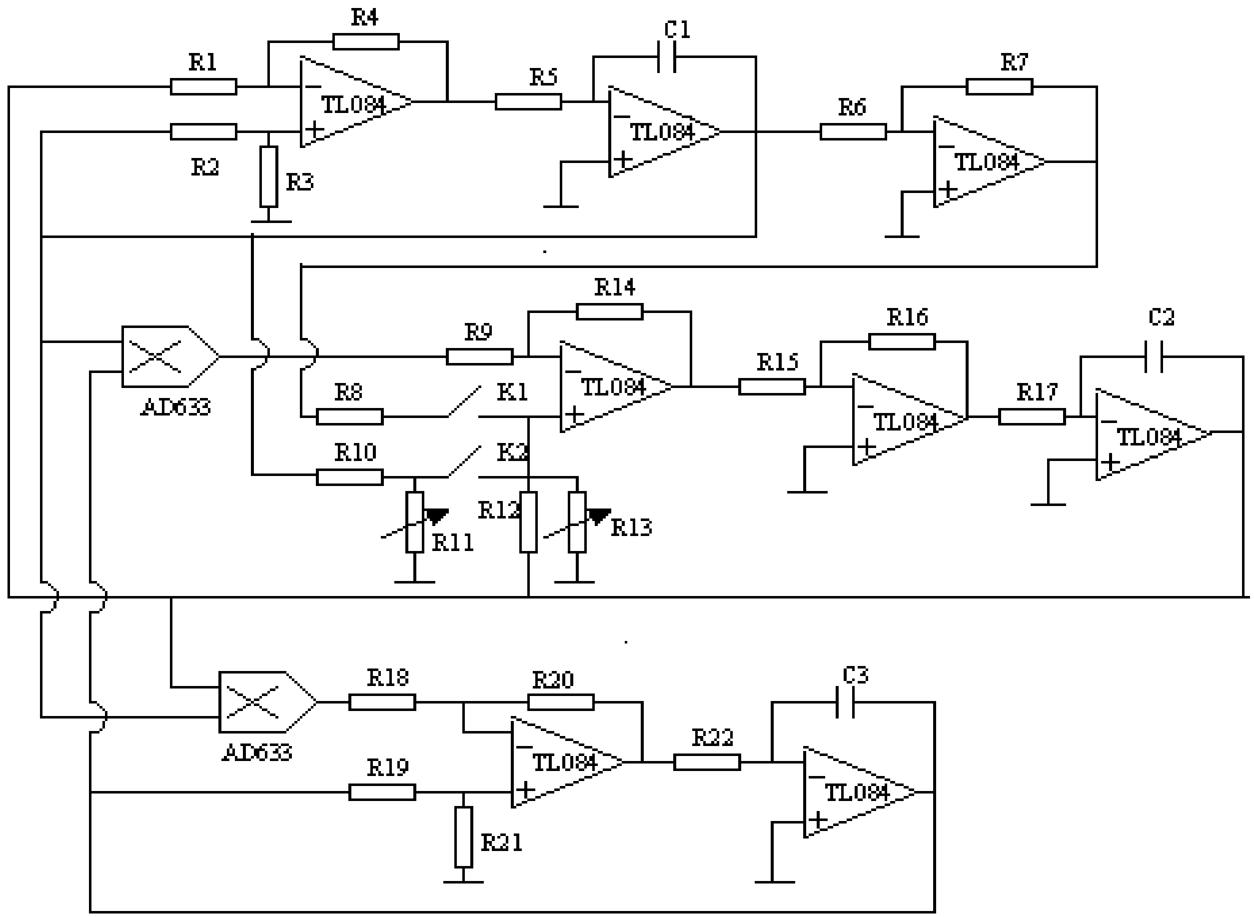
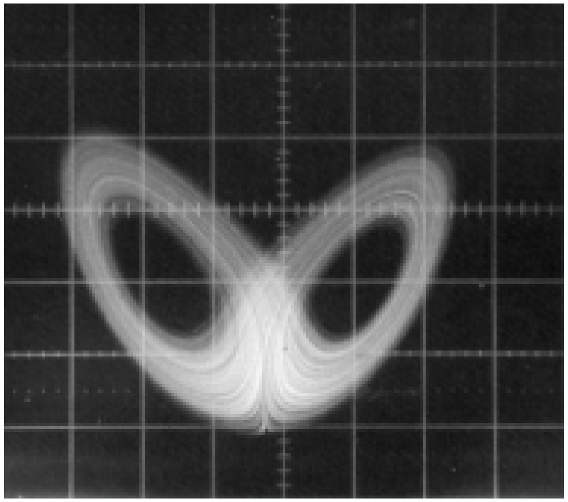
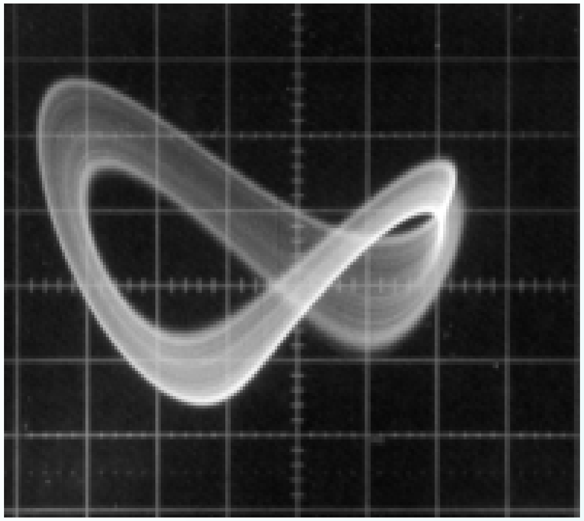
Circuit Implementation

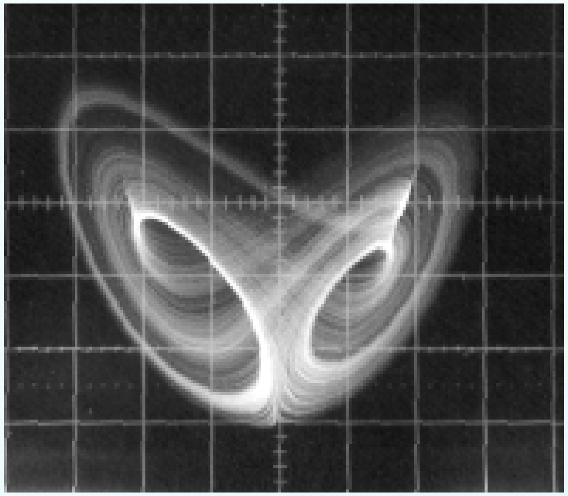


Y. X. Li, K. S. Tang and G. Chen, "Circuit design and implementation of a unified chaotic system," Proceedings, 2006 Int. Conference on Communications, Circuits and Systems, 25-28 June 2006, 2569 – 2572.

Experimental Observations







Generalized Lorenz Canonical Form

$$\dot{z} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} z + (cz) \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{bmatrix} z, \quad \lambda_1 > 0, \ \lambda_{2,3} < 0,$$

where

$$z = [z_1, \ z_2, \ z_3]^T, c = [1, \ -1, \ 0] \quad -\lambda_2 > \lambda_1 > -\lambda_3 > 0, \quad \tau \in R$$

Lorenz: $0 < \tau < +\infty$ **Lü:** $\tau = 0$ **Chen:** $-1 < \tau < 0$ **?:** $\tau \le -1$

S. Čelikovsky and G. Chen, "On a generalized Lorenz canonical form of chaotic systems," Int. J. of Bifurcation and Chaos, 12, 1789-1812, 2002.

T. S. Zhou, G. Chen, S. Čelikovsky, "Si'lnikov chaos in the generalized Lorenz canonical form of dynamics systems," Nonlinear Dynamics, 39(4): 319-334, 2005.

Proof

Show the existence of a heteroclinic orbit between two saddle-focus fixed points

(a constructive approach)

- **Start from a series expansion of the heteroclinic orbit**
- **Substituting it into the characteristic equation of the system**
- **Force it to satisfy the basic properties as a heteroclinic orbit**
- **Force it to satisfy the Shilnikov conditions**
- **Guarantee the uniform convergence of the series expansion**

Hyperbolic Generalized Lorenz Canonical Form

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} x + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\operatorname{sgn}(\tau+1) \\ 0 & 1 & 0 \end{bmatrix} x,$$

where $x = [x_1, \quad x_2, \quad x_3]^T, \ \lambda_1 > 0, \lambda_{2,3} < 0 \qquad \tau \leq -1$

Lorenz: $Eigenvalues = \{0, \quad \pm j\}$ **HGLC:** $Eigenvalues = \{0, \quad \pm 1\}$

S. Čelikovsky and G. Chen, "Hyperbolic-type generalized Lorenz system and its canonical form," in Proceedings of the 15th Triennial World Congress of IFAC, Barcelona, Spain, July 2002
S. Čelikovsky and G. Chen, "On the generalized Lorenz canonical form," Chaos, Solitons and Fractals, 26, 1271-1276, 2005.

The case of $\tau = -1$

Shimizu-Morioka Model (1976):

$$\begin{cases} \dfrac{dx}{d\theta} = y \\ \dfrac{dy}{d\theta} = x(1-z) - \lambda y \\ \dfrac{dz}{d\theta} = -\alpha z + x^2 \end{cases}$$

It is the case of the Generalized Lorenz Canonical Form with $\tau = -1$

$$\begin{cases} x = (z_1 - z_2)\sqrt{\dfrac{\lambda_1 - \lambda_2}{(-\lambda_1\lambda_2)^{3/2}}} \\ y = (\lambda_1 z_1 - \lambda_2 z_2)\sqrt{\dfrac{\lambda_1 - \lambda_2}{(-\lambda_1\lambda_2)^{5/2}}} \\ z = z_3 \dfrac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2} \\ \theta = t\sqrt{-\lambda_1\lambda_2}, \end{cases}$$

$$\lambda = -\frac{\lambda_1 + \lambda_2}{\sqrt{-\lambda_1\lambda_2}}, \quad \alpha = \frac{\lambda_3}{\sqrt{-\lambda_1\lambda_2}}.$$

T. Shimizu and N. Morioka, "On the bifurcation of a symmetric limit cycle to an asymmetric one in a simple model," Phys. Lett. A, 76(3-4), 201-204, 1976.

A Special Case

Liu-Liu-Liu-Liu Model (Xi'an JTU, 2004):

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = bx - kxz \\ \dot{z} = -cz + hx^2, \end{cases}$$

where $a, b, c, k, h > 0$

It is a special case of the Shimizu-Morioka Model (1976). Therefore,

It is a special case of the Generalized Lorenz Canonical Form with $\tau = -1$

C. Liu, T. Liu, L. Liu and K. Liu, "A new chaotic attractor," Chaos, Solitons and Fractals, 22, 1031-1038, 2004.

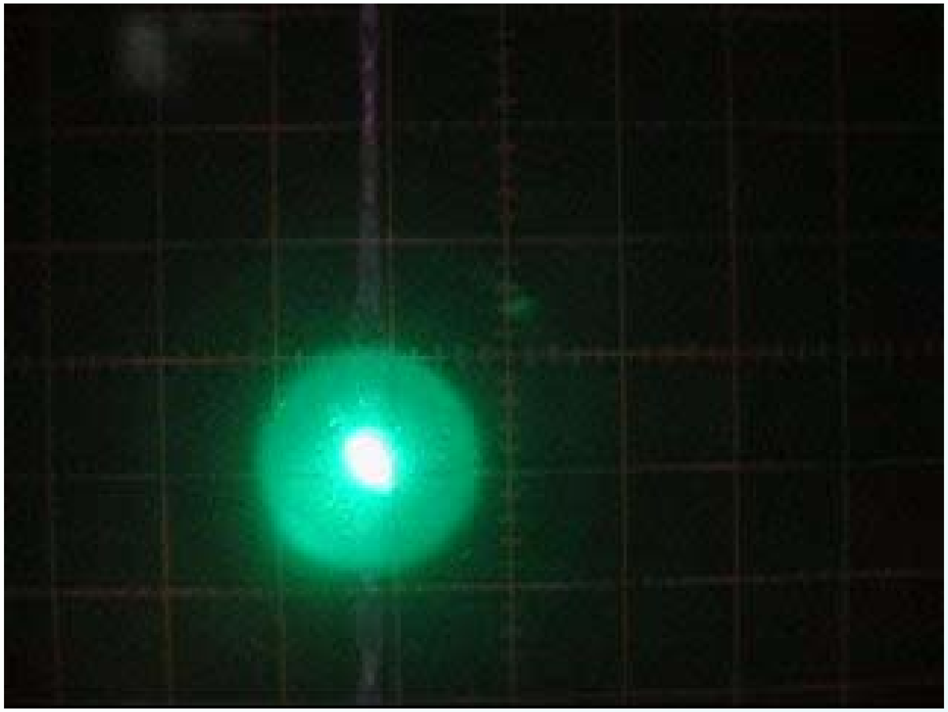
Summary

Generalized Lorenz Canonical Form (GLCF) and Its Special Realization

GLCF	Special Chaotic Systems
$\tau \in (-\infty, -1)$	Hyperbolic Generalized Lorenz System (2002)
$\tau = -1$	Shimizu-Morioka System (1979)
$\tau \in (-1, 0)$	$a_{21}a_{12} < 0$ Chen System (1999)
$\tau = 0$	$a_{21}a_{12} = 0$ Lü System (2002)
$\tau \in (0, \infty)$	$a_{21}a_{12} > 0$ Lorenz System (1963)

广义 Lorenz 系统族

Transition Between Lorenz and Chen Attractors



Controlling Chaotic Chen System To Hyperchaotic

- **Using a simple dynamical state-feedback controller**

- **Using a simple sinusoidal parameter perturbation input**

From Chaos
To Hyperchaos

Controlled Generalized Lorenz Canonical Form:

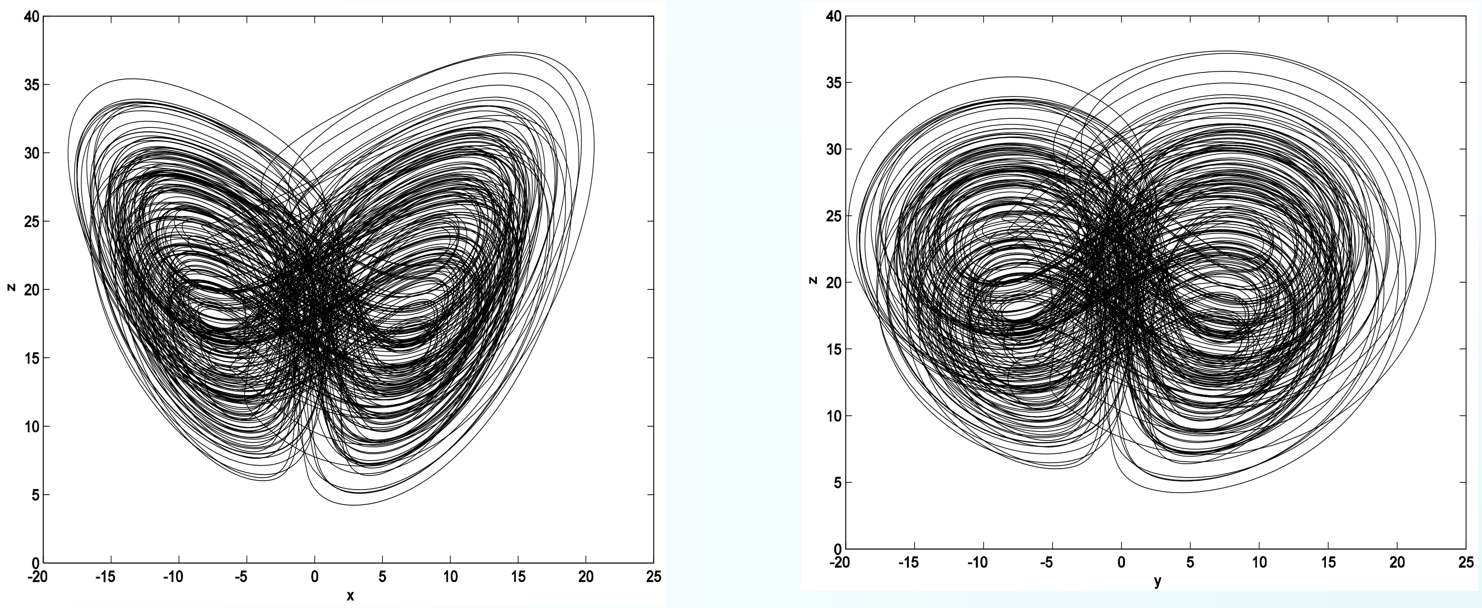
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 0 & 0 & a_{33} & 0 \\ -k & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} + x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$$

where *u* is the controller and *k* is the constant control gain to be determined (a very simple dynamical linear state feedback controller).

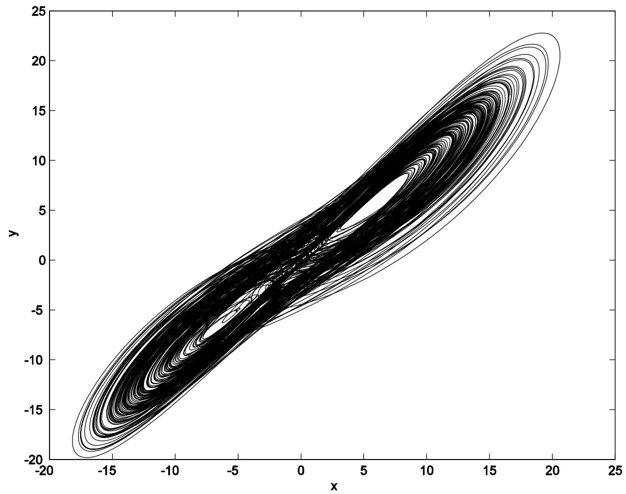
Controlling the Chen System

Parameters:

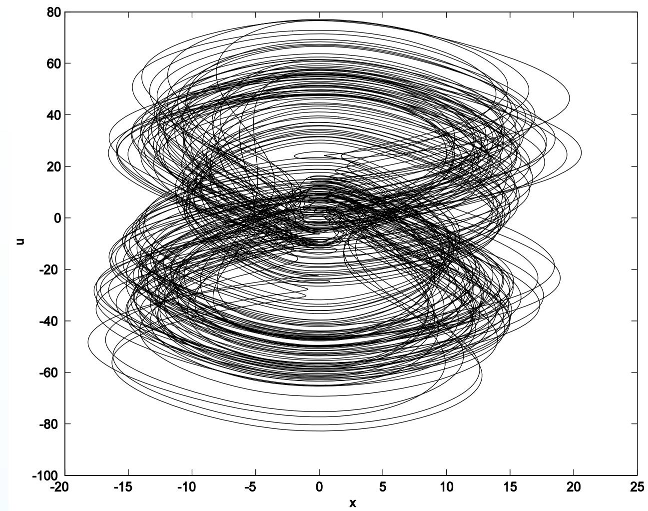
$$a_{11} = -a_{12} = -35, \quad a_{21} = 7, \quad a_{22} = 12,$$
$$a_{33} = -3, \quad k = 20$$



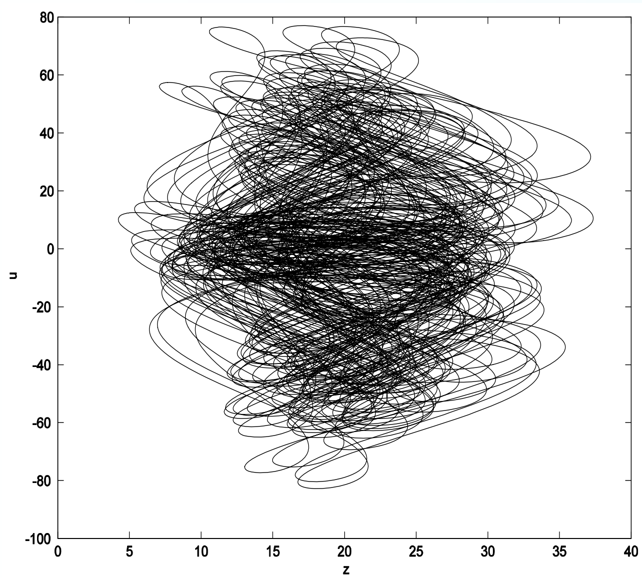
(a) *x vs z* (b) *yvs z*



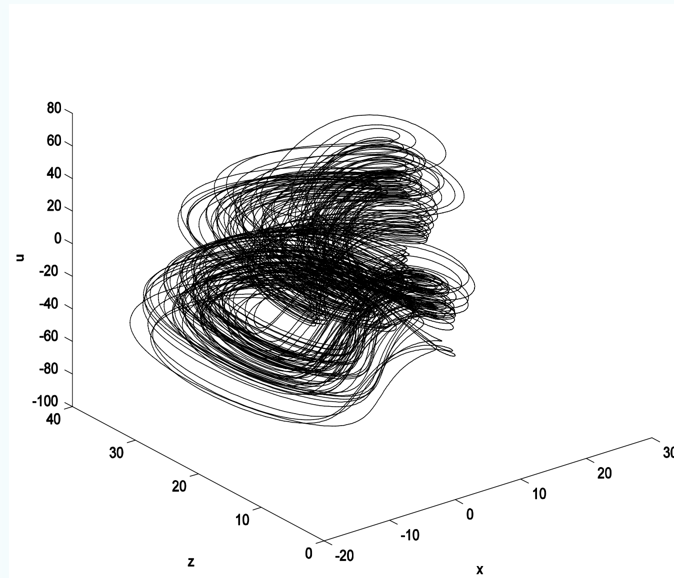
(c) *x vs y*



(d) *x vs u*

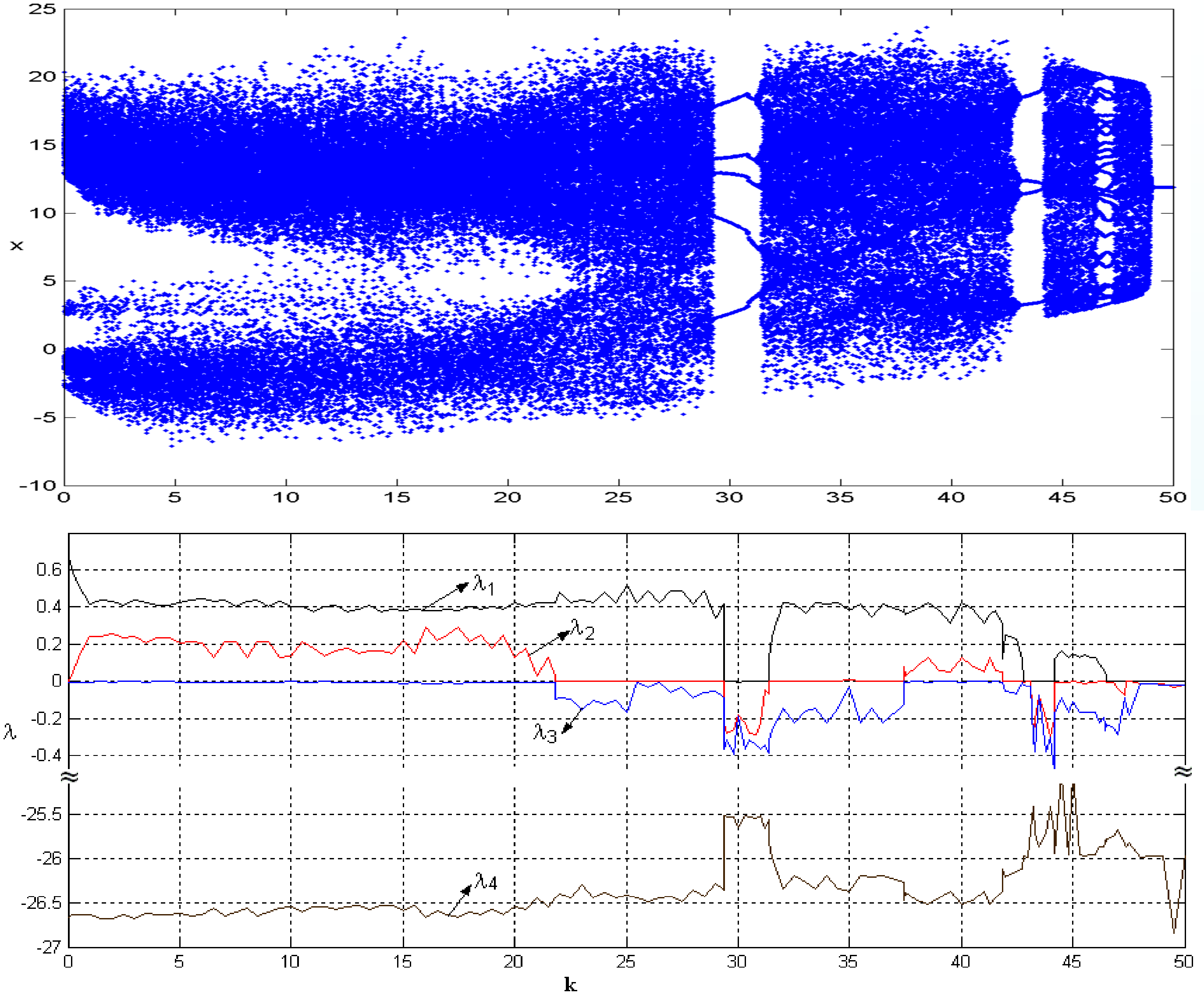


(e) *z vs u*

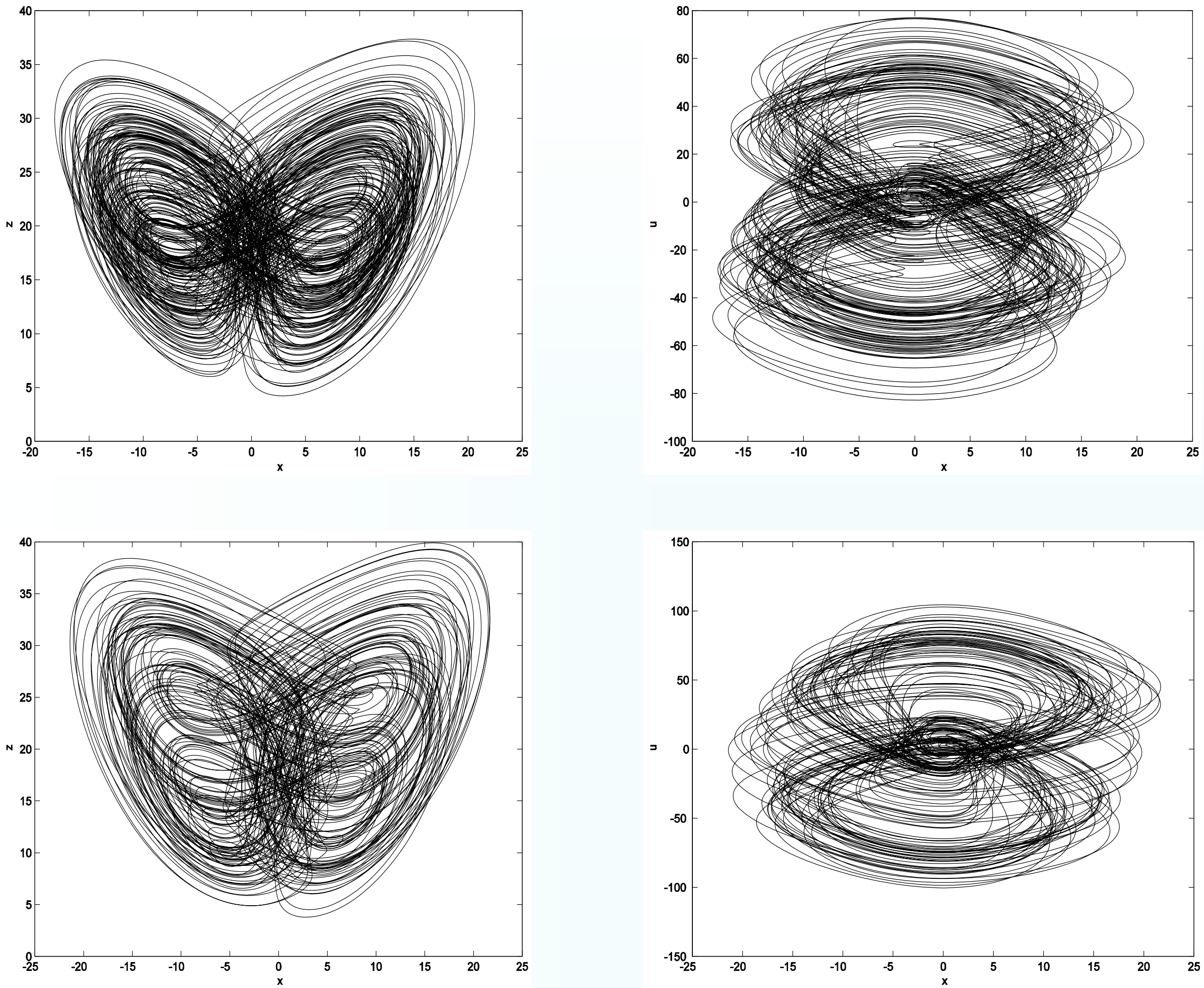


(f) x, *z vs u*

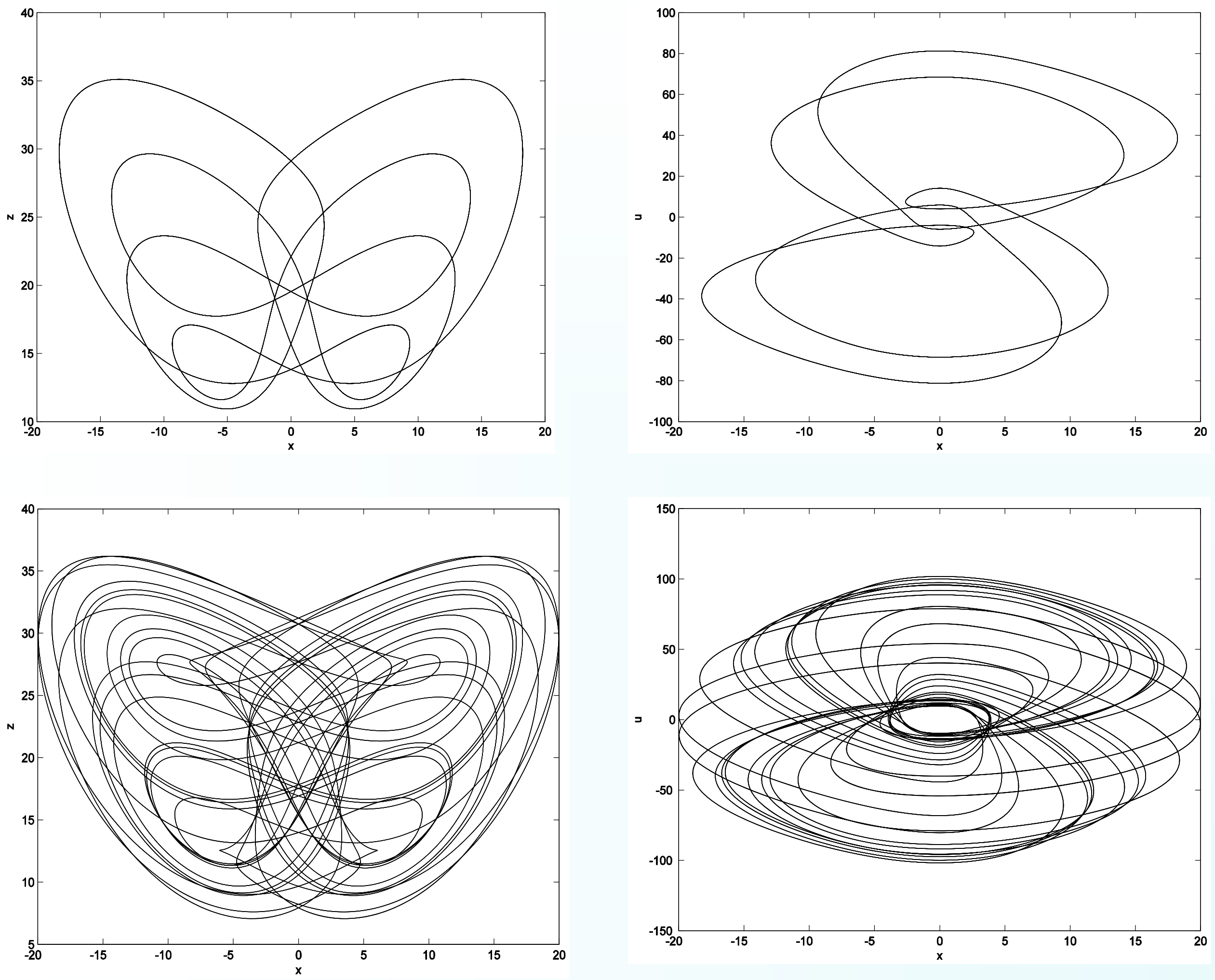
Bifurcation Analysis



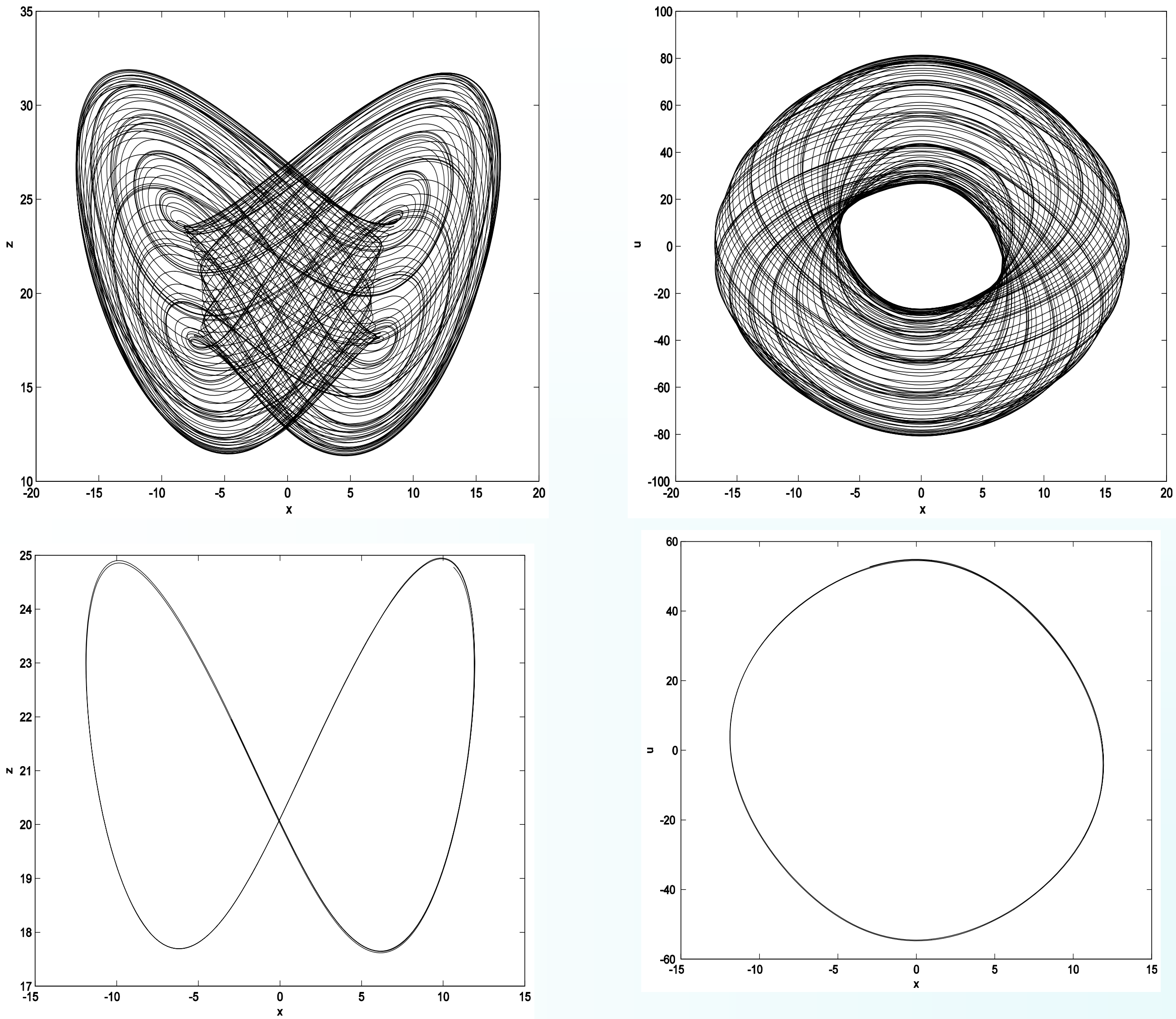
Visualizing the Bifurcation Process



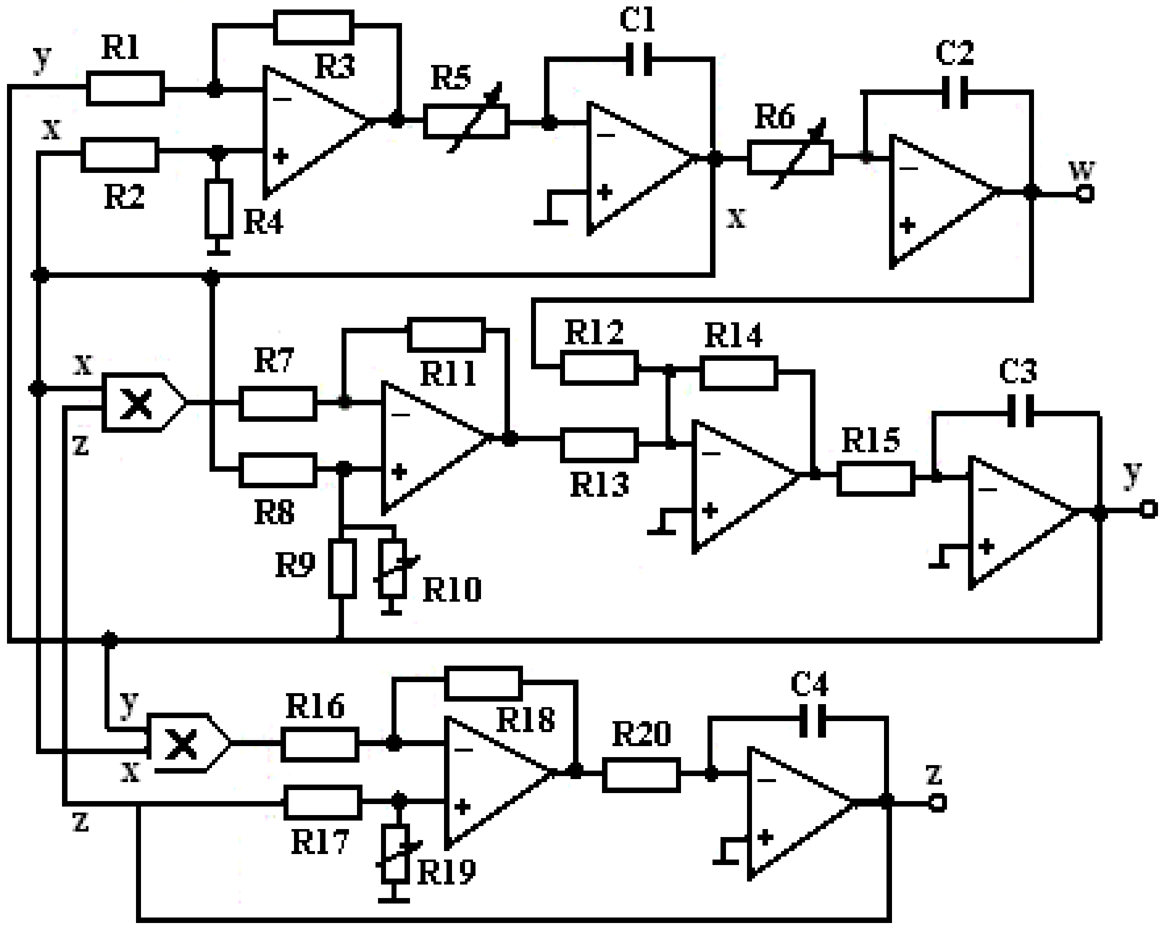
Visualizing the Bifurcation Process

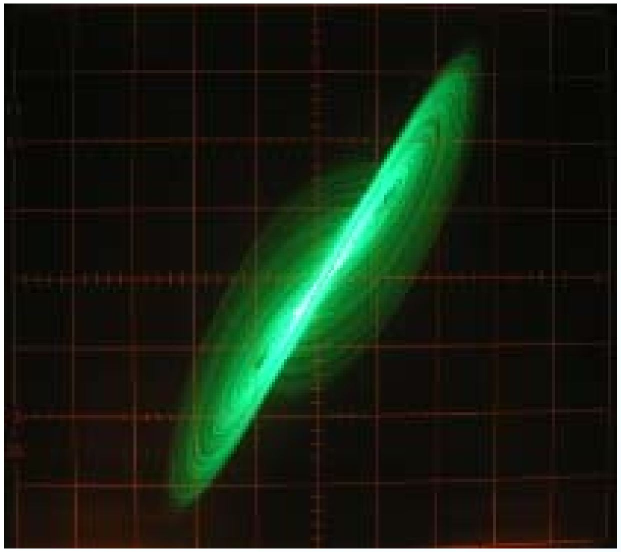


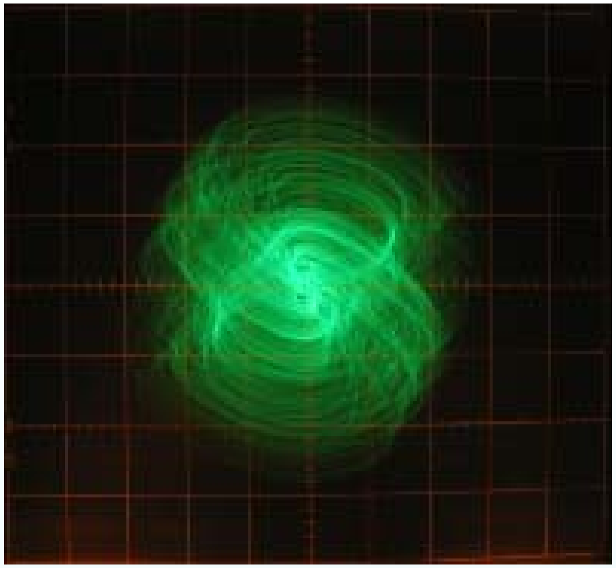
Visualizing the Bifurcation Process

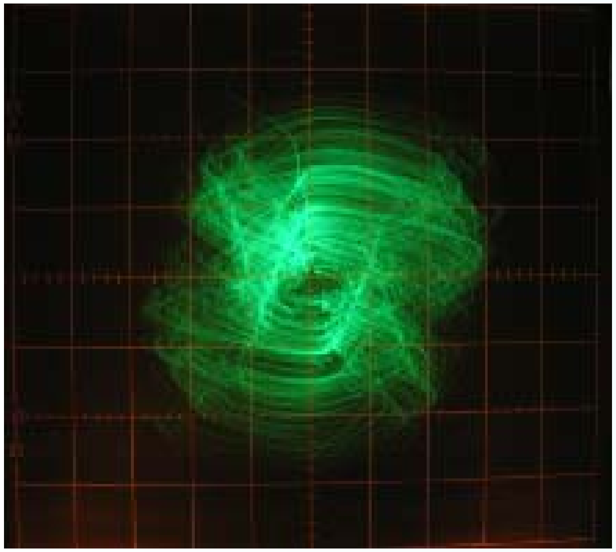


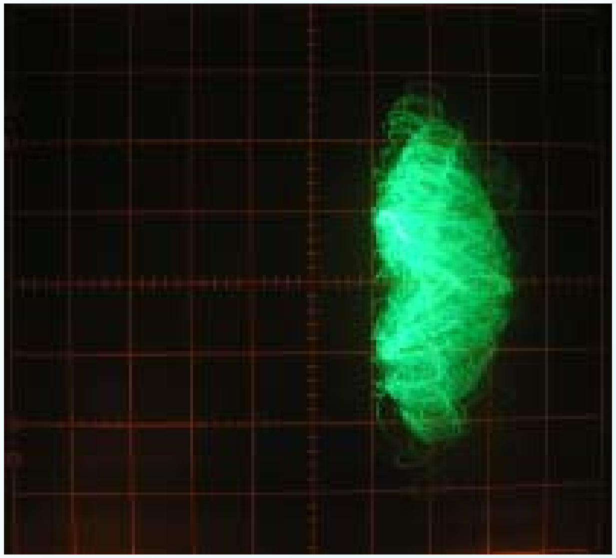
Circuit Implementation
(Hyperchaotic Chen system)

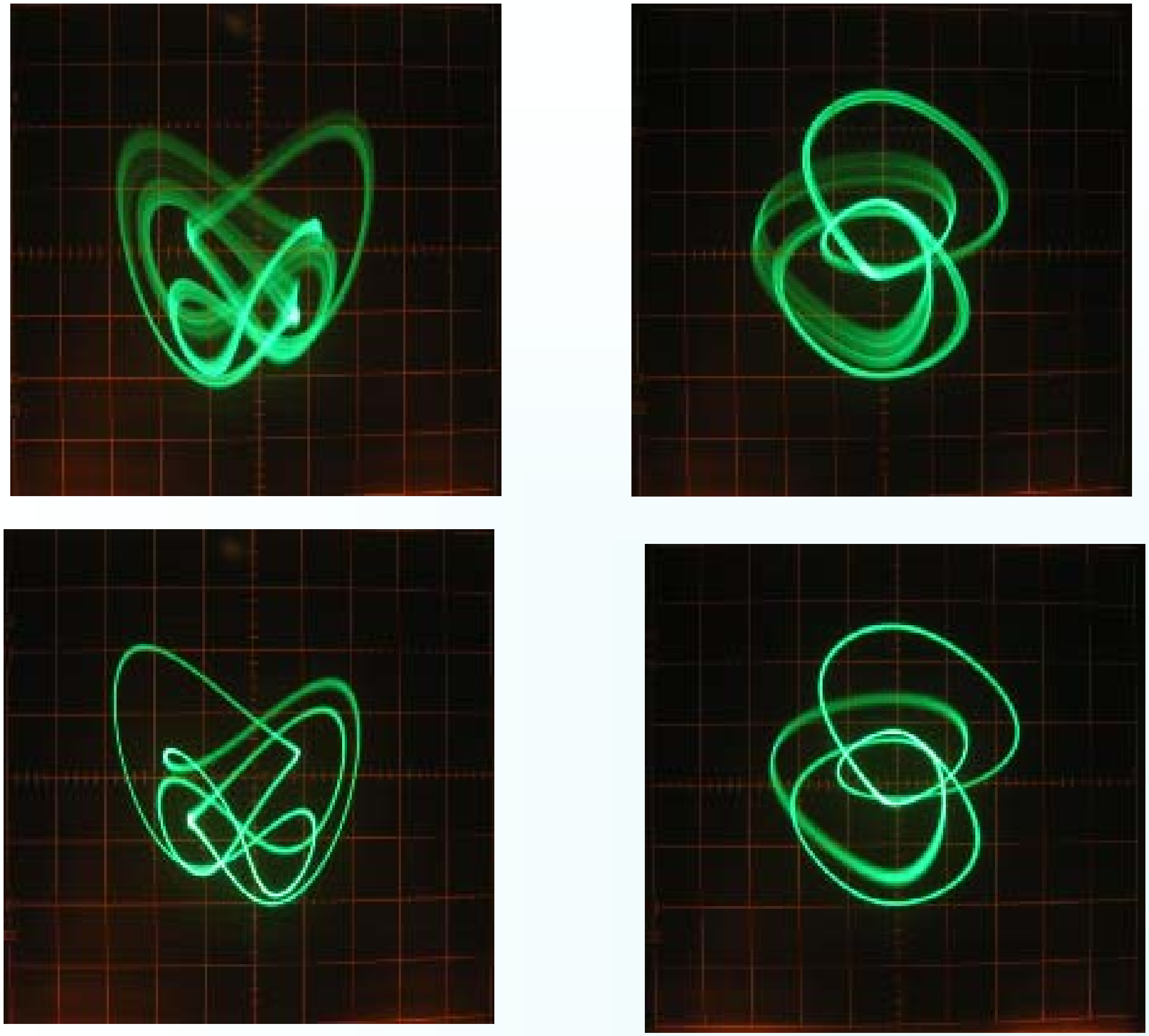


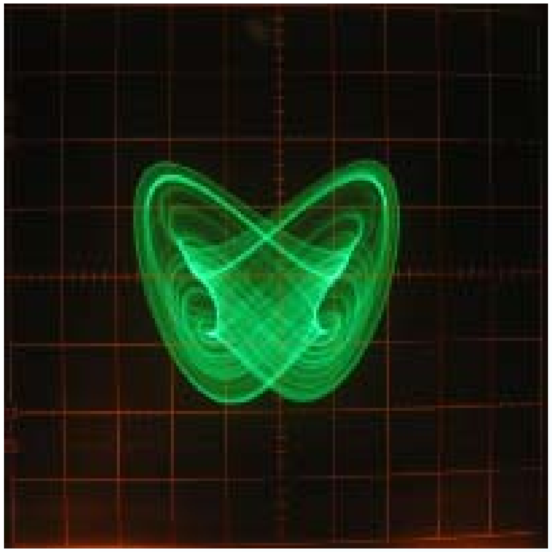


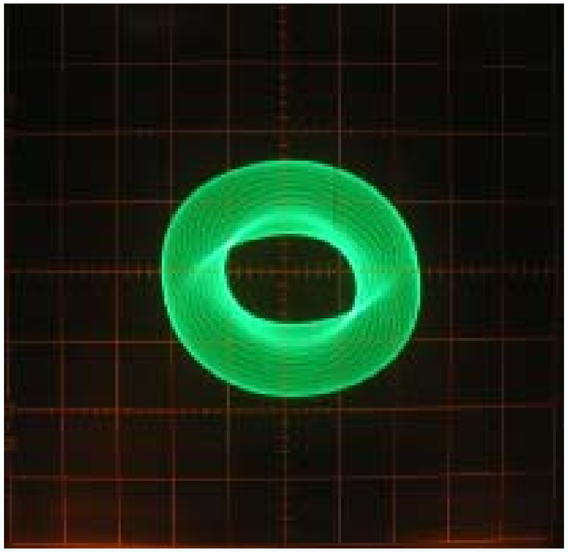


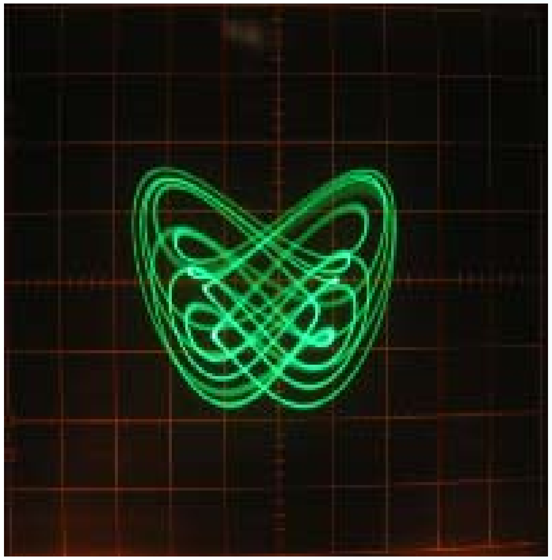












Sinusoidal Control Input

The Controlled Unified Chaotic Systems:

$$\dot{x} = (25 - 10a)\,(y - x)$$

$$\dot{y} = (17.5a + 10.5)x - sign(a)xz + (13.3 - 14a)y$$

$$\dot{z} = sign(a)xy - \frac{8}{3}z$$

where $a = \cos(\omega t) \in [-1,1]$ and $sign(u) = \begin{cases} 1 & u \geq 0 \\ -1 & u < 0 \end{cases}$

Under Sinusoidal Control:

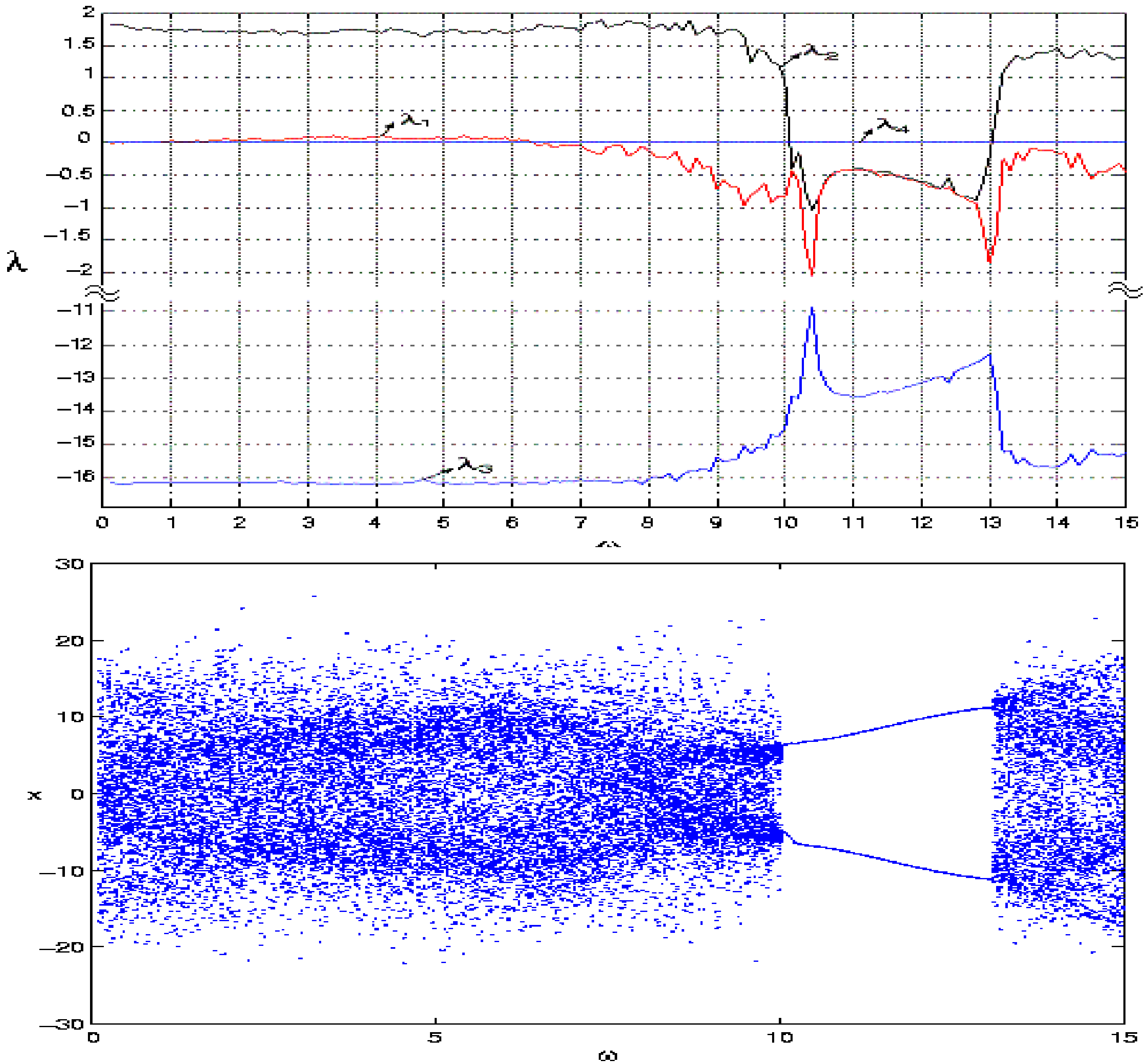
$$a = \cos(\omega t) \in [-1,1]$$

According to the canonical-form criterion:

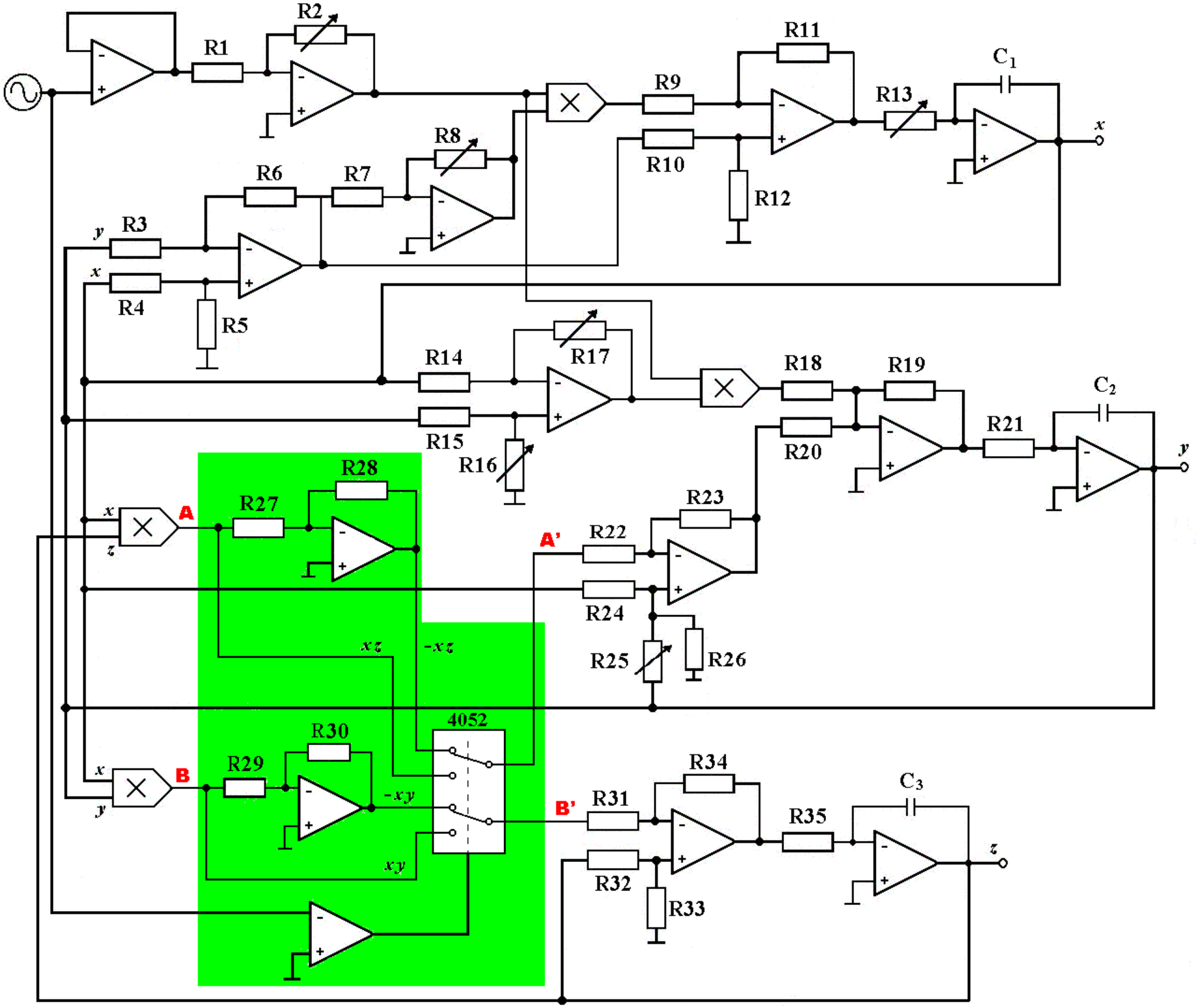
$$t \in \left[\frac{1}{\omega}\left(2n\pi + \frac{\pi}{2}\right), \frac{1}{\omega}\left(2n\pi + \frac{3\pi}{2}\right)\right)$$ → **Generalized Lorenz System**

$$t \in \left[\frac{1}{\omega}\left(2n\pi - \frac{\pi}{2}\right), \frac{1}{\omega}\left(2n\pi + \frac{\pi}{2}\right)\right]$$ → **Generalized Chen System**

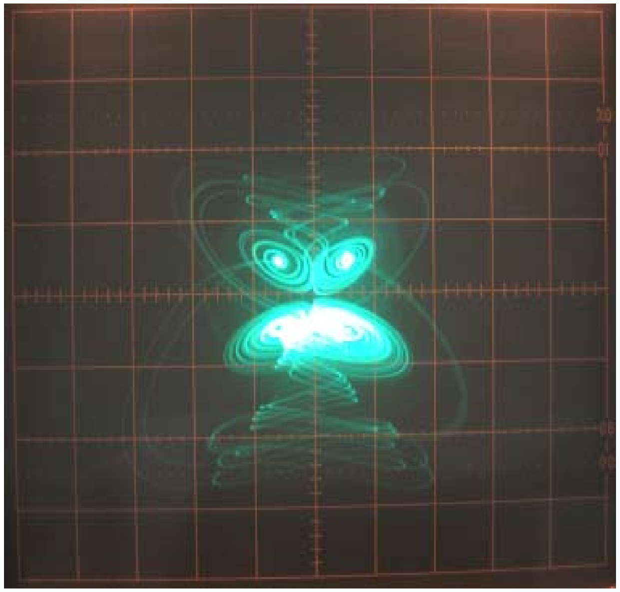
Bifurcation Analysis

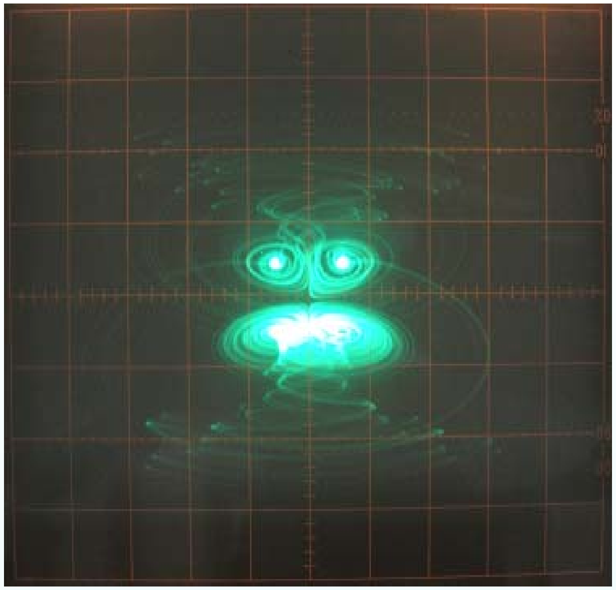


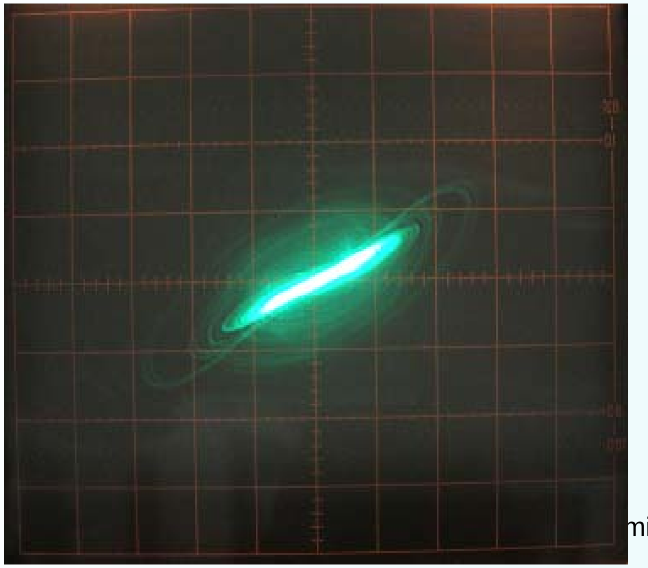
Circuit Design



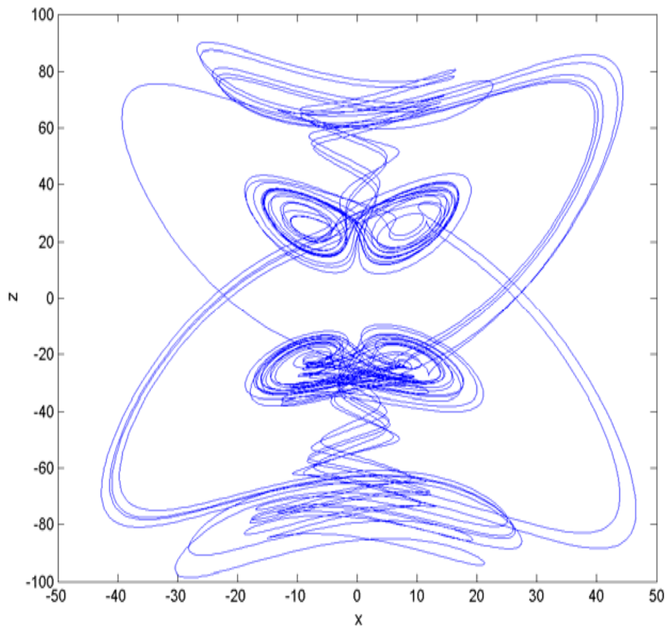
Experimental Results

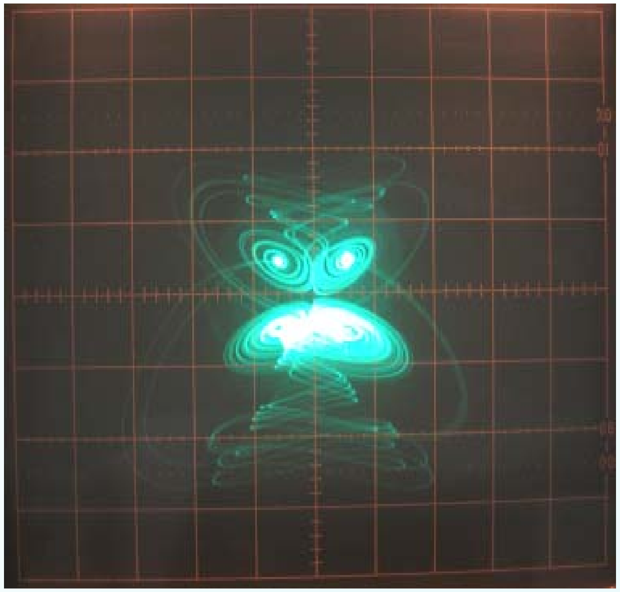






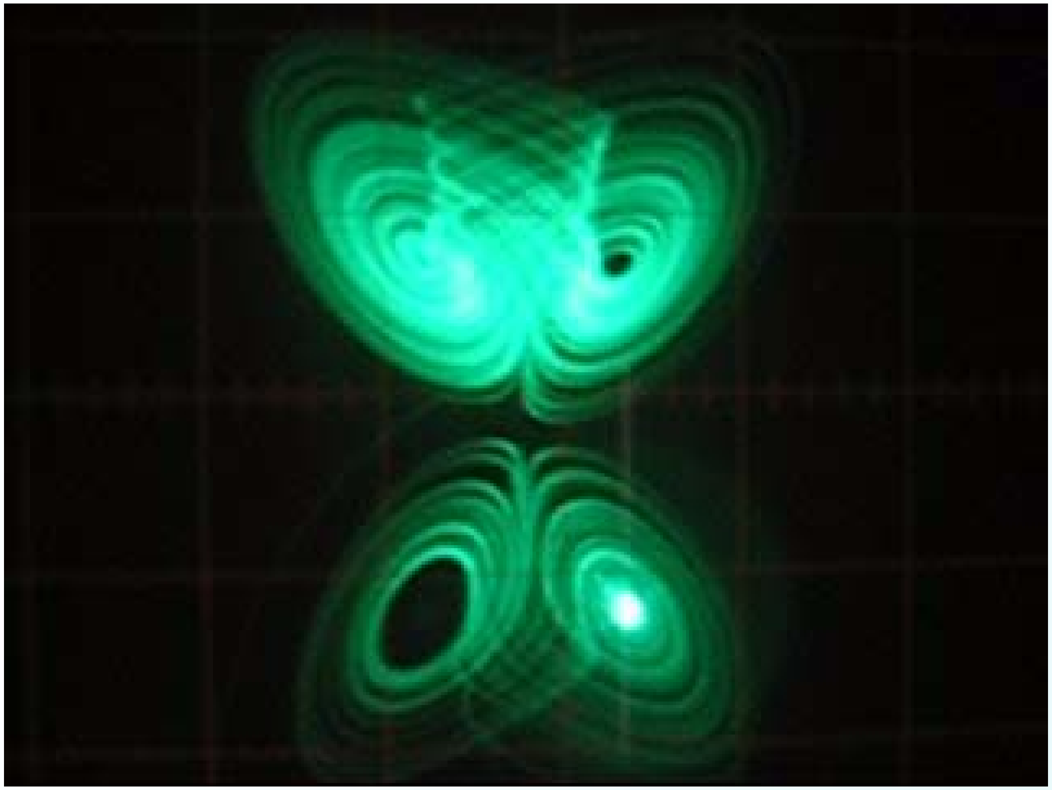
Observing Hyperchaotic Chen Attractor





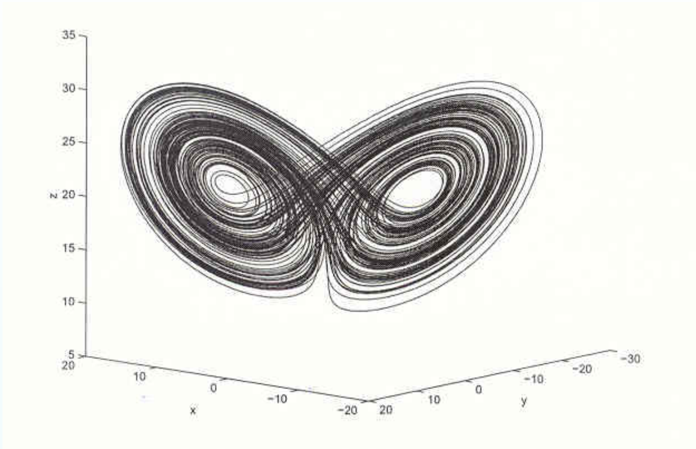
Y. X. Li, K. S. Tang and G. Chen, "Hyperchaotic Chen's system and its generation", DCDIS, 14, 97-102, 2007.

Hyperchaotic Chen Attractor



Fractional-Order Chen System

$$\begin{cases} \dfrac{d^\sigma x}{dt^\sigma} = a(y-x) \\ \dfrac{d^\sigma y}{dt^\sigma} = (c-a)x - xz + cy \\ \dfrac{d^\sigma z}{dt^\sigma} = xy - bz, \end{cases}$$



$$\frac{d^\sigma f(t)}{dt^\sigma} = \frac{1}{\Gamma(n-\sigma)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\sigma-n+1}} d\tau$$

When $\sigma = 0.6 \sim 0.7$ **this system is chaotic** [J. G. Lu and G. Chen (2006): $\sigma = 0.3$]

C. G. Li and G. Chen, "Chaos in the fractional order Chen system and its control," Chaos, Solitons and Fractals, 22, 549-554, 2004.

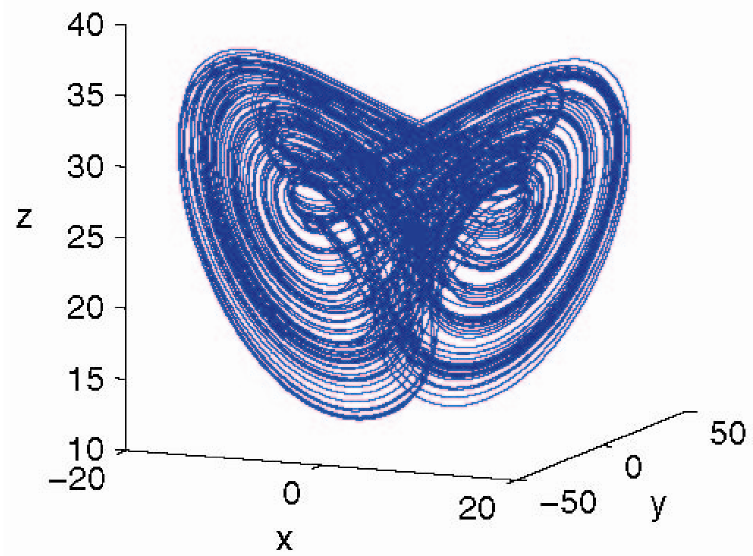
Fractional-Order Chen System

$$\frac{d^\alpha x}{dt^\alpha} = a(y-x) + \gamma \cos(u)\,,$$

$$\frac{d^\alpha y}{dt^\alpha} = (c-a)x - xz + cy\,,$$

$$\frac{d^\alpha z}{dt^\alpha} = xy - bz\,,$$

$$\frac{du}{dt} = \omega\,,$$

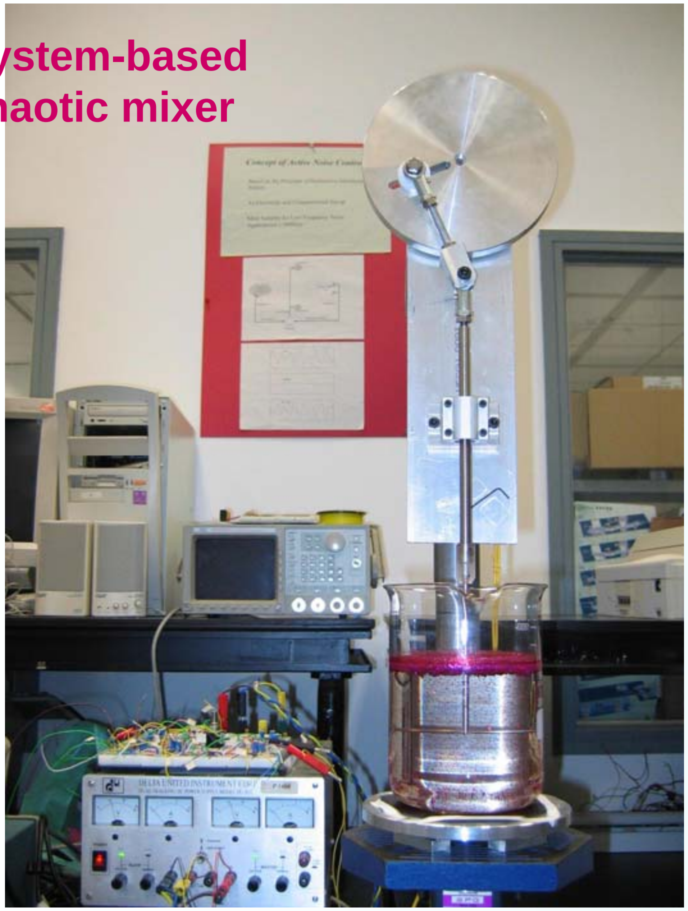


When $a = 35, b = 3, c = 32, \gamma = 35, \alpha = 0.8, \omega = 15$ this system is hyperchaotic

H. Zhang, C. G. Li, G. Chen, X. Gao, "Hyperchaos in the fractional-order nonautonomous Chen's system and its synchronization," Int J. Modern Phys. C, 16, 815-826, 2005.

Some Potential Applications

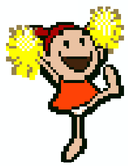
Chen-system-based hyperchaotic mixer



Related Technology:

- 吕金虎、禹思敏、陈关荣 (专利)：一种四阶网格状多环面混沌电路及其使用方法；2005年申请，2009年批准；专利号 ZL2005 1 0086638.2.
- 吕金虎、禹思敏、陈关荣 (专利)：一种涡卷混沌信号发生器及其使用方法；2005年申请，2009年批准；专利号ZL 2005 1 0086603.9.

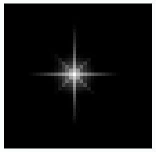
Concluding Remarks

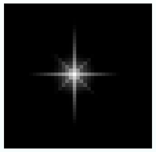


Lorenz

Done !

☺





Rössler

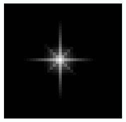
?

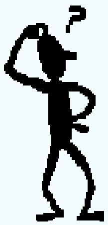
3-D Autonomous with

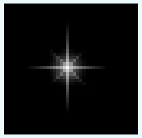
1 or 2 Quadratic Terms

Sprott

?







(Others)

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Thank You !

with the compliments of

Guanrong Chen 陈关荣

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City University of Hong Kong

http://www.ee.cityu.edu.hk/~gchen/