International Journal of Bifurcation and Chaos, Vol. 28, No. 1 (2018) 1850018 (8 pages)

DOI: 10.1142/S0218127418500189

On Lorenz and Chen Systems

Ruy Barboza
Department of Electrical and Computer Engineering,
São Carlos School of Engineering, University of São Paulo,
13566-590 São Carlos, SP, Brazil
rbz@usp.br

Received October 15, 2017

In this work, a simple comparative analysis on the Lorenz and the Chen systems is presented, in order to understand some aspects that make these two systems distinct from one another.

Keywords: Lorenz system; Chen system; Lorenz parameters; Chen attractor.

1. Introduction

The Lorenz system is considered the most important chaotic system of differential equations in the study of chaos theory. Since its discovery [Lorenz, 1963], thousands of papers have been dedicated to investigating its properties. These investigations make use of several methods and approaches. One such approach is the use of modified Lorenz systems, also called Lorenz-like systems, which are characterized either by removing terms from the equations (see e.g. [van der Schrier & Maas, 2000]) or by modifying some terms of the equations. An example of the latter is the Chen system which, along with the original Lorenz system, is the subject of this work.

The paper is organized as follows: in Sec. 2 the basic equations are presented for future reference; in Sec. 3 some distinctive features of the two systems are presented and discussed; in Sec. 4 a unified system describing both the Lorenz and the Chen systems is presented and analyzed.

2. Equations

The Lorenz system is given by three coupled first-order equations:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - xz - y, \quad \dot{z} = xy - bz, \tag{1}$$

where b, σ and r are positive real parameters. When investigating the Lorenz system, Chen and Ueta [1999] had the idea of modifying it by adding a control function $u(x, y, z)$ to the second equation of (1):

$$\dot{y} = rx - xz - y + u(x, y, z). \tag{2}$$

By using the anticontrol theory [Chen & Dong, 1998] and numerical experiments, they chose the linear function $u = -\sigma x + (1 + r)y$ as the one simple enough giving interesting results [Ueta & Chen, 2000]. Thereafter, the following Lorenz-like system became known as the Chen system:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= (r - \sigma)x - xz + ry, \\ \dot{z} &= xy - bz. \end{aligned} \tag{3}$$

By setting $\dot{x} = \dot{y} = \dot{z} = 0$ the equilibrium points of the two systems are obtained as:

$$\text{Lorenz: } x = y = z = 0; \quad x = y = \pm\sqrt{b(r - 1)}, \quad z = r - 1, \tag{4a}$$

$$\text{Chen: } x = y = z = 0; \quad x = y = \pm\sqrt{b(2r - \sigma)}, \quad z = 2r - \sigma. \tag{4b}$$

Note that nontrivial equilibria will exist for the Lorenz system in the r-range $(1, \infty)$ and, for the Chen system, $(\sigma/2, \infty)$.

3. Five Distinguishing Features of the Two Systems

3.1. *Positive feedback*

Despite their similarities, the Lorenz and the Chen systems can be regarded as distinct because the Chen system has a feature that is absent in the Lorenz system, namely positive feedback. Each one of the three first-order Lorenz equations is dissipative, characterized, respectively, by the terms $-\sigma x$, $-y$ and $-bz$. On the other hand, the second equation of the Chen system is regenerative, characterized by the positive-feedback term $+ry$, as a result of the control function $u(x, y, z)$ applied to the Lorenz system. This fundamental property of the Chen system is the main cause of other distinguishing features between the two systems, further discussed as follows.

3.2. *Dissipation*

An important quantity associated with system (1) is the trace of its Jacobian, given by $-(\sigma+b+1)$. If the trace is negative then any volume V_0 of initial conditions in the phase space evolves toward $V(t) = 0$, meaning that the Lorenz system is dissipative in the whole (b, σ, r)-space. The intervention (2) modified the trace, now given by $-(\sigma + b - r)$, implying that the Chen system is dissipative only in the region below the plane given by $r = \sigma + b$.

3.3. *Hopf bifurcation*

Associated with system (1) is the critical r-value given by the Hopf theorem [Sparrow, 1982]:

$$r_0 = \sigma\frac{\sigma + b + 3}{\sigma - b - 1} \tag{5}$$

which describes a surface separating the region of sinks and sources from that of periodic and chaotic dynamics. To derive (5) one first writes the characteristic equation for the nontrivial equilibria as $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. Then Eq. (5) is obtained from the relation $a_0 = a_1a_2$, where $a_0 = 2\sigma b(r-1)$, $a_1 = b(\sigma + r)$, and $a_2 = \sigma + b + 1$. In addition, the following condition for transversality is verified to hold at r_0 in order to guarantee a Hopf bifurcation: $da_0/dr \neq d(a_1a_2)/dr$.

Analogously, for system (3) the critical r-value is given by [Lü *et al.*, 2002]:

$$r_0 = \frac{1}{2}(\sqrt{17\sigma^2 - 6b\sigma + b^2} - 3\sigma + b). \tag{6}$$

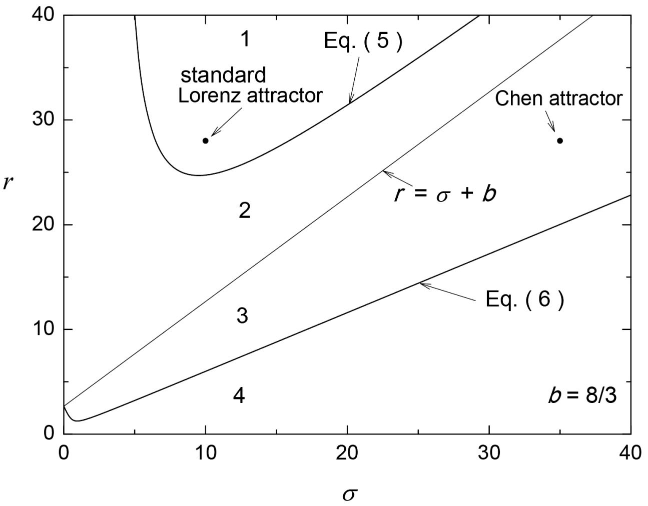

Fig. 1. Hopf bifurcation diagrams of Eqs. (1) and (3) for $b = 8/3$.

The importance of (5) and (6) for the sake of comparing (1) and (3) is better appreciated by displaying them graphically as shown in Fig. 1, with constant $b = 8/3$. For the Lorenz system, chaos and periodic solutions occur in region 1, while in regions 2–4 the solutions converge toward a sink (a more detailed explanation of the solutions' behavior in these regions is found in [Sparrow, 1982]). Analogously, for the Chen system, chaos and periodic solutions occur in region 3, while in region 4 the solutions tend toward a sink. In regions 1 and 2 the Chen system is nondissipative. Special attention is directed to the plane $r = \sigma + b$, which separates the Hopf surfaces given by (5) and (6):

$$\text{Lorenz: } r_0 > \sigma + b, \tag{7a}$$

$$\text{Chen: } r_0 < \sigma + b. \tag{7b}$$

Relations (7a) and (7b) can easily be obtained from (5) and (6), respectively. Therefore regions 1 and 3 never intersect each other, whatever the values of b, σ and r are. This result demonstrates the correctness of the Chen and Ueta [1999] approach, since they aimed at showing that by using even a linear controller, one can obtain chaos in a large region of the (b, σ, r)-space where no chaos exist in the unmodified Lorenz system — a technique called anticontrol of chaos that could be applicable to other chaotic systems [Chen & Dong, 1998].

3.4. *Trapping regions*

Several simple trapping regions for the Lorenz system have been reported, also by Lorenz himself.

On the other hand, a trapping region for the Chen system was not identified before the one shown in this paper, though with a description comparatively more involved than that for the Lorenz system.

3.4.1. *A simple trapping region for the Lorenz system*

► [Lorenz, 1979]: Consider the sphere

$$S : x^2 + y^2 + (z - \sigma - r)^2 = R^2 \tag{8a}$$

and the ellipsoid

$$E : \sigma x^2 + y^2 + b\left(z - \frac{\sigma}{2} - \frac{r}{2}\right)^2 = b\left(\frac{\sigma}{2} + \frac{r}{2}\right)^2. \tag{8b}$$

The sphere S is a trapping region for system (1) provided that the diameter of S exceeds the maximum diameter of E.

3.4.2. *A trapping region for the Chen system*

► [Barboza & Chen, 2011]: Consider the following transformation of variables:

$$X = \bar{x}, \quad Y = \bar{y} - C\bar{x}, \quad Z = D - \bar{z}, \quad \tau = bt, \tag{9}$$

where $\bar{x} = x/b$, $\bar{y} = y/b$ and $\bar{z} = z/b$. Then (3) can be rewritten as

$$\begin{aligned} \dot{X} &= AY - BX, \\ \dot{Y} &= -AX + XZ - Y, \\ \dot{Z} &= -XY - Z - CX^2 + D, \end{aligned} \tag{10}$$

where here the dot means $d/d\tau$. The parameters A, B, C and D are defined in Sec. 3.5. Now, for any $Z_0 \leq 0$ and $U_0 \geq \frac{1}{2}(D - Z_0)^2$, consider the ellipsoids

$$E_i : \frac{1}{2}\left(\frac{A - Z_i}{A}X^2 + Y^2 + (Z - Z_i)^2\right) = U_i, \qquad i = 0, 1, 2, \ldots \tag{11a}$$

where, for $i \geq 1$,

$$Z_i = Z_{i-1}\left(1 + \frac{B}{AC}\right) - \frac{B - 1}{C}, \tag{11b}$$

$$U_i = \frac{A - Z_i}{A - Z_{i-1}}\left[U_{i-1} - \frac{(Z_{i-1} - Z_i)^2}{2}\right] \tag{11c}$$

with $B > \frac{A}{A - Z_0}$. In the upper half-space $H_i : Z > Z_i$, let $\Omega_i = E_i^V \cap H_i$, where E_i^V is the region enclosed by E_i. Moreover, for $N \geq 0$, define

$$\Omega_{0,1,\ldots,N} = \bigcup_{i=0}^{N} \Omega_i. \tag{11d}$$

Then, there exists an integer N, $0 \leq N < \infty$, such that $\Omega_{0,1,\ldots,N}$ is a trapping region for system (3).

3.5. ***Number of parameters***

3.5.1. *The Chen system is exactly a two-parameter system*

Consider the Chen system as given by (10). It was pointed out in the "Appendix C" of [Barboza & Chen, 2011] that the following identities hold:

$$B = A - AC, \quad D = 2AC - C - 1 \tag{12}$$

where $A = \sigma/b$ and $C = (b/\sigma)(1 + r/b)$. Relations (12) reveal that the only independent parameters present in (10) are A and C. Clearly, this implies that when using Eq. (3) one shall have in mind that any of the three parameters r, σ and b can be expressed as a function of the other two. Accordingly, since the above referenced work is dated 2011, formula (12) was the first proof that the Chen system can be described by only two parameters.[1] Obviously (9), (10) and (12) rely on the following basic transformation:

$$\bar{x} = \frac{x}{b}, \quad \bar{y} = \frac{y}{b}, \quad \bar{z} = \frac{z}{b}, \quad \tau = bt \tag{13}$$

and on the following two-parameter b-scaled system obtained from (3) and (13):

$$\begin{gathered} \dot{\bar{x}} = \bar{\sigma}(\bar{y} - \bar{x}), \quad \dot{\bar{y}} = (\bar{r} - \bar{\sigma})\bar{x} - \bar{x}\bar{z} + \bar{r}\bar{y}, \\ \dot{\bar{z}} = \bar{x}\bar{y} - \bar{z} \end{gathered} \tag{14}$$

where $\bar{\sigma} = \sigma/b$ and $\bar{r} = r/b$. Note that (9) is the same as (13) after translation and rotation, thus the preliminary Eqs. (13) and (14) are implicit in (9), (10) and (12).[2] Also note that there is no special

[1] Such two-parameter property of the Chen system was decisive for the proof of its boundedness in [Barboza & Chen, 2011].

[2] Equations (13) and (14) were omitted in [Barboza & Chen, 2011] in order to avoid redundancy and save space.

reason to choose b as the scaling parameter, since choosing either σ or r would work equally well for the proof of the trapping region in Sec. 3.4.2. It should be mentioned that scaled equations such as Eqs. (13) and (14) were later derived independently first by Leonov [2013] and then by Algaba *et al.* [2013]. See also [Leonov & Kusnetzov, 2015].

3.5.2. *The Lorenz system is essentially a two-parameter system*

Consider a plot of the standard Lorenz butterfly attractor with $r = 28$, $\sigma = 10$ and $b = 8/3$. Clearly the same attractor is obtained (though magnified) with $r = k \cdot 28$, $\sigma = k \cdot 10$ and $b = k \cdot 8/3$, where $1 \leq k < \infty$. More precisely, all the qualitative properties of the system would be preserved after multiplying all the parameters by k. This is because in the usual range of parameters found in the literature, the Lorenz system behaves essentially as a two-parameter system. To understand this more clearly, first write (1) with parameters kr, $k\sigma$ and kb, then apply transformation (13) with b replaced by k. After redefining x, y, z and t, the resulting system is identical to (1) but with the last term of the second equation divided by k:

$$\dot{y} = rx - xz - \frac{y}{k}. \tag{15}$$

Combining (15) with the first equation $\dot{x} = \sigma(y - x)$ one obtains:

$$\ddot{x} + \left(\sigma + \frac{1}{k}\right)\dot{x} - \sigma\left(r - \frac{1}{k}\right)x = -\sigma xz. \tag{16}$$

Finally, by integrating (16) in order to return to a first-order system one obtains:

$$\begin{aligned} \dot{x} &= \sigma y - \left(\sigma + \frac{1}{k}\right)x, \\ \dot{y} &= \left(r - \frac{1}{k}\right)x - xz, \\ \dot{z} &= xy - bz - \frac{x^2}{k\sigma}. \end{aligned} \tag{17}$$

Figure 2 shows the one-dimensional bifurcation diagrams for $k = 1$ and $k = 1000$. The ordinates in Fig. 2 correspond to successive maxima of the variable z. Observe that apart from a scale factor the two diagrams have identical structures. Thus one can say that the Lorenz system is essentially a two-parameter system in the sense that it has the same properties as the following reduced system:

$$\begin{gathered} \dot{x} = \sigma(y - x), \quad \dot{y} = rx - xz, \\ \dot{z} = xy - bz - \frac{x^2}{k\sigma} \end{gathered} \tag{18}$$

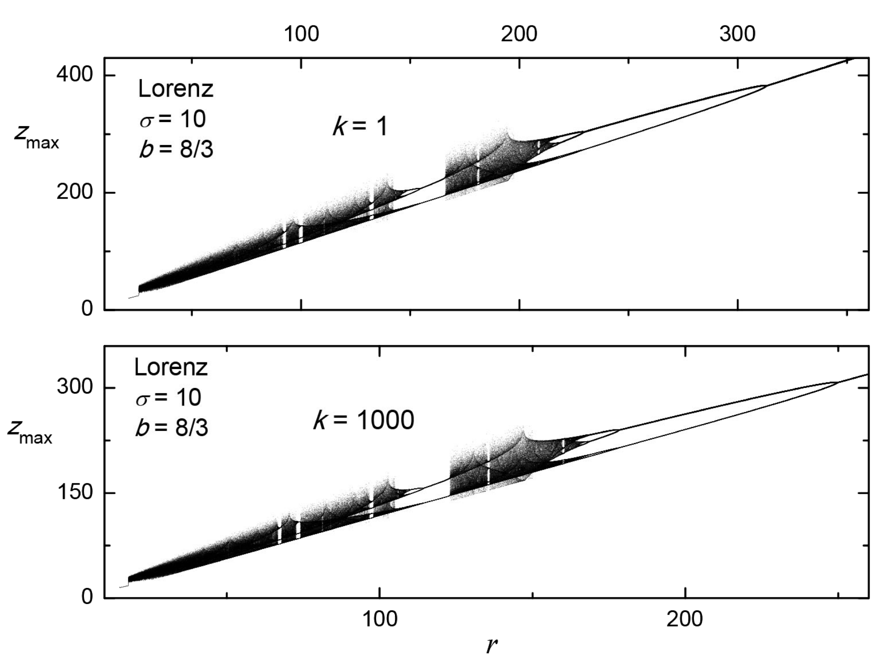

Fig. 2. One-dimensional bifurcation diagrams of the Lorenz system (17) for two values of k.

which for any arbitrary $k \geq 1$ can be recast as a two-parameter system via (13). Note that in general one can assume that $|xy| \gg x^2/(k\sigma)$, therefore the last term of the $\dot{z}$-equation can be removed from (18).

4. Joint Analysis Using a Unified System

4.1. *Three-parameter Chen system*

An r-scaled Chen system can be obtained analogously as (14) by applying (13) to (3), with b replaced by r, thus

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{\sigma}(\tilde{y} - \tilde{x}), \\ \dot{\tilde{y}} &= \tilde{r}\tilde{x} - \tilde{x}\tilde{z} + \tilde{y}, \\ \dot{\tilde{z}} &= \tilde{x}\tilde{y} - \tilde{b}\tilde{z}, \end{aligned} \tag{19}$$

where $\tilde{r}$ is defined as *a new independent parameter*, with $\tilde{x} = x/r$, $\tilde{y} = y/r$, $\tilde{z} = z/r$, $\tilde{\sigma} = \sigma/r$, $\tilde{b} = b/r$ and $\tilde{t} = rt$, with the overdot meaning $d/d\tilde{t}$. Hence, for convenience in the task of comparing it with the Lorenz system (1), Eq. (19) is introduced here as a generic *three-parameter* Chen system, having Eq. (3) as an important *special case* for which one has $\tilde{r} = 1 - \sigma/r$.

4.2. *Dropping the tildes*

For ease of comparison with (1) let the tildes of (19) be dropped and from now on consider both (1) and (19) as controlled by the same parameters b, σ and r, along with the same time-scale t. Therefore the symbols b, σ, r and t here will not be confounded with those previously used for the original Chen system (3).

4.3. *Simplest unified system*

Systems (1) and (19) can be unified as

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - xz - my, \\ \dot{z} &= xy - bz \end{aligned} \tag{20}$$

which describes either the Lorenz system ($m = 1$) or the Chen system ($m = -1$). Note that Eq. (20) with $m = -1$ is the simplest version of the Chen system since it differs from the Lorenz equations by the single term $+y$.

4.4. *Equilibria and parameter ranges*

The equilibrium points of (20) are

$$x = y = z = 0; \quad x = y = \pm\sqrt{b(r - m)}, \quad z = r - m. \tag{21}$$

In the present section the restriction imposed in Sec. 2 is removed, allowing each parameter r, σ and b to take any real value. Note in (21) that in the case of $b > 0$ the range of r for the Chen system ($m = -1$) can be extended to negative values still guaranteeing nontrivial equilibria:

$$r > m \quad \begin{cases} r \in (1, \infty) & \text{Lorenz} \\ r \in (-1, \infty) & \text{Chen.} \end{cases} \tag{22}$$

4.5. *The regenerative term*

By adopting Eq. (20) for both the Lorenz and the Chen systems it is crucial to explain their distinct behaviors, as already detailed in Sec. 3.1. Indeed, in the Lorenz system the last term $-y$ of the second equation merely adds dissipation and can in fact be removed from system (1) without affecting the main qualitative properties of the system — see Sec. 3.5.2. In contrast, in the Chen system the term $+y$ makes the $\dot{y}$-equation regenerative, thus making the Chen system potentially unstable.

4.6. *Hopf bifurcation*

The Jacobian corresponding to (20) has the trace $-(\sigma + b + m)$, therefore the system is dissipative only if $\sigma + b > -m$. Proceeding as in Sec. 3.3, the critical r-value for Eq. (20) is obtained as

$$r_0 = \sigma\frac{m(\sigma + b) + 3}{m(\sigma - b) - 1}. \tag{23}$$

This equation is illustrated in Fig. 3, which is analogous to Fig. 1, showing r_0 as a function of σ with constant $b = 3/28$ for $m = \pm 1$. Referring in the figure only to positive parameters b and σ, for the Lorenz system, chaos and periodic solutions occur in region 1, while in regions 2–4 the solutions converge toward a sink (see [Sparrow, 1982] for a better explanation). Analogously, for the Chen system chaos and periodic solutions occur in regions 1 and 3, while in region 4 the solutions tend toward a sink. In region 2 the Chen system is nondissipative.

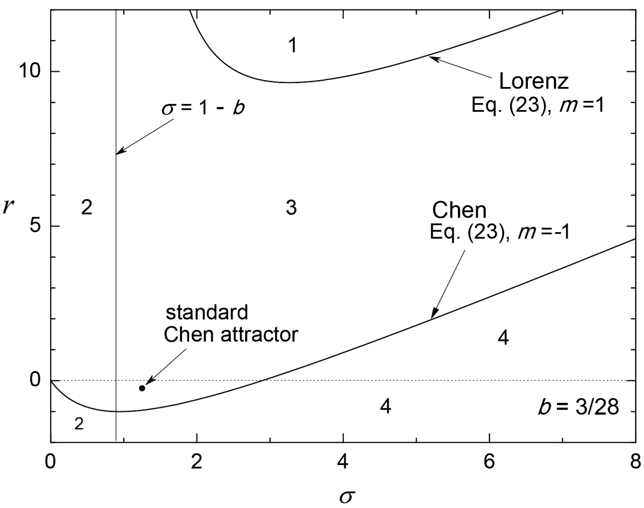

Fig. 3. Hopf bifurcation diagram given by (23) for $b = 3/28$.

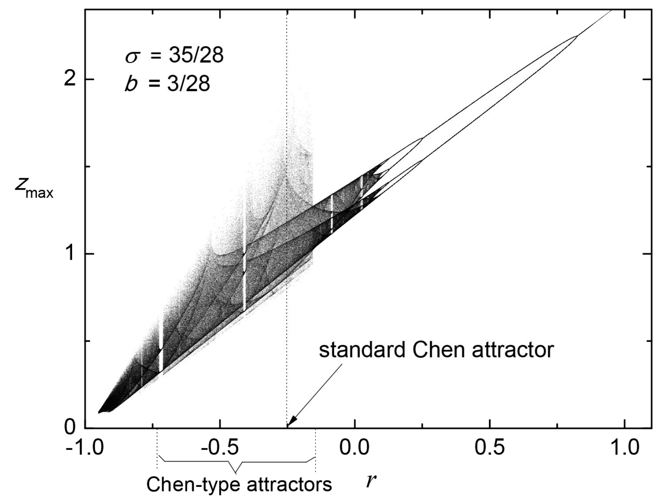

Fig. 4. One-dimensional bifurcation diagram of the Chen system (20), $m = -1$, for $b = 3/28$ and $\sigma = 35/28$.

4.7. *One-dimensional bifurcation diagrams*

By using the same procedure as in Sec. 3.5.2 with $k = 1$, the unified system (20) can be expressed as

$$\begin{aligned} \dot{x} &= \sigma y - (\sigma + m)x, \\ \dot{y} &= (r - m)x - xz, \\ \dot{z} &= xy - bz - \frac{mx^2}{\sigma} \end{aligned} \tag{24}$$

which indicates that for large σ and r the bifurcation diagrams of (20) are qualitatively the same for both $m = 1$ and $m = -1$. On the other hand, for small values of σ and r the term my becomes important, and the distinct behavior of the Chen system when compared with the Lorenz system takes place. In fact, for small r the Lorenz system ($m = 1$) is nonchaotic, in contrast with the Chen system ($m = -1$), as shown in Fig. 4, which displays a one-dimensional bifurcation diagram of the Chen system with r as the varying parameter along with small fixed values of σ and b. Note that for positive b and σ, chaotic solutions of the Chen system can be obtained for both positive and negative r.

4.8. *Symmetry*

4.8.1. *Parameters space*

Consider the parameters space (b, σ, r). Since dissipation does not depend on r, in Fig. 5 only the plane (b, σ) is shown. The region where the Lorenz system is dissipative is given by $\sigma + b > -1$, shown as regions 1 and 2. In region 3 the Lorenz system is nondissipative. The Chen system (20) with $m = -1$ is dissipative only in region 1, given by $\sigma + b > +1$, being nondissipative in regions 2 and 3. Also shown in Fig. 5 is a point $P_1 = (b, \sigma, r)$ and its symmetric image $P_2 = (-b, -\sigma, -r)$.

4.8.2. *Equations*

The basic and important symmetry property of the Lorenz equations is well known [Sparrow, 1982; Dullin *et al.*, 2007], which is also valid for the Chen system. It shows that the system equations are invariant under replacement of the variables x and y by $-x$ and $-y$, respectively. Applying it to (20)

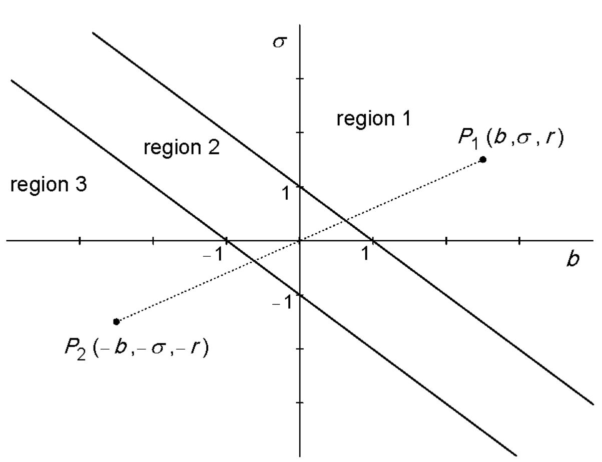

Fig. 5. Dissipative regions $\sigma + b > -m$ of the Lorenz and Chen systems.

yields

$$\begin{aligned}(-\dot{x}) &= \sigma[(-y) - (-x)],\\ (-\dot{y}) &= r(-x) - (-x)z - m(-y),\\ +\dot{z} &= (-x)(-y) - bz.\end{aligned} \tag{25}$$

As usual, the above operation can be denoted as $(x, y, z; t; \sigma, b, r; m) \to (-x, -y, z; t; \sigma, b, r; m)$. Now, observe that Eq. (25) remains the same if the minus signs before the variables x and y are displaced to z, t, b, σ, r and m:

$$\begin{aligned}-\dot{x} &= (-\sigma)(y - x),\\ -\dot{y} &= -rx - x(-z) - (-m)y,\\ -(-\dot{z}) &= xy - (-b)(-z).\end{aligned} \tag{26}$$

Here the minus signs on the left-hand side indicate that the integration of (26) has to be done backward in time, because $t \to -t$, thus $d/dt \to d/d(-t)$. The above operation can be denoted as

$$(x, y, z; t; \sigma, b, r; m) \to (x, y, -z; -t; -\sigma, -b, -r; -m). \tag{27}$$

For simplicity, assume $\sigma > 0$ and $b > 0$. Then, with $z' = -z$ and $m = \pm 1$, consider the Lorenz and the Chen systems listed separately, along with their symmetric versions:

$$\begin{aligned}
&\text{A} \begin{cases}\dot{x} = \sigma(y - x)\\ \dot{y} = rx - xz - y \quad \text{Lorenz,}\\ \dot{z} = xy - bz\end{cases}\\
&\text{B} \begin{cases}-\dot{x} = -\sigma(y - x)\\ -\dot{y} = -rx - xz' + y \quad \text{symmetric Lorenz,}\\ -\dot{z}' = xy + bz'\end{cases}\\
&\text{C} \begin{cases}\dot{x} = \sigma(y - x)\\ \dot{y} = rx - xz + y \quad \text{Chen,}\\ \dot{z} = xy - bz\end{cases}\\
&\text{D} \begin{cases}-\dot{x} = -\sigma(y - x)\\ -\dot{y} = -rx - xz' - y \quad \text{symmetric Chen.}\\ -\dot{z}' = xy + bz'\end{cases}
\end{aligned} \tag{28}$$

Note that on comparing the solutions of the above systems one has A = B $\neq$ C = D. However, because of the conversion $m \to -m$ the right-hand side of B looks like that of the Chen system at the point P_2 in Fig. 5, and the right-hand side of D looks like that of the Lorenz system at P_2. This allows the interpretation that the forward-time solutions of the Lorenz system corresponding to the point P_1 in Fig. 5 are the same as the backward-time solutions of the Chen system corresponding to the point P_2 — although with an inversion of the z-axis — and vice versa. Therefore the Lorenz and the Chen systems are *mutually symmetric*. Note that such solutions show identical stability properties. For example, a stable limit cycle obtained with A corresponds to a stable limit cycle obtained with B, and similarly for C and D.

Now, consider Eq. (29). Following Algaba *et al.* [2013], the solutions of the Lorenz system E are also the same as those of D, and thus of C, but with opposite stability properties (implying an unstable limit cycle in the above example) because $-t \to t$. Analogous considerations can be made involving A, B and the Chen system F.

$$\begin{aligned}
&\text{E} \begin{cases}\dot{x} = -\sigma(y - x)\\ \dot{y} = -rx - xz - y \quad \text{Lorenz, at } P_2,\\ \dot{z} = xy + bz\end{cases}\\
&\text{F} \begin{cases}\dot{x} = -\sigma(y - x)\\ \dot{y} = -rx - xz + y \quad \text{Chen, at } P_2.\\ \dot{z} = xy + bz\end{cases}
\end{aligned} \tag{29}$$

Therefore, the Lorenz system in the nondissipative region 3 of Fig. 5 can be analyzed using the Chen system in the dissipative region 1, and the Chen system in region 3 can be analyzed using the Lorenz system in region 1. This symmetry property is conceptually and aesthetically pleasing and makes the Lorenz and the Chen systems much more strongly related. One can say that the Chen system is symmetrically contained in the Lorenz system and vice versa. Despite this, however, the Lorenz and the Chen systems remain distinct. For example, as referred to in Sec. 3.4, a simple nonintricate trapping region continues to be difficult to find for the Chen system using forward- or backward-time integration, while it is very easy to find for the Lorenz system in forward time.

From the above, it seems clear that what really matters in practice is the investigation of the forward-time Lorenz and Chen systems in their respective dissipative regions of parameters. Moreover, as pointed out in [Leonov & Kusnetzov, 2015], a number of the main available tools for studying dissipative dynamical systems usually work well only in forward time, which makes the study of the Chen system meaningful besides the Lorenz system.

5. Conclusion

In this paper it was shown that at least five distinctive features can be identified in order to illustrate the differences between the Lorenz and the Chen systems. It was emphasized that the regenerative action of the term $+ry$ on the second equation of (3) is the essential particularity that is the source of all the main distinctive aspects of the Chen system in comparison with the Lorenz system. In addition it was shown that the Lorenz system behaves essentially as a two-parameter system. Finally, by using a simple unified system a symmetry property connecting the Lorenz and the Chen systems was established.

References

Algaba, A., Fernandez-Sanchez, F., Merino, M. & Rodriguez-Luis, A. [2013] "Chen's attractor exists if Lorenz repulsor exists: The Chen system is a special case of the Lorenz system," *Chaos* **23**, 033108.

Barboza, R. & Chen, G. [2011] "On the boundedness of the Chen system," *Int. J. Bifurcation and Chaos* **21**, 3373–3385.

Chen, G. & Dong, X. [1998] *From Chaos to Order: Methodologies, Perspectives and Applications* (World Scientific, Singapore); reprinted [2014].

Chen, G. & Ueta, T. [1999] "Yet another chaotic attractor," *Int. J. Bifurcation and Chaos* **9**, 1465–1466.

Dullin, H. R., Schmidt, S., Richter, P. H. & Grossmann, S. K. [2007] "Extended phase diagram of the Lorenz model," *Int. J. Bifurcation and Chaos* **17**, 3013–3033.

Leonov, G. [2013] "Criteria for the existence of homoclinic orbits of systems Lu and Chen," *Dokl. Math.* **87**, 220–223.

Leonov, G. A. & Kusnetzov, N. V. [2015] "On differences and similarities in the analysis of the Lorenz, Chen, and Lu systems," *Appl. Math. Comp.* **256**, 334–343.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmos. Sci.* **20**, 130–141.

Lorenz, E. N. [1979] "On the prevalence of aperiodicity in simple systems," *Global Analysis*, Lecture Notes in Mathematics, Vol. 755, eds. Grmela, M. & Marsden, J. E. (Springer, Berlin), pp. 53–75.

Lü, J., Zhou, T., Chen, G. & Zhang, S. [2002] "Local bifurcations of the Chen system," *Int. J. Bifurcation and Chaos* **12**, 2257–2270.

Sparrow, C. [1982] *The Lorenz Equations: Bifurcation, Chaos, and Strange Attractors* (Springer-Verlag, NY).

Ueta, T. & Chen, G. [2000] "Bifurcation analysis of Chen's equation," *Int. J. Bifurcation and Chaos* **10**, 1917–1931.

van der Schrier, G. & Maas, L. R. M. [2000] "The diffusionless Lorenz equations; Shil'nikov bifurcations and reduction to an explicit map," *Physica D* **141**, 19–36.