# Concluding Remarks

## Signals in Time Domain

For signals which are functions of time, there are two main types: continuous-time and discrete-time.

A continuous-time signal $x(t)$ is defined on a continuous range of time $t \in [T_1, T_2]$, i.e., $x(t)$ has a value for any $t \in [T_1, T_2]$. It can be observed in real world and examples include speech, music, power line and ECG.

A discrete-time signal $x[n]$ is defined only at discrete instants of time where $n$ is integer. It can be obtained from sampling a continuous-time signal or generated using computer.

# Continuous-Time and Discrete-Time Signal Conversion

$x[n]$ can be obtained from a continuous-time signal $x(t)$ via sampling:

$$x[n] = x(t)|_{t=nT} = x(nT), \quad n = \cdots - 1, 0, 1, 2, \cdots \qquad (10.1)$$

If $x(t)$ is bandlimited such that $X(j\Omega) = 0$ for $|\Omega| \geq \Omega_b$ and if the sampling frequency $\Omega_s > 2\Omega_b$, then $x(t)$ can be reconstructed from $x[n]$:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \text{sinc}\left(\frac{t - nT}{T}\right) \qquad (10.2)$$

# Signal Representation in other Domains

Apart from the time domain, we can also study signals in other domains.

For $x(t)$, it can be converted to $X(s)$ and $X(j\Omega)$.

In the Laplace transform domain, the conversion is:

$$x(t) \leftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{10.3}$$

Together with the region of convergence (ROC), $x(t)$ and $X(s)$ correspond to a one-to-one mapping. That is, both $x(t)$ and $X(s)$ with ROC are equivalent.

There are at least two advantages of Laplace transform:

- It generalizes the Fourier transform, that is, substituting $s = j\Omega$ yields $X(j\Omega)$. We can see whether the ROC includes the $j\Omega$-axis to check the existence of Fourier transform. The inverse Laplace transform techniques can be applied to convert $X(j\Omega)$ back to $x(t)$.

- It facilitates the analysis of linear time-invariant (LTI) systems. In the time domain, the input $x(t)$, output $y(t)$ and impulse response $h(t)$ are characterized by convolution but in the Laplace transform, they have simpler relation:

$$y(t) = x(t) \otimes h(t) \leftrightarrow Y(s) = X(s)H(s) \tag{10.4}$$

If $x(t)$ is periodic, then it can be represented as Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \tag{10.5}$$

which is a linear combination of harmonically related complex sinusoids. The Fourier series coefficients are:

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt, \quad k = \cdots -1, 0, 1, 2, \cdots \tag{10.6}$$

where $T_p$ is the fundamental period and $\Omega_0 = 2\pi/T_p$ is the fundamental frequency.

We can write this pair as:

$$x(t) \leftrightarrow X(j\Omega) \quad \text{or} \quad x(t) \leftrightarrow a_k \qquad (10.7)$$

because $\{a_k\}$ contain the amplitude information of all frequency components of $x(t)$. For example, we know the strength of $e^{jk\Omega_0 t}$ from $|a_k|$.

If $x(t)$ is aperiodic, then it can be represented as Fourier transform as:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega = x(t) \leftrightarrow X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \qquad (10.8)$$

where $X(j\Omega)$ indicates the amplitude at frequency $\Omega$.

Even if $x(t)$ is periodic, it can also be represented using Fourier transform as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0) \quad (10.9)$$

Nevertheless, we still see that $X(j\Omega)$ is characterized by $\{a_k\}$ as in the Fourier series in (10.7).

The Fourier transform is related to Laplace transform via:

$$X(j\Omega) = X(s)|_{s=j\Omega} \quad (10.10)$$

Hence we can use the techniques in Laplace transform to compute Fourier transform and inverse Fourier transform.

It is worth mentioning that although $X(j\Omega)$ does not naturally arise in real world, its magnitude $|X(j\Omega)|$ can be observed using electronic equipment, namely, spectrum analyzer.

Example 10.1

Given the frequency response of a continuous-time LTI system:

$$H(j\Omega) = \frac{1}{j\Omega + a}, \quad a > 0$$

Find the system impulse response $h(t)$.

Although inverse Fourier transform in (10.8) can be employed to determine $h(t)$, integration is needed.

Another approach which is computationally simpler is to make use of Laplace transform. Via the substitution of $j\Omega = s$, the system transfer function is:

$$H(s) = \frac{1}{s+a}$$

As $H(j\Omega)$ exists, we know that the ROC should include the $j\Omega$ -axis and hence is $\Re\{s\} > -a$. From Table 9.1, we easily obtain:

$$h(t) = e^{-at}u(t)$$

This is consistent with Examples 5.3, 5.6 and 9.2.

## Example 10.2

Determine the continuous-time signal $x(t)$ if its Fourier transform has the form of:

$$X(j\Omega) = \frac{j\Omega + 4}{-\Omega^2 + 5j\Omega + 6}$$

Via substitution of $j\Omega = s$, the Laplace transform of $x(t)$ is:

$$X(s) = \frac{s+4}{s^2+5s+6} = \frac{s+4}{(s+2)(s+3)}$$

As $X(j\Omega)$ exists, we know that the ROC should include the $j\Omega$ -axis and hence is $\Re\{s\} > -2$.

By means of partial fraction expansion, we obtain:

$$X(s) = \frac{2}{s+2} - \frac{1}{s+3}, \quad \Re\{s\} > -2$$

Taking the inverse Laplace transform yields:

$$x(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

As the Laplace transform generalizes the Fourier transform, the properties of the Laplace transform are similar to those of Fourier transform and Fourier series.

For $x[n]$, it can be converted to $X(z)$ and $X(e^{j\omega})$.

In the $z$ transform domain, the conversion is:

$$x[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad (10.11)$$

Together with the ROC, $x[n]$ and $X(z)$ correspond to a one-to-one mapping. That is, both $x[n]$ and $X(z)$ with ROC are equivalent.

There are at least two advantages of $z$ transform:

- It generalizes the discrete-time Fourier transform, (DTFT), that is, substituting $z = e^{j\omega}$ yields $X(e^{j\omega})$. We can see whether the ROC includes the unit circle or $|z| = 1$ to check the existence of DTFT. The inverse $z$ transform techniques can be applied to convert $X(e^{j\omega})$ back to $x[n]$.

- It facilitates the analysis of LTI systems. In the time domain, the input $x[n]$, output $y[n]$ and impulse response $h[n]$ are characterized by convolution but in the $z$ transform, they have simpler relation:

$$y[n] = x[n] \otimes h[n] \leftrightarrow Y(z) = X(z)H(z) \quad (10.12)$$

We use DTFT to convert $x[n]$ to frequency domain:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = x[n] \leftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (10.13)$$

where $X(e^{j\omega})$, which is periodic with a period of $2\pi$, indicates the amplitude at frequency $\omega$.

The DTFT is related to $z$ transform via:

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} \qquad (10.14)$$

Hence we can use the techniques in $z$ transform to compute DTFT and inverse DTFT.

<u>Example 10.3</u>

Given the frequency response of a discrete-time LTI system:

$$H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - 0.1e^{-j\omega}}$$

Find the system impulse response $h[n]$.

Although inverse DTFT in (10.13) can be employed to determine $h[n]$, integration is needed.

Another approach which is computationally simpler is to make use of $z$ transform. Via the substitution of $e^{j\omega} = z$, the system transfer function is:

$$H(z) = \frac{1 + z^{-1}}{1 - 0.1z^{-1}} = \frac{1}{1 - 0.1z^{-1}} + \frac{z^{-1}}{1 - 0.1z^{-1}}$$

As $H(e^{j\omega})$ exists, we know that the ROC should include the unit circle and hence is $|z| > 0.1$. Using Table 8.1 and time-shifting property, we easily obtain:

$$h[n] = (0.1)^n u[n] + (0.1)^{n-1} u[n-1]$$

Example 10.4

Find the discrete-time signal $x[n]$ if its DTFT has the form of:

$$X(e^{j\omega}) = \frac{1}{20}\sum_{n=0}^{19} e^{-j\omega n}$$

Via substitution of $e^{j\omega} = z$, the $z$ transform of $x[n]$ is:

$$X(z) = \frac{1}{20}\sum_{n=0}^{19} z^{-n}$$

Clearly there is only one ROC, which is $|z| > 0$. Applying inverse $z$ transform on $X(z)$ yields:

$$\begin{aligned} x[n] &= \frac{1}{20}\left(\delta[n] + \delta[n-1] + \cdots + \delta[n-19]\right) \\ &= \frac{1}{20}\sum_{k=0}^{19}\delta[n-k] = \frac{1}{20}\left(u[n] - u[n-20]\right) \end{aligned}$$

which aligns with Example 6.7.

As the $z$ transform generalizes the DTFT, the properties of the $z$ transform are similar to those of DTFT.

# LTI System Analysis with Transforms

In the time domain, LTI system is characterized by convolution:

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad (10.15)$$

or

$$y[n] = x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \qquad (10.16)$$

In the Laplace (or Fourier) transform and $z$ transform (or DTFT) domains, (10.15) and (10.16) become multiplication:

$$Y(s) = X(s)H(s) \qquad (10.17)$$

and

$$Y(z) = X(z)H(z) \quad (10.18)$$

Equations (10.17) and (10.18) indicate that we may obtain $Y(s)$ (or $Y(z)$), $X(s)$ (or $X(z)$) and $H(s)$ (or $H(z)$) in an easier manner.

Note that even if the LTI systems are not stable, $H(s)$ and $H(z)$ still exist and their ROCs will not include the $j\Omega$-axis and unit circle, respectively, while $H(j\Omega)$ and $H(e^{j\omega})$ do not converge.

<u>Example 10.5</u>

Determine the transfer functions of the continuous-time and discrete-time LTI systems with impulse responses:

$$h(t) = e^{2t}u(t)$$

and

$$h[n] = 2^n u[n]$$

It is clear from (3.20) and (3.21) that the systems are unstable because they are not absolutely summable and integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

and

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Taking Laplace transform on $h(t)$ yields:

$$H(s) = \frac{1}{s-2}, \quad \Re\{s\} > 2$$

As $\Re\{s\} > 2$ does not include the $j\Omega$-axis, $H(j\Omega)$ does not exist. This conclusion can also be obtained because (9.9) is not satisfied.

Taking $z$ transform on $h[n]$ yields:

$$H(z) = \frac{1}{1 - 2z^{-1}}, \quad |z| > 2$$

As $|z| > 2$ does not include the unit circle, $H(e^{j\omega})$ does not exist. This conclusion can also be obtained because (8.9) is not satisfied.

## Example 10.6

Consider a continuous-time LTI system with impulse response $h(t)$, input $x(t)$ and output $y(t)$. Calculate $y(t)$ when $x(t) = h(t) = e^{at}u(t)$.

The Laplace transforms of both $x(t)$ and $h(t)$ are

$$X(s) = H(s) = \frac{1}{s-a}, \quad \Re\{s\} > a$$

As a result, we have:

$$Y(s) = X(s)H(s) = \frac{1}{(s-a)^2}, \quad \Re\{s\} > a$$

According to Table 9.1, we obtain:

$$y(t) = te^{at}u(t)$$

## Example 10.7

Consider a discrete-time LTI system with impulse response $h[n]$ , input $x[n]$ and output $y[n]$ . Calculate $y[n]$ when $x[n] = h[n] = a^n u[n]$.

The $z$ transforms of both $x[n]$ and $h[n]$ are

$$X(z) = H(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

As a result, we have:

$$Y(z) = X(z)H(z) = \frac{1}{(1 - az^{-1})^2}, \quad |z| > |a|$$

According to Table 8.1 and time-shifting property, we obtain:

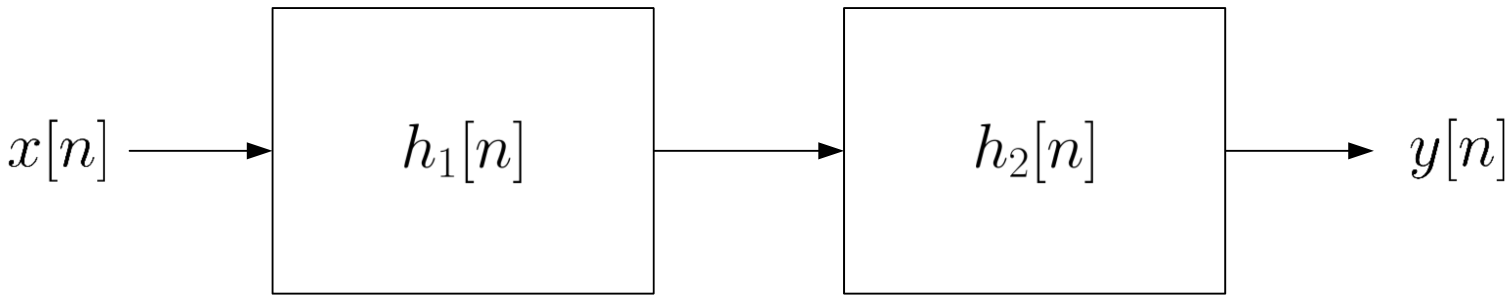
$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \Rightarrow (n+1)a^{n+1}u[n+1] \leftrightarrow \frac{a}{(1-az^{-1})^2}$$

Finally, we have:

$$y[n] = (n+1)a^n u[n+1] = (n+1)a^n u[n]$$

<u>Example 10.8</u>

Consider a cascade system of two discrete-time LTI systems with impulse responses $h_1[n]$ and $h_2[n]$. Let the system input and output be $x[n]$ and $y[n]$, respectively. Determine the overall impulse response $h[n]$ and transfer function $H(z)$ if $h_1[n] = h_2[n] = a^n u[n]$. Find the difference equation that relates $x[n]$ and $y[n]$.




The overall impulse response is:

$$h[n] = h_1[n] \otimes h_2[n]$$

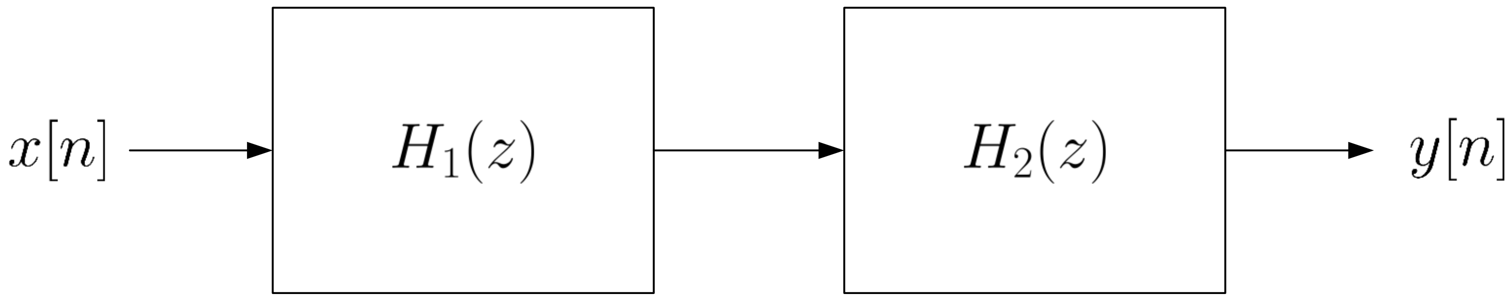
Using the result in Example 10.7, we have:

$$h[n] = (n+1)a^n u[n]$$

From Example 10.7 again, the overall transfer function is:

$$H(z) = H_1(z)H_2(z) = \frac{1}{(1-az^{-1})^2}, \quad |z| > |a|$$

Note that it is equivalent to use $H_1(z)$ and $H_2(z)$ in the block diagram:

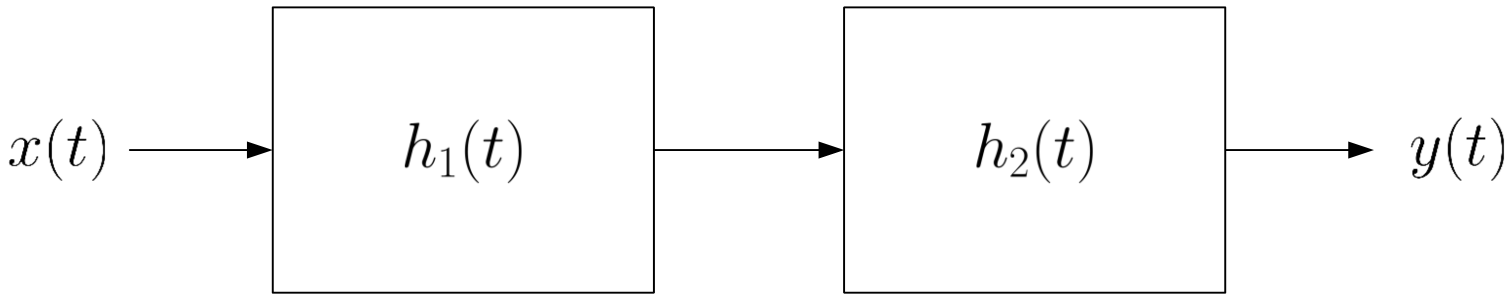



As $H(z) = Y(z)/X(z)$, we perform cross-multiplication and inverse $z$ transform to obtain:

$$\left(1 - az^{-1}\right)^2 Y(z) = X(z)$$
$$\Rightarrow \left(1 - 2az^{-1} + a^2 z^{-2}\right) Y(z) = X(z)$$
$$\Rightarrow y[n] - 2ay[n-1] + a^2 y[n-2] = x[n]$$

## Example 10.9

Consider a cascade system of two continuous-time LTI systems with impulse responses $h_1(t)$ and $h_2(t)$. Let the system input and output be $x(t)$ and $y(t)$, respectively. Determine the overall impulse response $h(t)$ and transfer function $H(s)$ if $h_1(t) = h_2(t) = e^{at}u(t)$. Find the differential equation that relates $x(t)$ and $y(t)$.



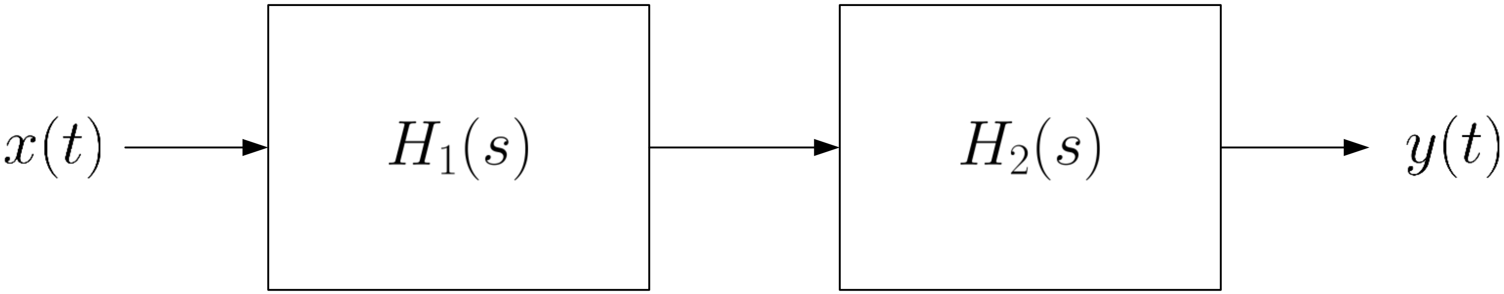

Using Example 10.6, the overall impulse response is:

$$h(t) = h_1(t) \otimes h_2(t) = te^{at}u(t)$$

and the overall transfer function is:

$$H(s) = H_1(s)H_2(s) = \frac{1}{(s-a)^2}, \quad \Re\{s\} > a$$

We can also use $H_1(s)$ and $H_2(s)$ in the block diagram:




As $H(s) = Y(s)/X(s)$, we have:

$$(s-a)^2 Y(s) = X(s) \Rightarrow \left(s^2 - 2as + a^2\right) Y(s) = X(s)$$
$$\Rightarrow \frac{d^2y(t)}{dt^2} - 2a\frac{dy(t)}{dt} + a^2 y(t) = x(t)$$