Signals in Time Domain

Chapter Intended Learning Outcomes:

(i) Classify different types of signals

(ii) Perform basic operations on signals

(iii) Recognize basic continuous-time and discrete-time signals and understand their properties

(iv) Generate and visualize discrete-time signals using MATLAB

Classification of Signals

There are many ways of classifying signals. Common examples are provided as follows.

Continuous-Time versus Discrete-Time

$x(t)$: take a value at every instant of time t.
$x[n]$: defined only at particular instants of time n.

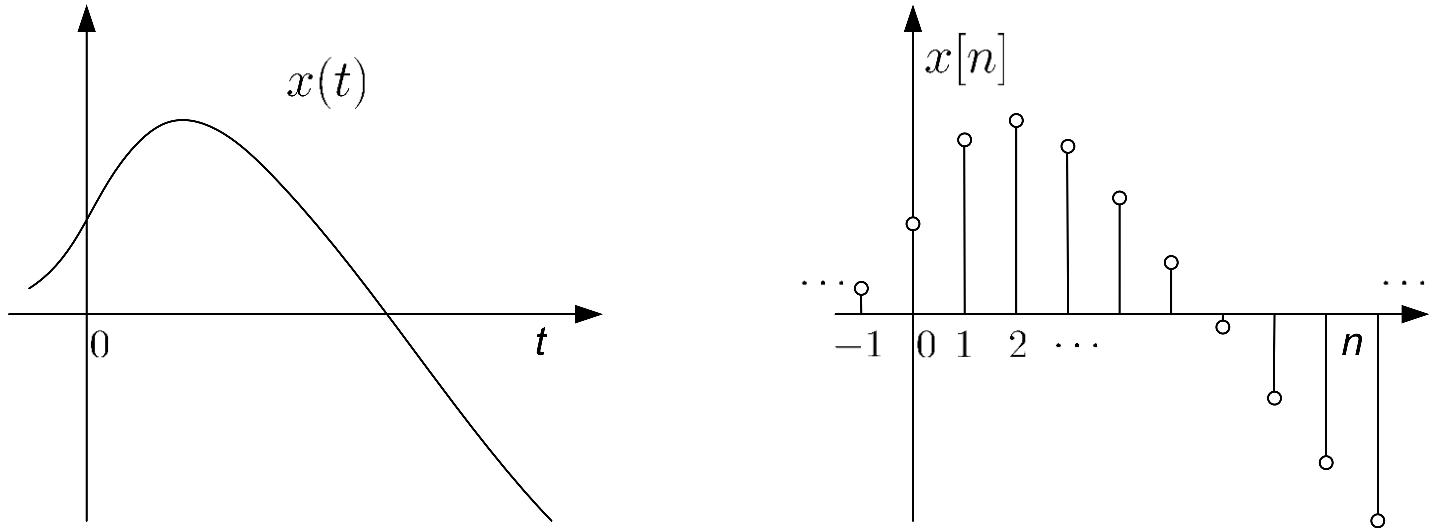

Fig. 2.1: Continuous-time versus discrete-time signals

Real versus Complex

Real-valued signal means that $x(t)$ or $x[n]$ is real for all t or n.

Complex-valued signal means that $x(t)$ or $x[n]$ can be decomposed as:

$$x(t) = \Re\{x(t)\} + j\Im\{x(t)\} \text{ or } x[n] = \Re\{x[n]\} + j\Im\{x[n]\} \quad (2.1)$$

where $\Re\{\}$ and $\Im\{\}$ represent the real and imaginary parts, respectively, while the latter is nonzero, and $j = \sqrt{-1}$.

Note that for a complex number x, we can also use magnitude $|x|$ and phase $\angle(x)$ for its representation:

$$|x| = \sqrt{(\Re\{x\})^2 + (\Im\{x\})^2} \quad (2.2)$$

and

$$\angle(x) = \tan^{-1}\left(\frac{\Im\{x\}}{\Re\{x\}}\right) \quad (2.3)$$

The magnitude can also be computed as:

$$|x| = \sqrt{x \cdot x^*} \tag{2.4}$$

where

$$x^* = \Re\{x\} - j\Im\{x\} \tag{2.5}$$

is the complex conjugate of x.

We may say that complex signals generalize real signals.

<u>Example 2.1</u>

Determine if the following signals are real or complex.

(a) $x(t) = 1 + 2t + 3t^2$

(b) $x(t) = jt$

(a)It is real-valued signal as $x(t)$ is real for all t.

(b)It is complex as $x(t)$ has nonzero imaginary component.

Periodic versus Aperiodic

$x(t)$ is said to be periodic if there exists $T > 0$ such that

$$x(t) = x(t + T) \tag{2.6}$$

for all t. The smallest T for which (2.6) holds is called the fundamental period.

$x[n]$ is said to be periodic if there exists a positive integer N such that

$$x[n] = x[n + N] \tag{2.7}$$

for all n. The smallest N for which (2.7) holds is called the fundamental period.

If a signal is not periodic, then it is aperiodic.

Example 2.2

Determine if the following signals are periodic or not. If it is periodic, compute the fundamental period.

(a) $x(t) = 1 + 2t + 3t^2$

(b) $x(t) = \cos(10\pi t)$

(c) $x(t) = \begin{cases} \cos(100t), & t \in [1, 100] \\ 0, & \text{otherwise} \end{cases}$

(d) $x[n] = \cos\left(\frac{2\pi n}{3}\right)$

(e) $x[n] = \cos\left(\frac{2n}{3}\right)$

(a) A quadratic function should not be periodic.

(b) As cosine function has a period of 2π, we can write:

$$x(t) = \cos(10\pi t) = \cos(10\pi t + 2\pi) = x(t + 2\pi/(10\pi)) = x(t + 1/5)$$

Hence it is periodic with $T = 1/5$.

(c) $x(t)=x(t+T)$ is not fulfilled for all t and it is aperiodic.

(d) Again, we can write:

$$x[n]=\cos\left(\frac{2\pi n}{3}\right)=\cos\left(\frac{2\pi n}{3}+2\pi\right)=\cos\left(\frac{2\pi(n+3)}{3}\right)=x[n+3]$$

Hence it is periodic with $N=3$.

(e)It is aperiodic because we cannot find an integer N to fulfil (2.7).

Even versus Odd

A signal is called an even function if

$$x_e(t)=x_e(-t) \quad \text{or} \quad x_e[n]=x_e[-n] \tag{2.8}$$

A signal is called an odd function if

$$x_o(t)=-x_o(-t) \quad \text{or} \quad x_o[n]=-x_o[-n] \tag{2.9}$$

Any signal can be represented by a sum of even and odd signals:

$$x(t) = x_e(t) + x_o(t) \quad \text{or} \quad x[n] = x_e[n] + x_o[n] \tag{2.10}$$

where

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \quad x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] \tag{2.11}$$

or

$$x_e[n] = \frac{1}{2}\left[x[n] + x[-n]\right] \quad \text{and} \quad x_o[n] = \frac{1}{2}\left[x[n] - x[-n]\right] \tag{2.12}$$

<u>Example 2.3</u>

Determine if the following signals are even or odd.

(a) $x(t) = \cos(\Omega \pi t)$

(b) $x(t) = \sin(\Omega \pi t)$

(c) $x(t) = \sin(\Omega \pi t + \theta)$

(d) $x[n] = 1 + 2n - 3n^2$

(a) It is even because $\cos(\Omega\pi t) = \cos(-\Omega\pi t)$.
(b) It is odd because $\sin(\Omega\pi t) = -\sin(-\Omega\pi t)$.

(c) It is neither odd nor even because:

$$\sin(\Omega\pi t + \theta) = \sin(\theta)\cos(\Omega\pi t) + \cos(\theta)\sin(\Omega\pi t)$$

which is a linear combination of even and odd functions.

(d) It is neither odd nor even. Applying (2.12) yields:

$$x_e[n] = \frac{1}{2}\left[(1+2n-3n^2) + (1+2(-n)-3(-n)^2)\right] = \frac{1}{2}\left[2-6n^2\right] = 1-3n^2$$

$$x_o[n] = \frac{1}{2}\left[(1+2n-3n^2) - (1+2(-n)-3(-n)^2)\right] = \frac{1}{2}\left[4n\right] = 2n$$

which are the even and odd components. Note that the same result can be easily obtained by inspection.

Energy versus Power

Energy of $x(t)$ or $x[n]$ is defined as:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{or} \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{2.13}$$

If the signal energy is infinite, it is meaningful to use power of $x(t)$ or $x[n]$ as the measure, which is defined as:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \quad \text{or} \quad \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{2.14}$$

Signal power is the time average of signal energy.

Note that for real signal, $|x(t)|^2 = x^2(t)$, while $|x(t)|^2 = x(t)x^*(t)$ for complex signal.

A signal is energy signal if $0 < E_x < \infty$, indicating its $P_x = 0$.

A signal is power signal if $0 < P_x < \infty$, indicating its $E_x = \infty$.

Example 2.4

Determine if the following signals are energy or power signals and then compute their energies or powers.

(a) $x(t) = \begin{cases} -2, & t \in [0, 10] \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = A\cos(\omega t + \theta)$

(c) $x[n] = 10e^{j2n}$

(a) $x^2(t) = 4$ only for $t \in [0, 10]$ and is zero otherwise. Thus it is an energy signal. Using (2.13), we get:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{10} 4dt = 40$$

(b) From Example 2.2(b), we know that $x(t)$ is periodic with $T = 2\pi/\omega$. Applying (2.14) with only one period:

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T} \int_0^T x^2(t) dt \\
&= \frac{\omega}{2\pi} \int_0^T A^2 \cos^2(\omega t + \theta) dt = \frac{A^2 \omega}{2\pi} \int_0^{2\pi/\omega} \frac{1}{2} [1 + \cos(2\omega t + 2\theta)] dt \\
&= \frac{\omega}{2\pi} \cdot \frac{A^2}{2} \cdot \frac{2\pi}{\omega} = \frac{A^2}{2} < \infty
\end{aligned}
$$

Hence it is a periodic signal with power $A^2/2$.

(c) $|x[n]|^2 = 10e^{j2n} \cdot 10e^{-j2n} = 100$. Summing $|x[n]|^2$ from $n = -\infty$ to $n = \infty$ is infinity and thus it is a power signal with P_x:

$$
\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 100 = 100
$$

Basic Signal Operations

Three basic operators on signals are described as follows.

Time Shift

Shift the signal to left or right:

$$x(t) \longrightarrow x(t-t_0) \quad \text{or} \quad x[n] \longrightarrow x[n-n_0] \qquad (2.15)$$

If t_0 or n_0 is positive, then it corresponds to time delay while it is a time advance for negative t_0 or n_0.

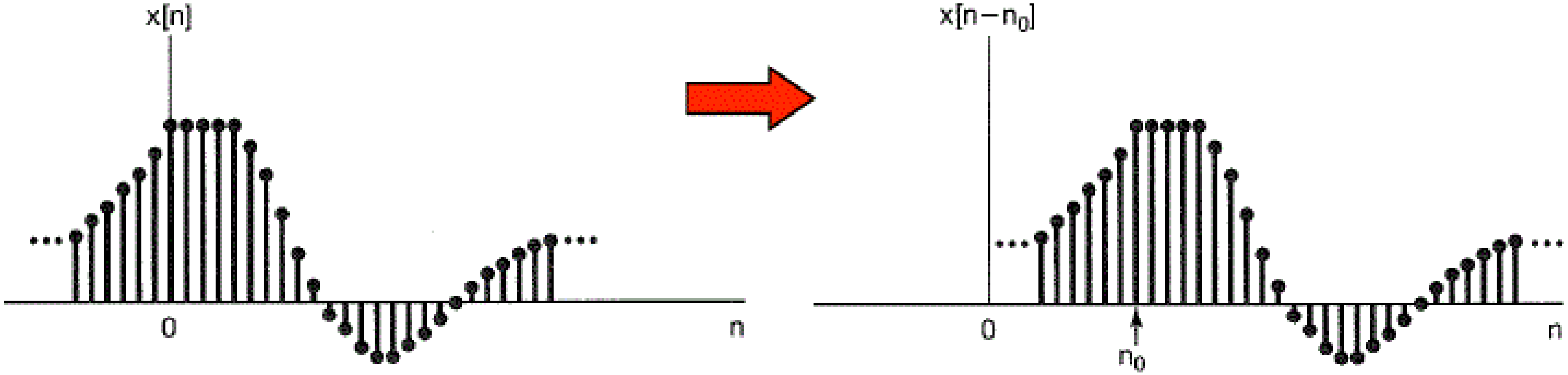

Fig. 2.2: Illustration of time shift

Time Reversal

Flip the signal around the vertical axis:

$$x(t) \longrightarrow x(-t) \quad \text{or} \quad x[n] \longrightarrow x[-n] \tag{2.16}$$

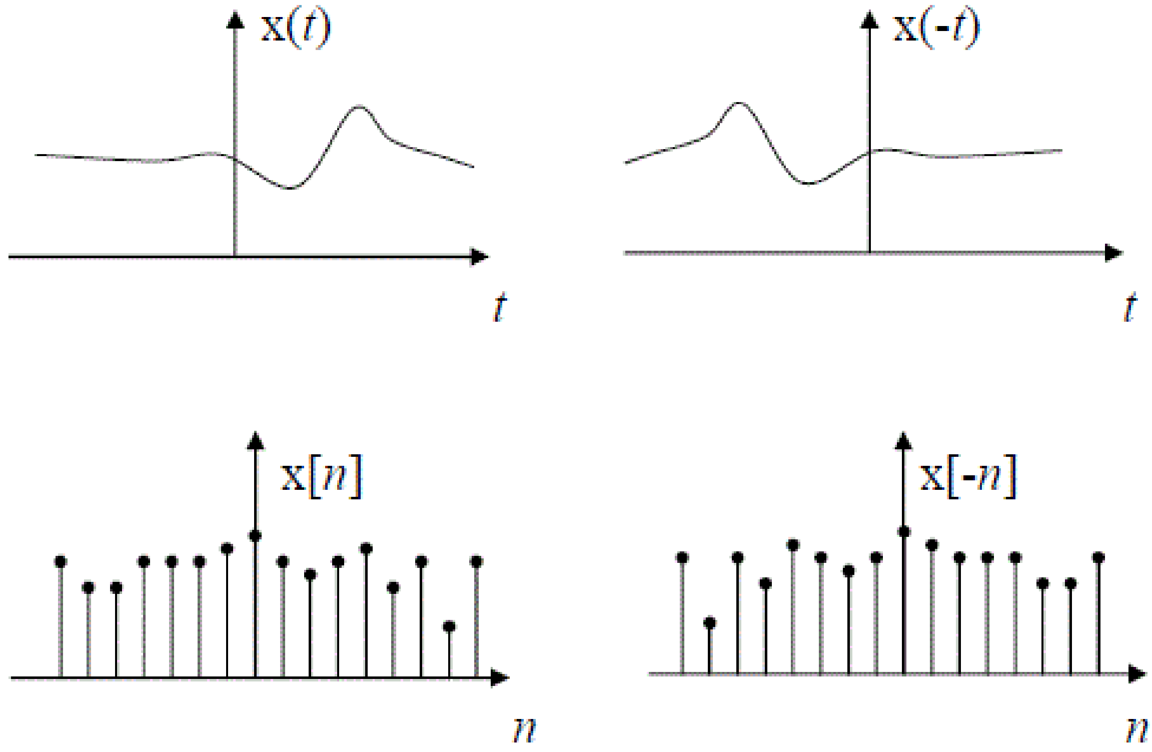

Fig. 2.3: Illustration of time reversal

Time Scaling

Linearly stretch or compress the signal:

$$x(t) \longrightarrow x(ct), \quad c > 0 \tag{2.17}$$

where $c < 1$ means stretch and $c > 1$ means compression. We do not discuss $x[n]$ as it is not defined for all time instants.

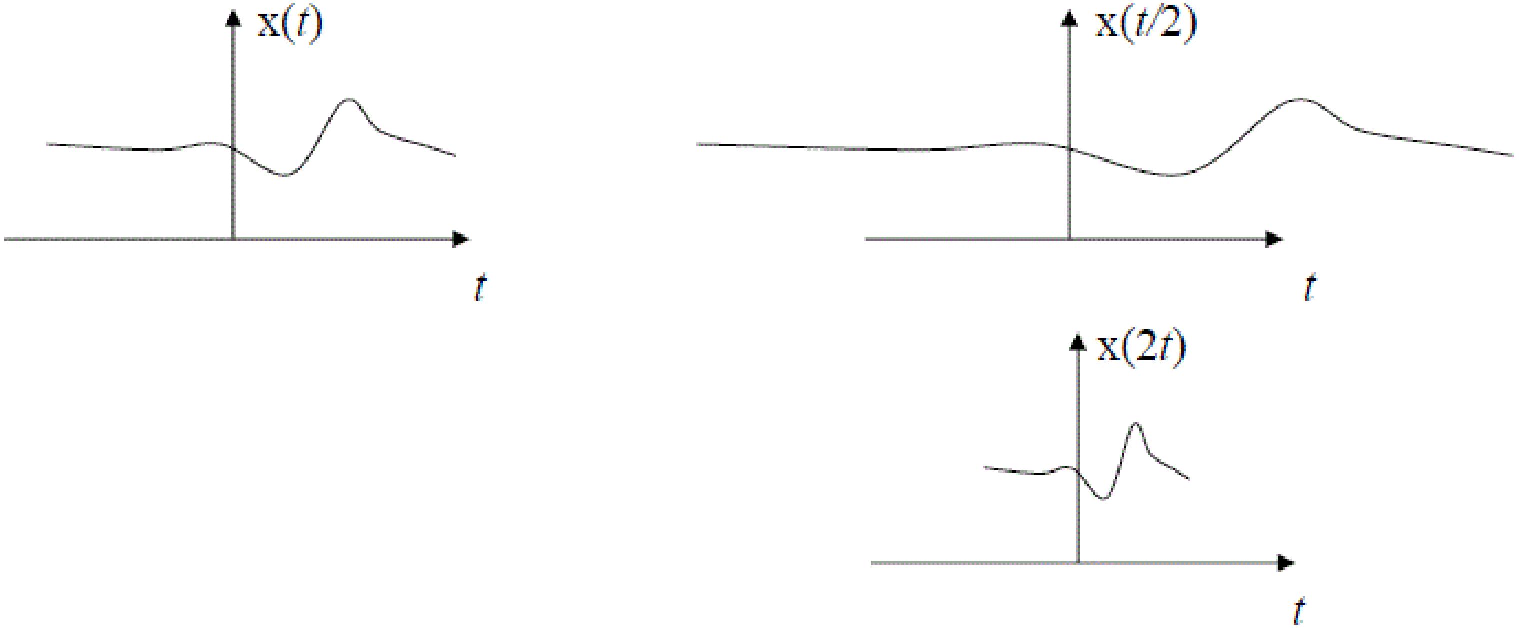

Fig. 2.4: Illustration of time scaling

Basic Continuous-Time Signals

Common and/or useful continuous-time signals are described as follows.

Unit Impulse

The unit impulse $\delta(t)$ has the following characteristics:

$$\delta(t) = 0, \quad t \neq 0 \tag{2.18}$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{2.19}$$

(2.18) and (2.19) indicate that $\delta(t)$ has a very large value or impulse at $t = 0$. That is, $\delta(t)$ is not well defined at $t = 0$.

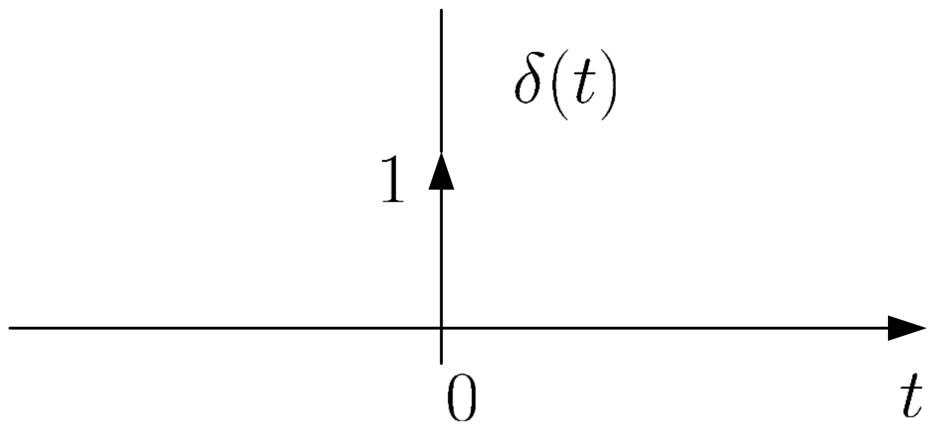

Fig. 2.5: Graphical representation of $\delta(t)$

From (2.18), multiplying a continuous-time signal $x(t)$ by an impulse $\delta(t-t_0)$ gives:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \tag{2.20}$$

That is, we only need to care about the value of $x(t)$ at the impulse location, namely, $t = t_0$.

We may consider $\delta(t)$ as the building block of any continuous-time signal, described by the sifting property:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad (2.21)$$

That is, imagining $x(t)$ as a sum of infinite impulse functions and each with amplitude $x(\tau)$.

Unit Step

The unit step function $u(t)$ has the form of:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \qquad (2.22)$$

As there is a sudden change from 0 to 1 at $t=0$, $u(0)$ is not well defined.

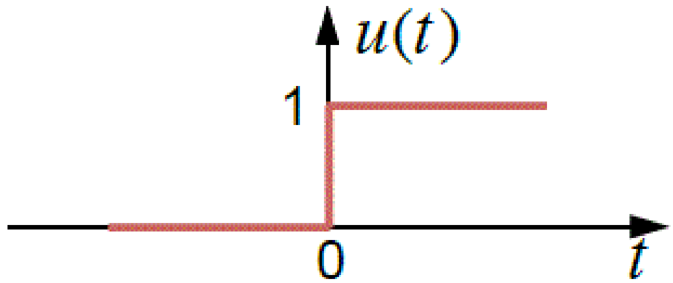

Fig. 2.6: Graphical representation of $u(t)$

$u(t)$ can be expressed in terms of $\delta(t)$ as:

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau \tag{2.23}$$

Conversely, we can use $u(t)$ to represent $\delta(t)$:

$$\delta(t) = \frac{du(t)}{dt} \tag{2.24}$$

Sinusoid

It is a sine (or cosine) wave of the form:

$$x(t) = A\cos(\omega t + \phi) \tag{2.25}$$

which is characterized by three parameters, amplitude $A > 0$, radian frequency ω and phase $\phi \in [0, 2\pi)$.

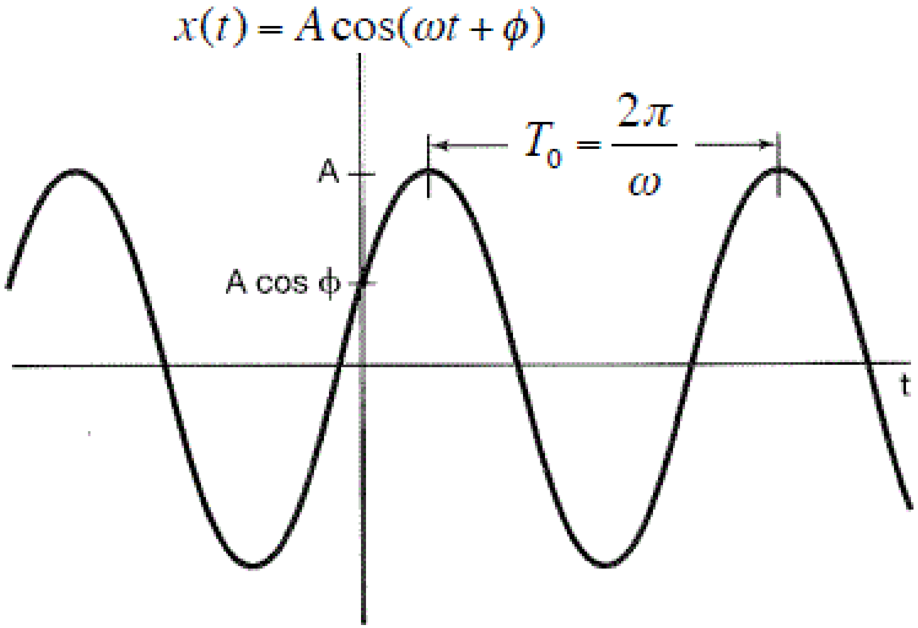

Fig. 2.7: Sinusoid

Rate of oscillation increases with ω.

Apart from ω, $f = \omega/(2\pi)$, the frequency in Hz can be used.

Fundamental period T_0 is determined as:

$$x(t) = x(t + T_0) = A\cos(\omega(t + T_0) + \phi) = A\cos(\omega t + 2\pi + \phi)$$

$$\Rightarrow \omega T_0 = 2\pi \Rightarrow T_0 = \frac{2\pi}{\omega} = \frac{1}{f} \tag{2.26}$$

For the complex-valued case, it has the form of:

$$x(t) = Ae^{j(\omega t + \phi)} \tag{2.27}$$

Using the Euler formula:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \tag{2.28}$$

It is seen that the real part of (2.27) is (2.25), while the imaginary part is $A\sin(\omega t+\phi)$ which is also a sinusoid.

A complex sinusoid is also periodic with radian frequency ω.

According to (2.28), we can obtain:

$$\cos(\phi)=\frac{e^{j\phi}+e^{-j\phi}}{2} \tag{2.29}$$

and

$$\sin(\phi)=\frac{e^{j\phi}-e^{-j\phi}}{2j} \tag{2.30}$$

Note that the general form of (2.25) or (2.27) is:

$$x(t)=\sum_{l=1}^{L} A_l\cos(\omega_l t+\phi_l) \quad \text{or} \quad x(t)=\sum_{l=1}^{L} A_l e^{j(\omega_l t+\phi_l)} \tag{2.31}$$

which is a sum of L tones.

Exponential

For the real-valued case, it has the form:

$$x(t) = Ae^{at} \tag{2.32}$$

where A and a are real numbers.

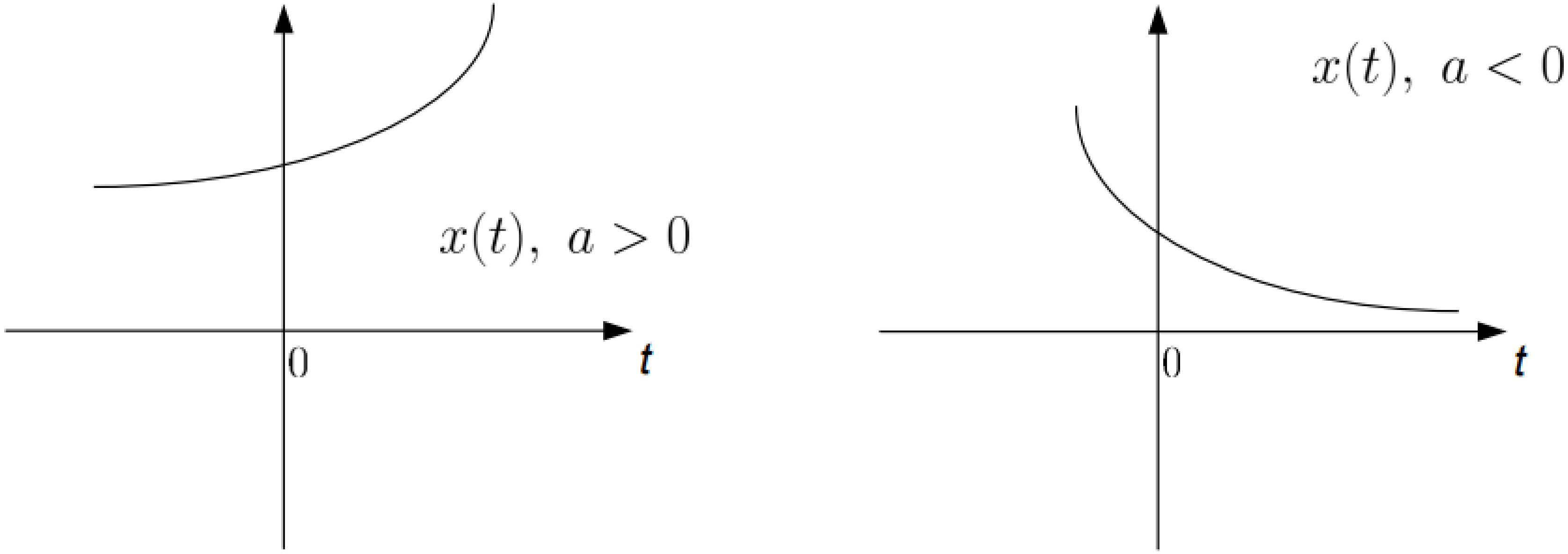

Fig. 2.8: Real exponential with $A > 0$

With complex A and a, (2.32) also represents complex case.

Basic Discrete-Time Signals

Typical examples of discrete-time signals are described as follows.

Unit Impulse (or Sample)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{2.33}$$

which is similar to $\delta(t)$ but $\delta[n]$ is simpler because it is well defined for all n while $\delta(t)$ is not defined at $t = 0$.

Unit Step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{2.34}$$

which is similar to $u(t)$ but $u[n]$ is well defined for all n while $u(t)$ is not defined $t = 0$.

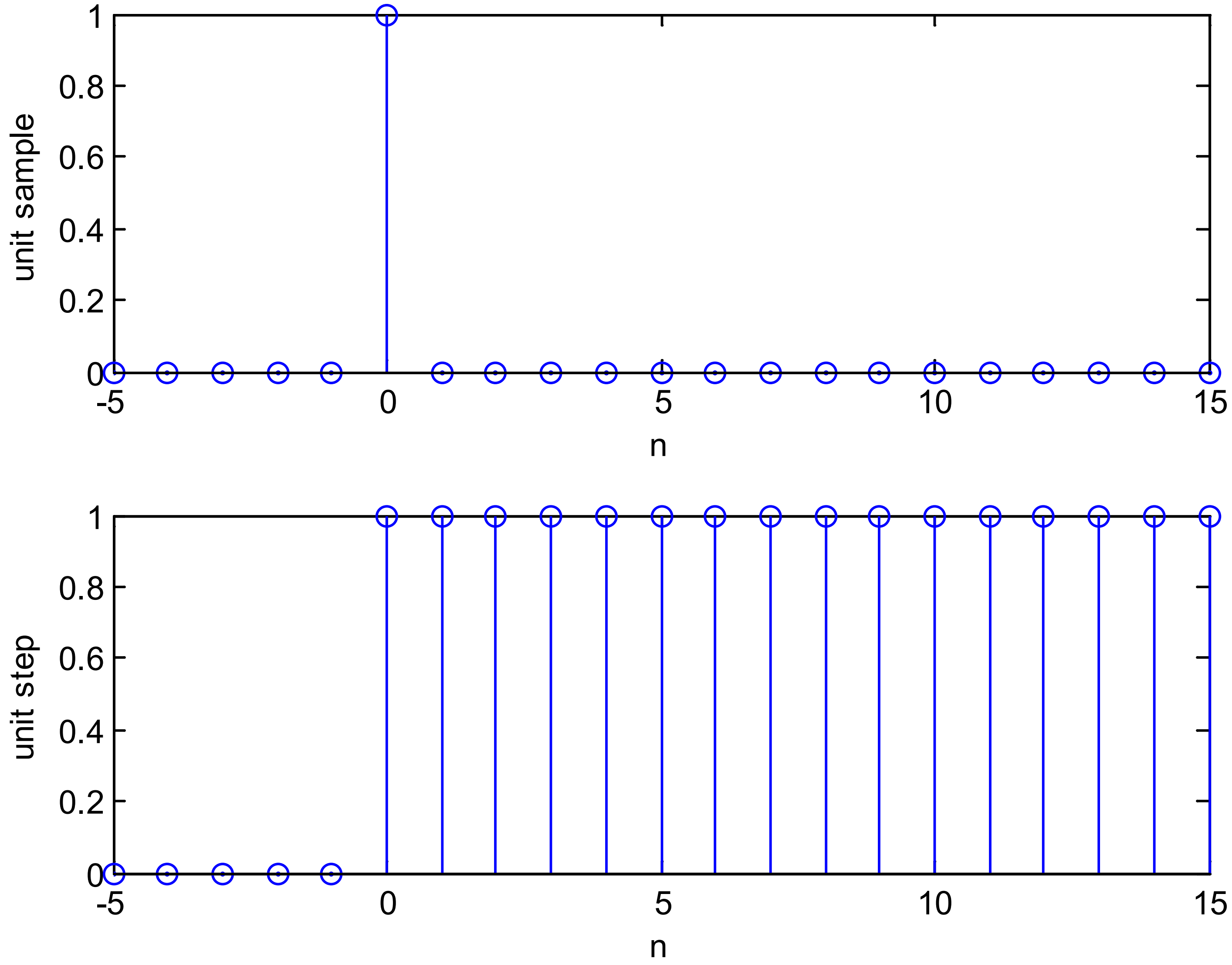

Fig. 2.9: Unit sample $\delta[n]$ and unit step $u[n]$

$\delta[n]$ is an important function because it serves as the building block of any discrete-time signal $x[n]$:

$$\begin{aligned} x[n] &= \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots \\ &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \end{aligned} \tag{2.35}$$

For example, $u[n]$ can be expressed in terms of $\delta[n]$ as:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \tag{2.36}$$

Conversely, we can use $u[n]$ to represent $\delta[n]$:

$$\delta[n] = u[n] - u[n-1] \tag{2.37}$$

which are analogous to (2.21), (2.23)-(2.24).

Sinusoid

For real-valued case, it has the form of:

$$x[n] = A\cos(\omega n + \theta) \qquad (2.38)$$

which is characterized by three parameters, amplitude $A > 0$, radian frequency $\omega \in (0, \pi)$ and phase $\phi \in [0, 2\pi)$.

(2.38) can be extended to the complex model as:

$$x[n] = Ae^{j(\omega n + \phi)}, \quad \omega \in (0, 2\pi) \qquad (2.39)$$

The general form of (2.38) or (2.39) is:

$$x[n] = \sum_{l=1}^{L} A_l \cos(\omega_l n + \phi_l) \quad \text{or} \quad x[n] = \sum_{l=1}^{L} A_l e^{j(\omega_l n + \phi_l)} \qquad (2.40)$$

which is a sum of L sinusoids.

Exponential

For the real-valued case, it has the form:

$$x[n] = Ae^{an} \qquad (2.41)$$

where A and a are real numbers.

(2.41) can also represent complex exponential by using complex A and a.

Introduction to MATLAB

MATLAB stands for "Matrix Laboratory".

It is an interactive matrix-based software for numerical and symbolic computation in scientific and engineering applications.

Its user interface is simple to use, e.g., we can use the `help` command to understand the usage and syntax of each MATLAB function.

Together with the availability of numerous toolboxes such as “5G”, “Bioinformatics”, “Deep Learning”, “Econometrics”, “Financial”, “Robotics System”, and “Statistics and Machine Learning”, it is useful in many disciplines.

MathWorks offers MATLAB to C conversion utility.

As it is not free, it is not as popular as Python. Nevertheless, CityU has subscribed to the MathWorks Total Academic Headcount (TAH) license for MATLAB:

https://www.cityu.edu.hk/csc/deptweb/facilities/central-sw-tah.htm

Individual License for Personally-owned Computers

The TAH license allows faculty, staff, and students to install MATLAB software on their personally-owned computers and laptops. Below are the steps for getting the installer and activating the individual license:

1. Go to CityU's MATLAB Portal.
2. Click "**Sign in to get started**" in the Get MATLAB and Simulink section.
3. Sign in with your CityU EID and password.
4. Create a MathWorks account using your CityU email address.
5. Download the installer from MathWorks website.
6. Run the installer.
7. In the installer, select **Log in with a MathWorks Account** and follow the online instructions.
8. When prompted to do so, select the **Academic – Total Headcount** license labeled **Individual**.
9. Select the products to download and install.
10. After downloading and installing your products, keep the **Activate MATLAB** checkbox selected and click **Next**.
11. When asked to provide a user name, verify that the displayed user name is correct. Continue with the process until activation is complete.

After installation, you can find it at Windows Start icon or click the MATLAB icon at Desktop.

Example 2.5

Use MATLAB to generate a discrete-time sinusoid of the form:

$$x[n] = A\cos(\omega n + \theta), \quad n = 0, 1, \cdots, N-1$$

with $A = 1$, $\omega = 0.3$, $\theta = 1$ and $N = 21$, which has a duration of 21 samples.

We can generate $x[n]$ by using the following MATLAB code:

```
N=21;                   %number of samples is 21
A=1;                    %tone amplitude is 1
w=0.3;                  %frequency is 0.3
p=1;                    %phase is 1
for n=1:N
x(n)=A*cos(w*(n-1)+p); %time index should be >0
end
```

Note that `x` is a vector and its index should be at least 1.

Alternatively, we can also use:

```
N=21;                   %number of samples is 21
A=1;                    %tone amplitude is 1
w=0.3;                  %frequency is 0.3
p=1;                    %phase is 1
n=0:N-1;                %define time index vector
x=A.*cos(w.*n+p);       %first time index is also 1
```

Note that "." denotes element-wise operation.

Both give

```
x =
Columns 1 through 7
0.5403    0.2675   -0.0292   -0.3233   -0.5885   -0.8011   -0.9422
Columns 8 through 14
-0.9991   -0.9668   -0.8481   -0.6536   -0.4008   -0.1122    0.1865
Columns 15 through 21
0.4685    0.7087    0.8855    0.9833    0.9932    0.9144    0.7539
```

Which approach is better? Why?

To plot $x[n]$, we can either use the commands `stem(x)` and `plot(x)`.

If the time index is not specified, the default start time is $n = 1$.

Nevertheless, it is easy to include the time index vector in the `stem`/`plot` command, e.g., we can use `stem` to plot $x[n]$ with the correct time index:

```
n=0:N-1;                    %n is vector of time index
stem(n,x)                   %plot x versus n
```

Similarly, `plot(n,x)` can be employed to show $x[n]$.

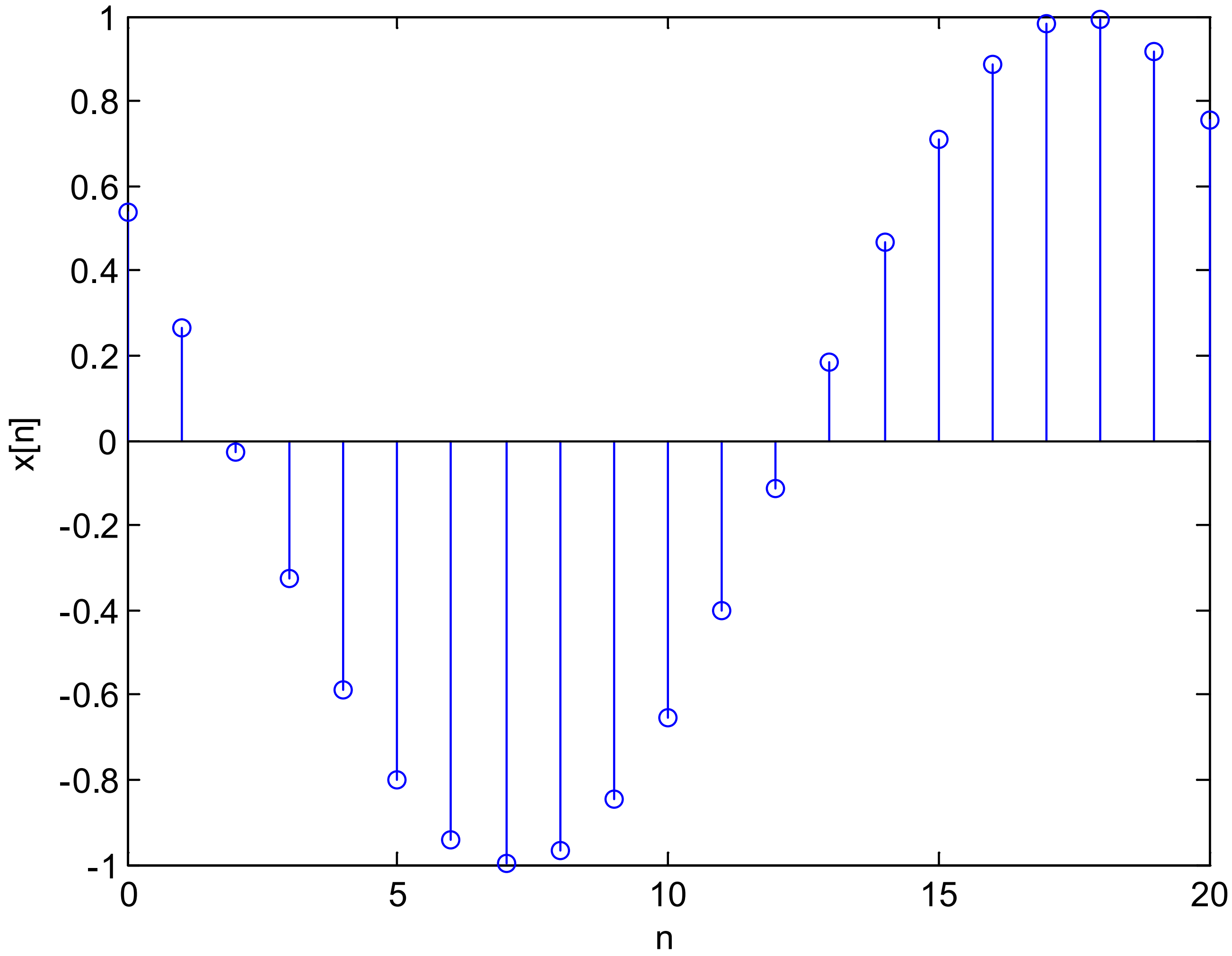

Fig. 2.10: Plot of discrete-time sinusoid using `stem`

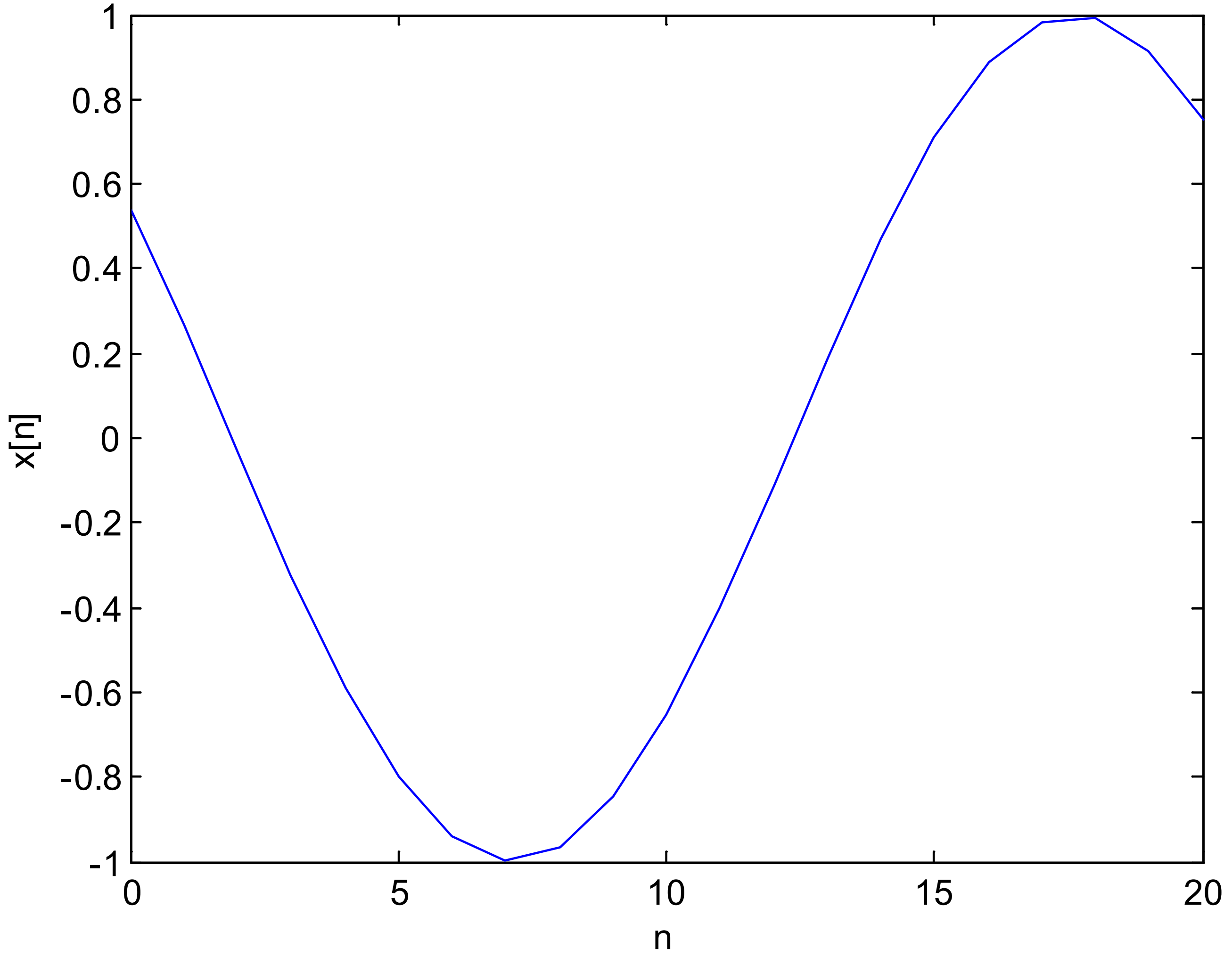

Fig. 2.11: Plot of discrete-time sinusoid using `plot`