Systems in Time Domain

Chapter Intended Learning Outcomes:

(i) Classify different types of systems

(ii) Understand the property of convolution and its relationship with linear time-invariant system

(iii) Understand the relationship between differential equation, difference equation and linear time-invariant system

(iv) Perform basic operations in systems

System Overview

It can be classified as continuous-time and discrete-time:

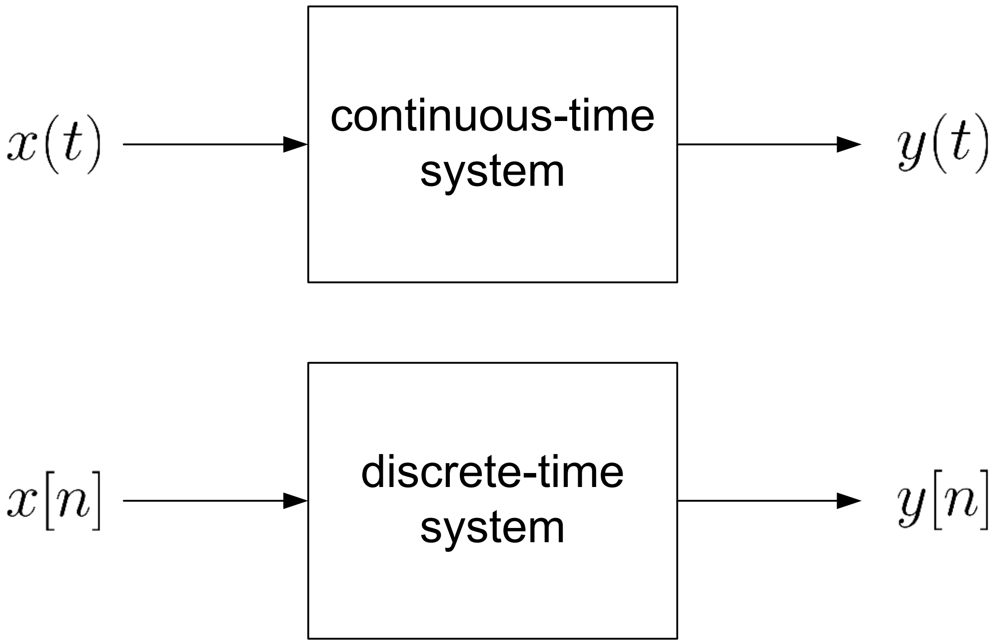


Fig. 3.1: Continuous-time and discrete-time systems

In a continuous-time (discrete-time) system, the input and output are continuous-time (discrete-time) signals.

A system is an operator $\mathcal{T}$ which maps input into output:

$$y(t) = \mathcal{T}\{x(t)\} \quad \text{or} \quad y[n] = \mathcal{T}\{x[n]\} \qquad (3.1)$$

Systems can be connected/combined to form a bigger/overall system, e.g., break down a big task into several sub-tasks and each system handles one sub-task.

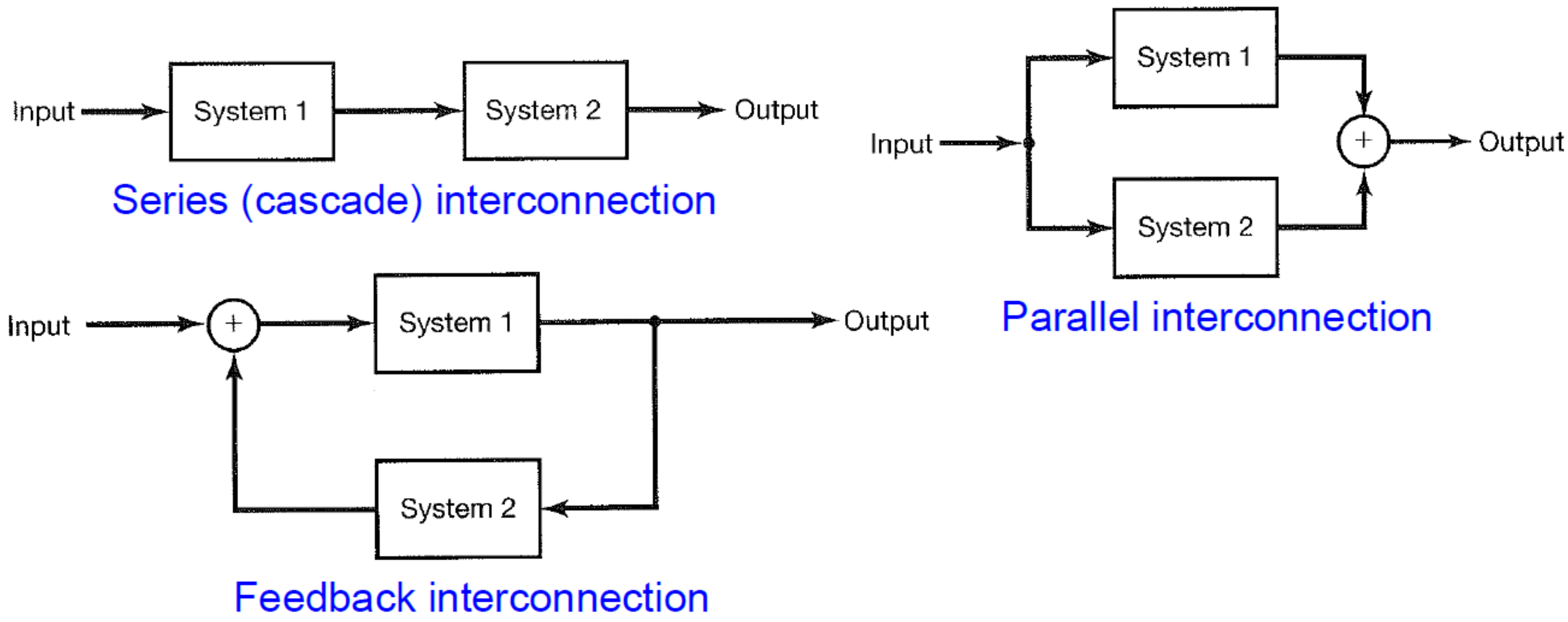


Fig. 3.2: Examples of system interconnections

Basic System Properties

Memoryless, invertibility, causality, stability, linearity, and time-invariance, are described as follows.

Memoryless

A system is memoryless if its output at a given time is dependent only on the input at that same time, i.e., $y(t)$ at time t depends only on $x(t)$ at time t; $y[n]$ at time n depends only on $x[n]$ at time n.

A memoryless system does not have memory to store any input values because it just operates on the current input.

If a system is not memoryless, we can call it a system with memory.

Example 3.1

Determine if the following systems are memoryless or not

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n-2]$

(a) The system is memoryless because the output at time t depends only on the input at time t.

(b) The system is not memoryless because $y[n]$ also depends only on $x[n-2]$, which is a previous input, and thus it needs memory to store $x[n-2]$ when processing the input at time n.

Invertibility

A system is invertible if distinct inputs lead to distinct outputs, or if an inverse system exists.

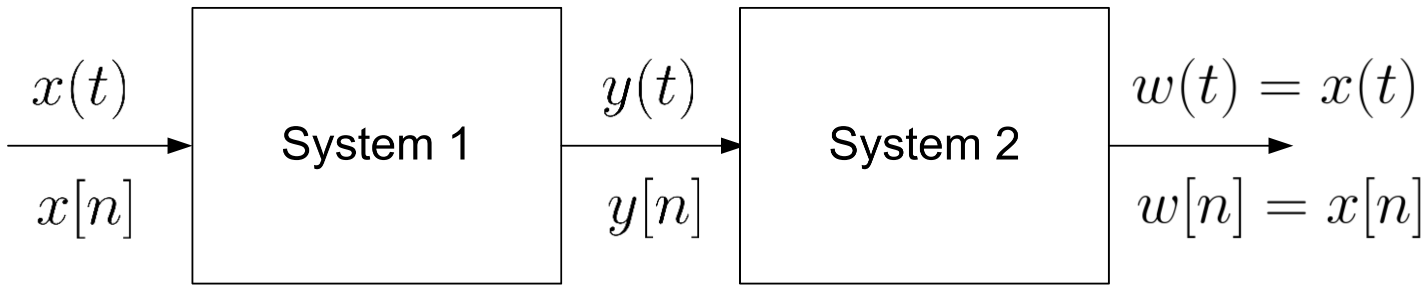


Fig. 3.3: Invertible system

That is, if we can get back the input $x(t)$ or $x[n]$ by passing the output $y(t)$ or $y[n]$ through another system, then the system is invertible, otherwise it is non-invertible.

Example 3.2

Determine if the following systems are invertible or not

(a) $y(t) = 2x(t)$

(b) $y(t) = x^2(t)$

(c) $y[n] = \sum_{k=-\infty}^{n} x[k]$

(d) $y[n] = 0$

(a) The system is invertible because we can pass $y(t)$ using another system to produce $w(t)=0.5y(t)=x(t)$.

(b) The system is not invertible because the sign information is lost in the system output. Even employing the square root function, there are two possibilities: $w(t)=\sqrt{y(t)}$ or $w(t)=-\sqrt{y(t)}$.

(c) $$y[n]=\sum_{k=-\infty}^{n} x[k]=[\cdots x[n-2]+x[n-1]]+x[n]=y[n-1]+x[n]$$

If we pass $y[n]$ using another system, $w[n]=y[n]-y[n-1]=x[n]$ can be obtained and hence the system is invertible.

(d) Any inputs will give the same output of zero and hence the system is not invertible.

Linearity

A system is linear if it obeys principle of superposition.

Given two pairs of inputs and outputs, linearity implies:

$$\mathcal{T}\{ax_1(t)+bx_2(t)\}=a\mathcal{T}\{x_1(t)\}+b\mathcal{T}\{x_2(t)\}=ay_1(t)+by_2(t) \quad (3.2)$$

and

$$\mathcal{T}\{ax_1[n]+bx_2[n]\}=a\mathcal{T}\{x_1[n]\}+b\mathcal{T}\{x_2[n]\}=ay_1[n]+by_2[n] \quad (3.3)$$

where $|a|<\infty$ and $|b|<\infty$.

If the system does not satisfy superposition, it is non-linear.

Example 3.3

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not:

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \cdots + x[n-1] + x[n]$$

A standard approach to determine the linearity of a system is given as follows. Let

$$y_i[n] = \mathcal{T}\{x_i[n]\}, \quad i = 1, 2, 3$$

with

$$x_3[n] = ax_1[n] + bx_2[n]$$

If $y_3[n] = ay_1[n] + by_2[n]$, then the system is linear. Otherwise, the system is non-linear. This also applies to continuous-time system.

Assigning $x_3[n] = ax_1[n] + bx_2[n]$, we have:

$$\begin{aligned} y_3[n] &= \sum_{k=-\infty}^{n} x_3[k] \\ &= \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]) \\ &= a \sum_{k=-\infty}^{n} x_1[k] + b \sum_{k=-\infty}^{n} x_2[k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Note that the outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = \sum_{k=-\infty}^{n} x_1[k]$ and $y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$.

As a result, the system is linear.

Example 3.4

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not.

$$y[n] = 3x^2[n] + 2x[n-3]$$

The system outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = 3x_1^2[n] + 2x_1[n-3]$ and $y_2[n] = 3x_2^2[n] + 2x_2[n-3]$. Assigning $x_3[n] = ax_1[n] + bx_2[n]$, its system output is then:

$$\begin{aligned} y_3[n] &= 3x_3^2[n] + 2x_3[n-3] \\ &= 3\left(ax_1[n] + bx_2[n]\right)^2 + 2ax_1[n-3] + 2bx_2[n-3] \\ &= 3\left(a^2x_1^2[n] + b^2x_2^2[n] + 2abx_1[n]x_2[n]\right) + 2ax_1[n-3] + 2bx_2[n-3] \\ &\neq a\left(3x_1^2[n] + 2x_1[n-3]\right) + b\left(3x_2^2[n] + 2x_2[n-3]\right) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

As a result, the system is non-linear.

Time-Invariance

A system is time-invariant if a time-shift of input causes a corresponding shift in output:

$$\text{if} \quad y(t) = \mathcal{T}\{x(t)\} \quad \text{then} \quad y(t-t_0) = \mathcal{T}\{x(t-t_0)\} \tag{3.4}$$

and

$$\text{if} \quad y[n] = \mathcal{T}\{x[n]\} \quad \text{then} \quad y[n-n_0] = \mathcal{T}\{x[n-n_0]\} \tag{3.5}$$

That is, the system response is independent of time.

Example 3.5

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

A standard approach to determine the time-invariance of a system is given as follows.

Let $y_1[n] = \mathcal{T}\{x_1[n]\}$ where $x_1[n] = x[n-n_0]$. If $y_1[n] = y[n-n_0]$, then the system is time-invariant. Otherwise, the system is time-variant. This also applies to continuous-time system.

From the given input-output relationship, $y[n-n_0]$ is:

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Let $x_1[n] = x[n-n_0]$, its system output is:

$$\begin{aligned} y_1[n] &= \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0] = \sum_{l=-\infty}^{n-n_0} x[l], \quad l = k - n_0 \\ &= y[n-n_0] \end{aligned}$$

As a result, the system is time-invariant.

Example 3.6

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not:

$$y[n] = 3x[3n]$$

From the given input-output relationship, $y[n-n_0]$ is of the form:

$$y[n-n_0] = 3x[3(n-n_0)] = 3x[3n-3n_0]$$

Let $x_1[n] = x[n-n_0]$, its system output is:

$$y_1[n] = 3x_1[3n] = 3x[3n-n_0] \neq y[n-n_0]$$

As a result, the system is time-variant.

Causality

A system is causal if the output $y(t)$ (or $y[n]$) at time t (or n) depends on input $x(t)$ (or $x[n]$) up to time t (or n).

In casual system, output does not depend on future input.

On the other hand, in a non-causal system, the output depends on future input.

Example 3.7

Determine if the following systems are causal or not.

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n+2]$

(c) $y[n] = \sum_{k=-\infty}^{n} x[k]$

(a) The system is causal because it does not depend on future input.

(b) The system is not causal because it depends on future input, namely, $x[n+2]$.

(c) $$y[n] = \sum_{k=-\infty}^{n} x[k] = \cdots x[n-2] + x[n-1] + x[n]$$

We see that the output $y[n]$ at time n depends on input $x[n]$ up to time n. Hence the system is causal.

Stability

A system is stable if every bounded input $x(t)$ or $x[n]$ produces a bounded output $y(t)$ or $y[n]$ for all time t or n, or if the bounded-input bounded-output criterion is satisfied.

That is:

$$|y(t)| < B \quad \text{if} \quad |x(t)| < A, \quad |A| < \infty, \ |B| < \infty \qquad (3.6)$$

and

$$|y[n]| < B \quad \text{if} \quad |x[n]| < A, \quad |A| < \infty, \ |B| < \infty \qquad (3.7)$$

If a bounded input produces an unbounded output, then the system is unstable.

<u>Example 3.8</u>

Determine if the following systems are stable or not.

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n+2]$

(c) $y[n] = \dfrac{1}{x[n]}$

(a) If $x(t)$ is bounded, say, $|x(t)| < A$ for all t, we easily get

$$|y(t)| < A^2$$

Hence the system is stable.

(b) The system is stable because:

$$|y[n]| = |x[n] + x[n+2]| \leq |x[n]| + |x[n+2]| < 2A$$

for a bounded input with $|x[n]| < A$ for all n.

(c) The system is not stable. It is because for a bounded input, namely, $x[n] = 0$, the output is unbounded.

Linear Time-Invariant System Characterization

In this course, we will mainly study systems which are both linear and time-invariant.

Apart from being fundamental, many practical applications correspond to linear time-invariant (LTI) system.

Impulse Response

The impulse response ($h(t)$ or $h[n]$) is the output of a LTI system when the input is the unit impulse ($\delta(t)$ or $\delta[n]$):

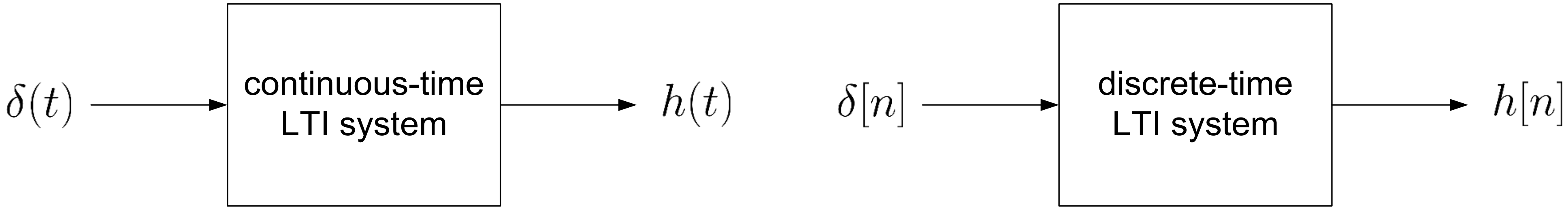


Fig. 3.4: Impulse response

For a continuous-time system, the impulse response is also continuous-time signal.

For a discrete-time system, the impulse response is also discrete-time signal.

A LTI system can be characterized by its impulse response, which indicates the system functionality.

Convolution

Convolution is used to describe the relationship between input, output and impulse response of a LTI in time domain.

We start with considering the discrete-time impulse response $h[n] = \mathcal{T}\{\delta[n]\}$ of a LTI system.

Recall (2.35) that a discrete-time signal is a linear combination of impulses with different time-shifts:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \tag{3.8}$$

Consider $x[n]$ as the system input with $y[n]$ being the output:

$$\begin{aligned} y[n] &= \mathcal{T}\{x[n]\} = \mathcal{T}\left\{ \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \right\} \\ &= \sum_{m=-\infty}^{\infty} x[m]\mathcal{T}\{\delta[n-m]\} \end{aligned} \tag{3.9}$$

due to the linearity property of (3.3).

Furthermore, using time-invariance property yields:

$$h[n-m] = \mathcal{T}\{\delta[n-m]\} \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] \otimes h[n] \tag{3.11}$$

which is called the convolution of $x[n]$ and $h[n]$, and $\otimes$ denotes the convolution operator.

According to (3.11), $h[n]$ completely characterizes the LTI system because for any input $x[n]$, the output $y[n]$ can be computed with the use of $h[n]$ via convolution where only multiplication and addition are involved.

There are three properties in convolution:

- Commutative

$$x[n] \otimes h[n] = h[n] \otimes x[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \quad (3.12)$$

- Associative

$$x[n] \otimes (h_1[n] \otimes h_2[n]) = (x[n] \otimes h_1[n]) \otimes h_2[n] \quad (3.13)$$

Combining (3.12) and (3.13) yields:

$$y[n] = x[n] \otimes h_1[n] \otimes h_2[n] = x[n] \otimes h_2[n] \otimes h_1[n] = x[n] \otimes (h_1[n] \otimes h_2[n]) \quad (3.14)$$

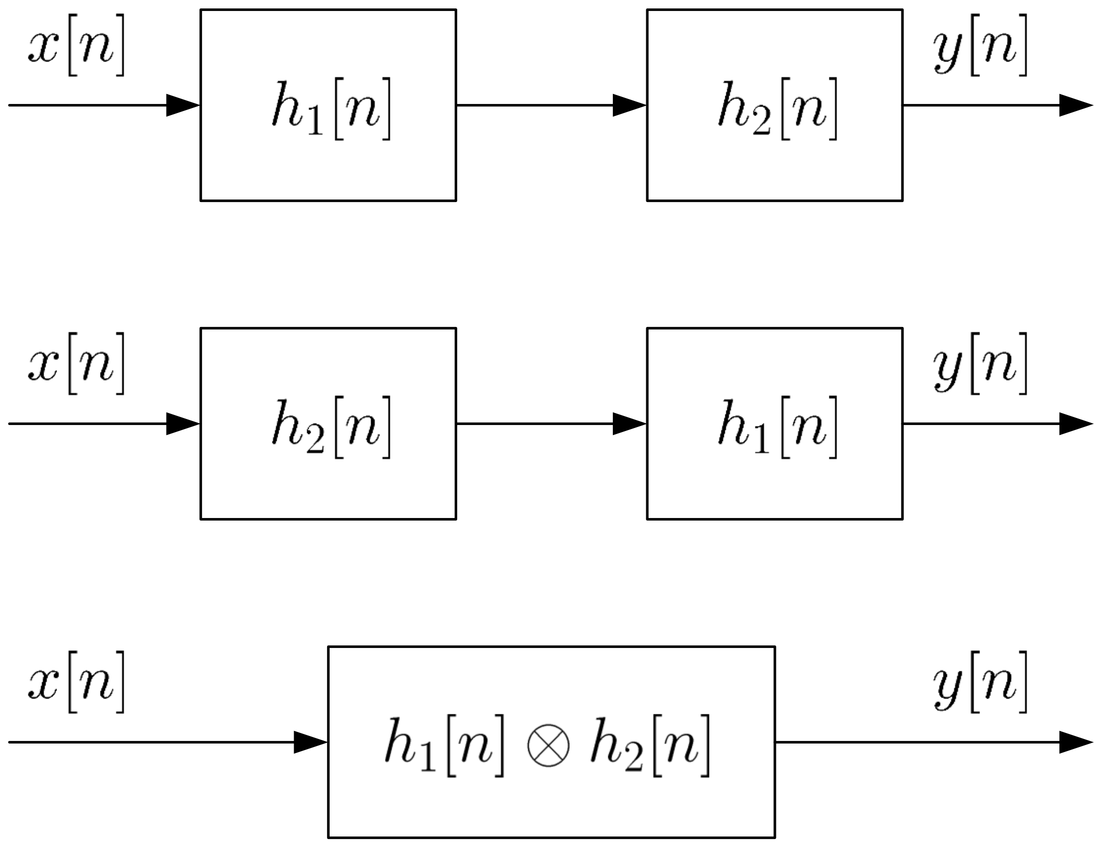


Fig. 3.5: Cascade interconnection and convolution

- Distributive

$$y[n] = x[n] \otimes (h_1[n] + h_2[n]) = x[n] \otimes h_1[n] + x[n] \otimes h_2[n] \quad (3.15)$$

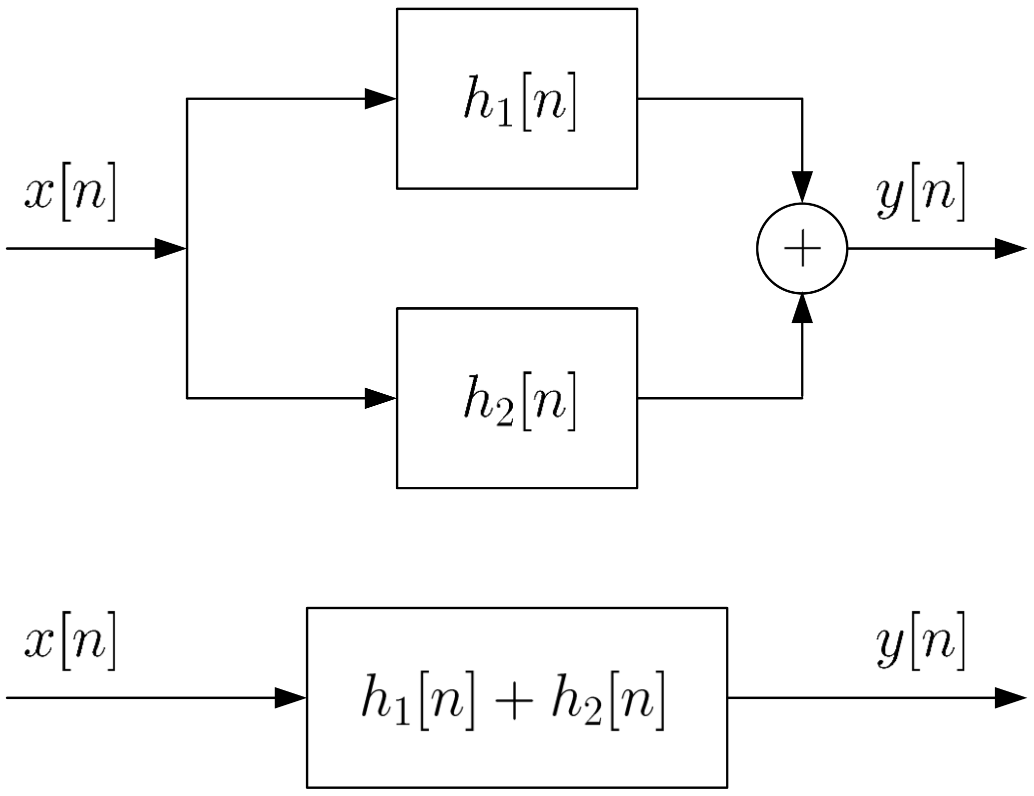


Fig. 3.6: Parallel interconnection and convolution

Example 3.9

Determine the function of a LTI discrete-time system if its impulse response is $h[n] = 0.5\delta[n] + 0.5\delta[n-1]$.

Using (3.11) and (3.8), we have:

$$
\begin{aligned}
y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} x[m]\left(0.5\delta[n-m] + 0.5\delta[n-1-m]\right) \\
&= 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] + 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-1-m] \\
&= 0.5(x[n] + x[n-1])
\end{aligned}
$$

The system computes the mean value of two input samples, i.e., current value and past value.

Similarly, for the continuous-time case, we start with considering $h(t) = \mathcal{T}\{\delta(t)\}$ of a LTI system.

Recall (2.21) that a continuous-time signal is considered as a linear combination of impulses with different time-shifts:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad (3.16)$$

Analogous to the development in (3.9)-(3.11), we get:

$$\begin{aligned} y(t) &= \mathcal{T}\{x(t)\} = \mathcal{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{T}\left\{\delta(t-\tau)\right\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)\otimes h(t) \end{aligned} \qquad (3.17)$$

Equation (3.17) is the convolution for the continuous-time case. However, the computation is more complicated than the discrete-time convolution because integration is needed.

Again, $h(t)$ characterizes the input-output relationship of LTI system.

Same as the discrete-time case, $h(t)$ specifies the system functionality and satisfies the commutative, associative as well as distributive properties.

<u>Example 3.10</u>

Determine the function of a LTI continuous-time system if its impulse response is $h(t) = \delta(t) + \delta(t-1)$.

Using (3.17) and (2.19)-(2.20), we obtain:

$$\begin{aligned} y(t) &= x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)[\delta(t-\tau) + \delta(t-1-\tau)] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau + \int_{-\infty}^{\infty} x(\tau)\delta(t-1-\tau) d\tau \\ &= \int_{-\infty}^{\infty} x(t)\delta(t-\tau) d\tau + \int_{-\infty}^{\infty} x(t-1)\delta(t-1-\tau) d\tau \\ &= x(t) \int_{-\infty}^{\infty} \delta(t-\tau) d\tau + x(t-1) \int_{-\infty}^{\infty} \delta(t-1-\tau) d\tau \\ &= x(t) + x(t-1) \end{aligned}$$

The system computes sum of inputs at two time instants: one at current time and the other at current time minus 1.

Example 3.11

Determine the function of a LTI continuous-time system if its impulse response is $h(t) = 0.1[u(t) - u(t-10)]$.

Using (3.17) and the commutative property, we get:

$$
\begin{aligned}
y(t) &= h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\
&= 0.1 \int_{-\infty}^{\infty} [u(\tau) - u(\tau - 10)] x(t-\tau) d\tau \\
&= \frac{1}{10} \int_{0}^{10} x(t-\tau) d\tau
\end{aligned}
$$

Note that $[u(\tau) - u(\tau - 10)]$ is a rectangular pulse for $\tau \in (0, 10)$.

The system computes average input value from the current time minus 10 to current time.

For LTI systems, we can also use the impulse response to check the system causality and stability.

A LTI system is causal if its impulse response satisfies:

$$h(t) = 0, \quad t < 0 \tag{3.18}$$

$$h[n] = 0, \quad n < 0 \tag{3.19}$$

A LTI system is stable if its impulse response satisfies:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \tag{3.20}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{3.21}$$

Example 3.12

Show that for a LTI discrete-time system, the causality definition in (3.19) is identical to the universal definition, i.e., $y[n]$ at time n depends on $x[n]$ up to time n.

Expanding the convolution formula in (3.12):

$$y[n] = x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$
$$= \cdots h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots$$

If $y[n]$ does not depend on future inputs $x[n+1], x[n+2], \cdots$, we must have $h[-1] = h[-2] = \cdots = 0$ or $h[n] = 0$ for $n < 0$.

Hence the two definitions regarding causality are identical.

Example 3.13

Compute the output $y[n]$ if the input is $x[n] = u[n]$ and the LTI system impulse response is $h[n] = \delta[n] + 0.5\delta[n-1]$. Discuss the stability and causality of system.

Using (3.11), we have:

$$
\begin{aligned}
y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} u[m]\left(\delta[n-m] + 0.5\delta[n-1-m]\right) \\
&= \sum_{m=0}^{\infty} \left(\delta[n-m] + 0.5\delta[n-1-m]\right) \\
&= \sum_{m=0}^{\infty} \delta[n-m] + 0.5\sum_{m=0}^{\infty} \delta[n-1-m] = u[n] + 0.5u[n-1]
\end{aligned}
$$

Alternatively, we can first establish the general relationship between $y[n]$ and $x[n]$ with the specific $h[n]$ as in Example 3.9:

$$
\begin{aligned}
y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\delta[n-m] + 0.5\delta[n-1-m]\right) \\
&= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] + 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-1-m] \\
&= x[n] + 0.5x[n-1]
\end{aligned}
$$

Substituting $x[n] = u[n]$ yields the same $y[n]$.

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{1} |h[n]| = 1.5 < \infty$ and $h[n] = 0$ for $n < 0$ the system is stable and causal.

Example 3.14

Compute the output $y[n]$ if the input is $x[n] = a^n u[n]$ and the LTI system impulse response is $h[n] = u[n] - u[n-10]$. Discuss the stability and causality of system.

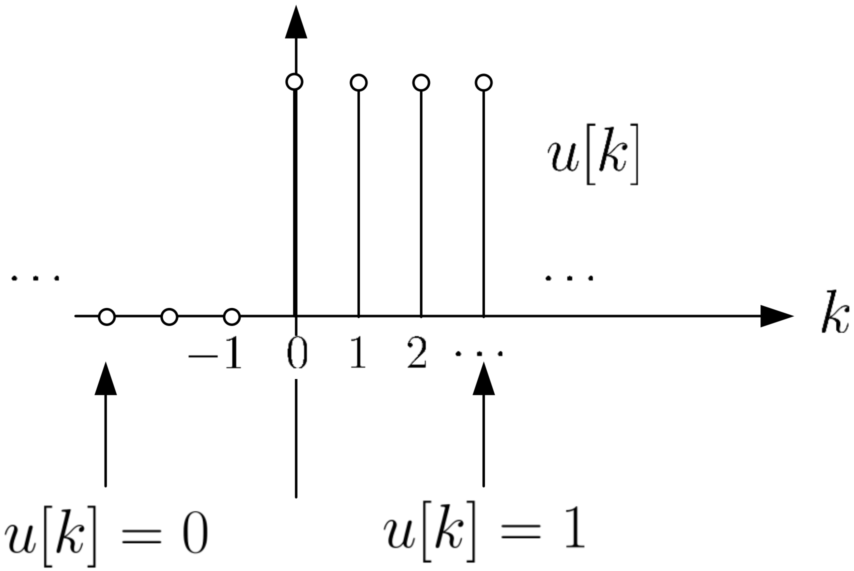
Using (3.11), we have:

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} a^m u[m] \left(u[n-m] - u[n-10-m]\right) \\
&= \sum_{m=0}^{\infty} a^m \left(u[n-m] - u[n-10-m]\right) \\
&= \sum_{m=0}^{\infty} a^m u[n-m] - \sum_{m=0}^{\infty} a^m u[n-10-m]
\end{aligned}
$$

Let $y_1[n]=\sum_{m=0}^{\infty} a^m u[n-m]$ and $y_2[n]=\sum_{m=0}^{\infty} a^m u[n-10-m]$ such that $y[n]=y_1[n]-y_2[n]$. By employing a change of variable, $y_1[n]$ is expressed as

$$y_1[n] = \sum_{m=0}^{\infty} a^m u[n-m] = \sum_{k=n}^{-\infty} a^{n-k} u[k], \quad k=n-m$$

$$= \sum_{k=-\infty}^{n} a^{n-k} u[k]$$



For $n < 0$, $y_1[n] = 0$ because $u[k] = 0$ for $k < 0$. For $n \geq 0$, $y_1[n]$ is:

$$y_1[n] = \sum_{k=0}^{n} a^{n-k} = 1 + a + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

where the geometric sum formula is applied:

$$\alpha + \alpha r + \cdots + \alpha r^{N-1} = \alpha \frac{1 - r^N}{1 - r}$$

That is,

$$y_1[n] = \frac{1 - a^{n+1}}{1 - a} u[n]$$

Similarly, $y_2[n]$ is:

$$y_2[n] = \sum_{m=0}^{\infty} a^m u[n-10-m]$$
$$= \sum_{k=-\infty}^{n-10} a^{n-10-k} u[k], \quad k = n-10-m$$

Since $u[k]=0$ for $k<0$, $y_2[n]=0$ for $n<10$. For $n \geq 10$, $y_2[n]$ is:

$$y_2[n] = \sum_{k=0}^{n-10} a^{n-10-k} = 1 + a + \cdots + a^{n-10} = \frac{1-a^{n-9}}{1-a}$$

That is,

$$y_2[n] = \frac{1-a^{n-9}}{1-a} u[n-10]$$

Combining the results, we have:

$$y[n] = \frac{1-a^{n+1}}{1-a}u[n] - \frac{1-a^{n-9}}{1-a}u[n-10]$$

or

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \leq n < 10 \\ \dfrac{a^{n-9}\left(1-a^{10}\right)}{1-a}, & 10 \leq n \end{cases}$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{9} |h[n]| = 10 < \infty$, the system is stable. Moreover, the system is causal because $h[n] = 0$ for $n < 0$.

Example 3.15

Determine $y[n] = x[n] \otimes h[n]$ where $x[n]$ and $h[n]$ are

$$x[n] = \begin{cases} n^2 + 1, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} n + 1, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Here, the lengths of both $x[n]$ and $h[n]$ are finite. More precisely, $x[0] = 1$, $x[1] = 2$, $x[2] = 5$, $x[3] = 10$, $h[0] = 1$, $h[1] = 2$, $h[2] = 3$ and $h[3] = 4$ while all other $x[n]$ and $h[n]$ have zero values.

We still use (3.11) but now it reduces to a finite summation:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$
$$= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3]$$

By considering the non-zero values of $h[n]$, we obtain:

$$y[n] = \begin{cases} 1, & n=0 \\ 4, & n=1 \\ 12, & n=2 \\ 30, & n=3 \\ 43, & n=4 \\ 50, & n=5 \\ 40, & n=6 \\ 0, & \text{otherwise} \end{cases}$$

Alternatively, for finite-length discrete-time signals, we can use the MATLAB command `conv` to compute the convolution of finite-length sequences:

```
n=0:3;
x=n.^2+1;
h=n+1;
y=conv(x,h)
```

The results are

```
y = 1     4     12     30     43     50     40
```

As the default starting time indices in both `h` and `x` are 1, we need to determine the appropriate time index for `y`.

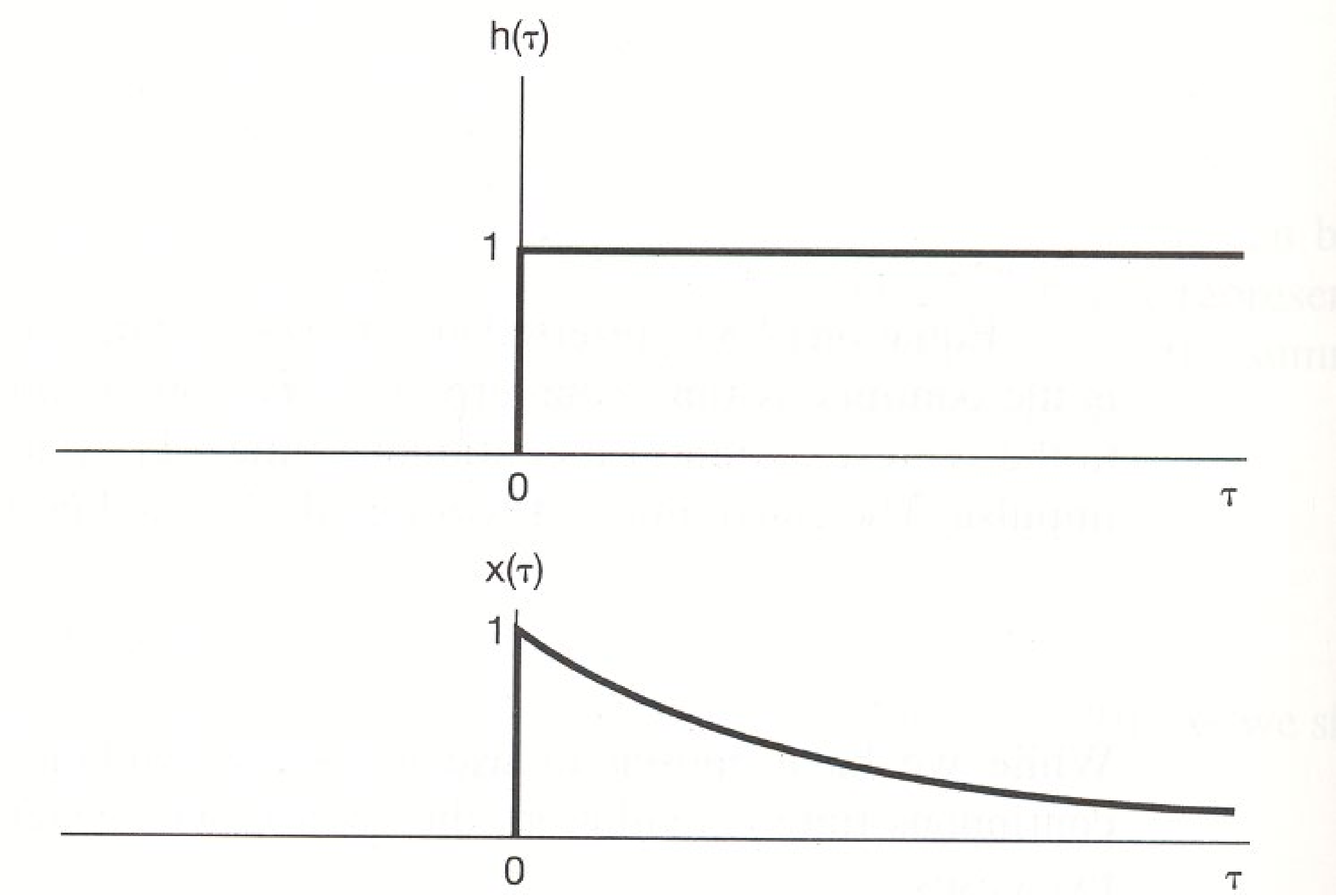
The correct index can be obtained by computing one value of $y[n]$ using (3.11). For simplicity, we may compute $y[0]$:

$$\begin{aligned} y[0] &= \sum_{m=-\infty}^{\infty} x[m]h[-m] \\ &= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + \cdots \\ &= x[0]h[0] \\ &= 1 \end{aligned}$$

In general, if the lengths of $x[n]$ and $h[n]$ are M and N, respectively, the length of $y[n] = x[n] \otimes h[n]$ is $(M+N-1)$.

Example 3.16

Compute the output $y(t)$ if the input is $x(t) = e^{-at}u(t)$ with $a > 0$ and the LTI system impulse response is $h(t) = u(t)$. Discuss the stability and causality of system.



Using (3.17), we have:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-a\tau}u(t-\tau)d\tau, \quad \lambda = t-\tau \\ &= \int_{t}^{-\infty} e^{-a(t-\lambda)}u(\lambda)\cdot -d\lambda \\ &= e^{-at}\int_{-\infty}^{t} e^{a\lambda}u(\lambda)d\lambda \end{aligned}$$

Similar to Example 3.14, when $t<0$, the integral will only involve the zero part of $u(\lambda)$ because $u(\lambda)=0$ for $\lambda<0$. Hence

$$y(t) = e^{-at}\int_{-\infty}^{t} e^{a\lambda}u(\lambda)d\lambda = 0, \quad t<0$$

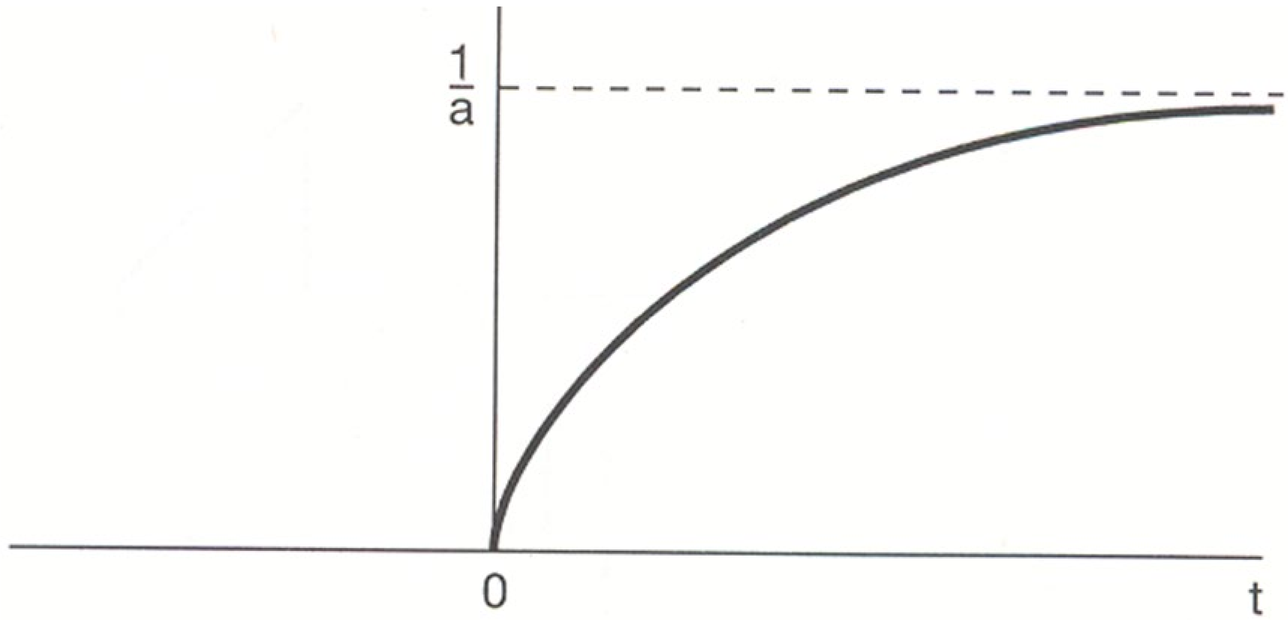
When $t>0$, the integral will involve the non-zero part of $u(\lambda)$ because $u(\lambda)=1$ for $0<\lambda\leq t$. The output is then:

$$y(t) = e^{-at} \int_{-\infty}^{t} e^{a\lambda} u(\lambda) d\lambda = e^{-at} \int_{0}^{t} e^{a\lambda} d\lambda$$

$$= e^{-at} \cdot \frac{1}{a} e^{a\lambda} \Big|_0^t = e^{-at} \cdot \frac{1}{a} \left(e^{at} - 1\right) = \frac{1}{a} \left(1 - e^{-at}\right)$$

We can combine the results for $t < 0$ and $t > 0$ to yield:

$$y(t) = \frac{1}{a} \left(1 - e^{-at}\right) u(t)$$



Linear Constant Coefficient Difference Equation

For a LTI discrete-time system, its input $x[n]$ and output $y[n]$ are related via a N th-order linear constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (3.22)$$

which is useful to check whether a system is both linear and time-invariant or not.

Example 3.17

Determine if the following input-output relationships correspond to LTI systems.

(a) $y[n] = 0.1y[n-1] + x[n] + x[n-1]$

(b) $y[n] = x[n+1] + x[n]$

(c) $y[n] = 1/x[n]$

(a) It corresponds to a LTI system with $N = M = 1$, $a_0 = 1$, $a_1 = -0.1$ and $b_0 = b_1 = 1$

(b) We reorganize the equation as:

$$y[n] = x[n+1] + x[n] \Rightarrow y[n-1] = x[n] + x[n-1]$$

which agrees with (3.22) when $N = M = 1$, $a_0 = 0$ and $a_1 = b_0 = b_1 = 1$. Hence it also corresponds to a LTI system.

(c) It does not correspond to a LTI system because $x[n]$ and $y[n]$ are not linear in the equation.

Note that if a system cannot be fitted into (3.22), there are three possibilities: linear and time-variant; non-linear and time-invariant; or non-linear and time-variant.

Example 3.18

Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n-1]$$

Using (3.12), we have:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots \end{aligned}$$

we can easily deduce that only $h[0]$ and $h[1]$ are nonzero. That is, the impulse response is:

$$h[n] = \delta[n] - \delta[n-1]$$

The difference equation can be used to generate the system output and even the system input.

Assuming that $a_0 \neq 0$, $y[n]$ is computed as:

$$y[n] = \frac{1}{a_0}\left(-\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]\right) \qquad (3.23)$$

Assuming that $b_0 \neq 0$, $x[n]$ can be obtained from:

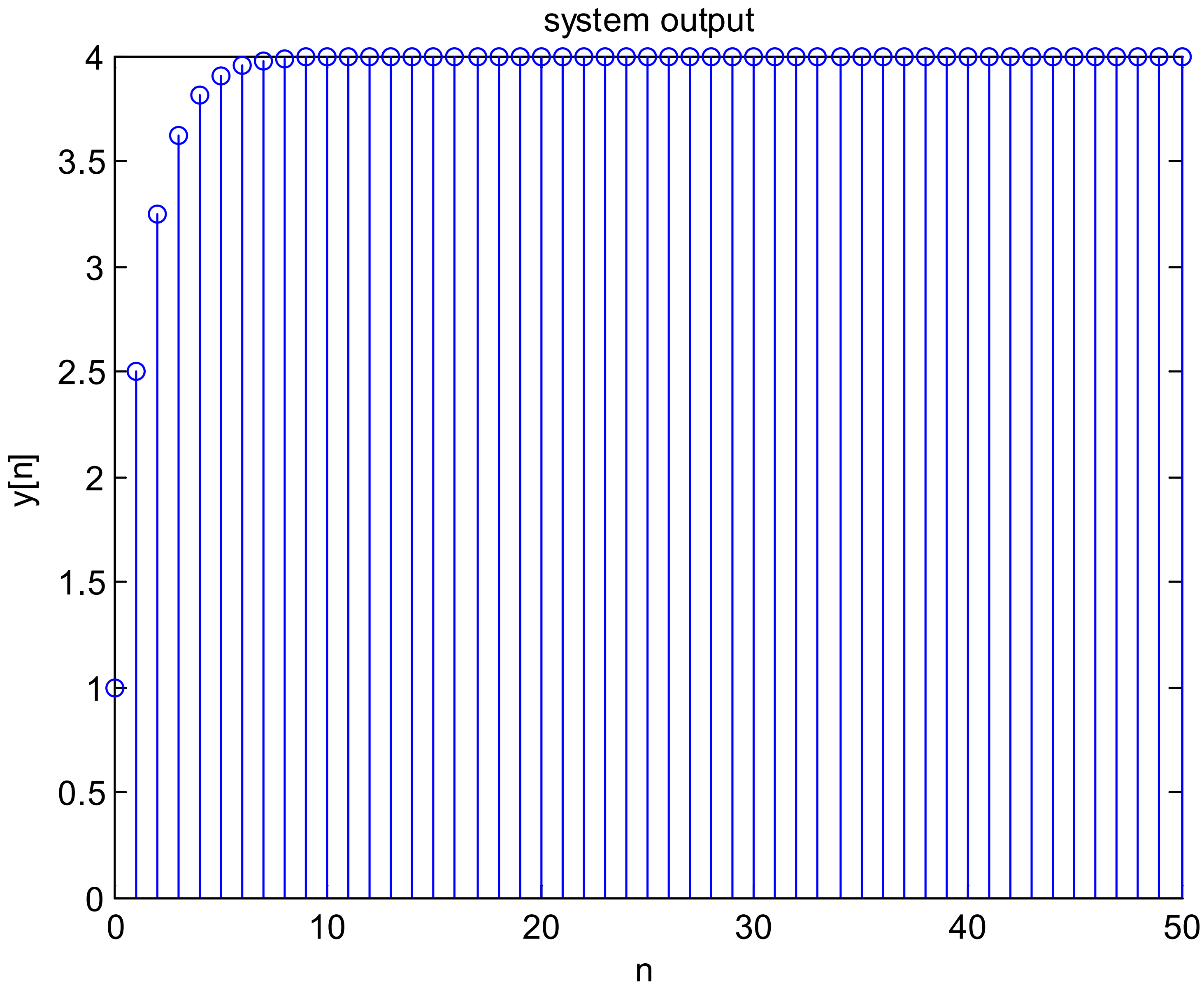
$$x[n] = \frac{1}{b_0}\left(\sum_{k=0}^{N} a_k y[n-k] - \sum_{k=1}^{M} b_k x[n-k]\right) \qquad (3.24)$$

Example 3.19

Given a LTI system described by difference equation of $y[n]=0.5y[n-1]+x[n]+x[n-1]$, compute the system output $y[n]$ for $0 \leq n \leq 50$ with an input of $x[n]=u[n]$. It is assumed that $y[-1]=0$.

The MATLAB code is:

```
N=50;               %data length is N+1
y(1)=1;             %compute y[0], only x[n] is nonzero
for n=2:N+1
y(n)=0.5*y(n-1)+2; %compute y[1],y[2],…,y[50]
                    %x[n]=x[n-1]=1 for n>=1
end
n=[0:N];            %set time axis
stem(n,y);
```


system output
y[n]
n
0
0.5
1
1.5
2
2.5
3
3.5
4
0
10
20
30
40
50

Alternatively, we can use the MATLAB command `filter` by rewriting the equation as:

$$y[n] - 0.5y[n-1] = x[n] + x[n-1]$$

The corresponding MATLAB code is:

```
x=ones(1,51);               %define input
a=[1,-0.5];                 %define vector of a_k
b=[1,1];                    %define vector of b_k
y=filter(b,a,x);            %produce output
stem(0:length(y)-1,y)
```

The `x` is the input which has a value of 1 for $0 \leq n \leq 50$, while a and b are vectors which contain $\{a_k\}$ and $\{b_k\}$, respectively.

The MATLAB programs for this example are provided as `ex3_19.m` and `ex3_19_2.m`.

Linear Constant Coefficient Differential Equation

For a continuous-time LTI system, its input $x(t)$ and output $y(t)$ are related via a Nth-order linear constant coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{3.25}$$

which is useful to check whether a system is both linear and time-invariant or not.

Analogous to the discrete-time case, we can use (3.25) to compute system input, output and impulse response.