Fourier Transform

Chapter Intended Learning Outcomes:

(i) Represent continuous-time aperiodic signals using Fourier transform

(ii) Understand the properties of Fourier transform

(iii) Understand the relationship between Fourier transform and linear time-invariant system

Aperiodic Signal Representation in Frequency Domain

For a periodic continuous-time signal, it can be represented in frequency domain using Fourier series.

But in general, signals are not periodic. To address this, we use Fourier transform:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad (5.1)$$

where $X(j\Omega)$ is a function of frequency Ω, also known as spectrum, and we can study the signal frequency components from it.

Unlike Fourier series, $X(j\Omega)$ is continuous in frequency, i.e., defined on a continuous range of Ω.

The inverse transform is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \qquad (5.2)$$

As in (4.7), we may write:

$$x(t) \leftrightarrow X(j\Omega) \qquad (5.3)$$

That is, both $x(t)$ and $X(j\Omega)$ represent the same signal: $x(t)$ is the time domain representation while $X(j\Omega)$ is the frequency domain representation.

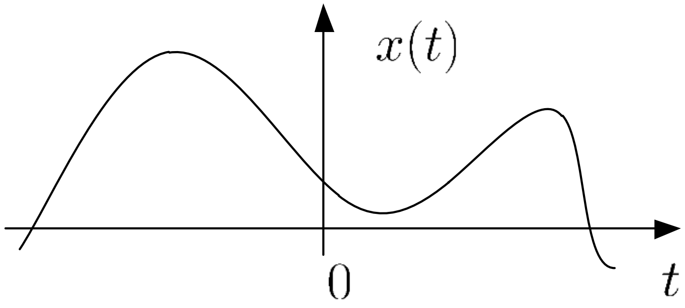
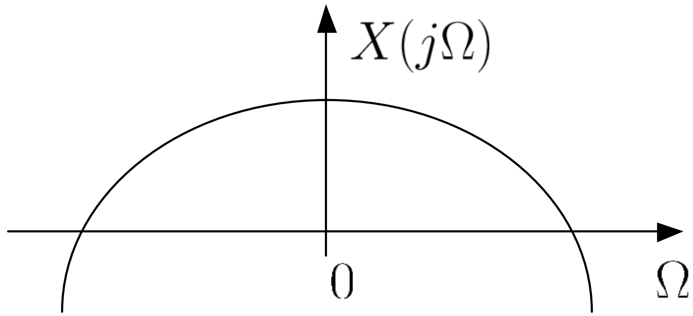
time domain	frequency domain
	
$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \Longrightarrow$	$\Longleftarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
continuous and aperiodic	continuous and aperiodic

Fig.5.1: Illustration of Fourier transform

Derivation of Fourier Transform

Fourier transform can be derived from Fourier series as follows.

We start with an aperiodic $x(t)$ and then construct its periodic version $\tilde{x}(t)$ with period T.

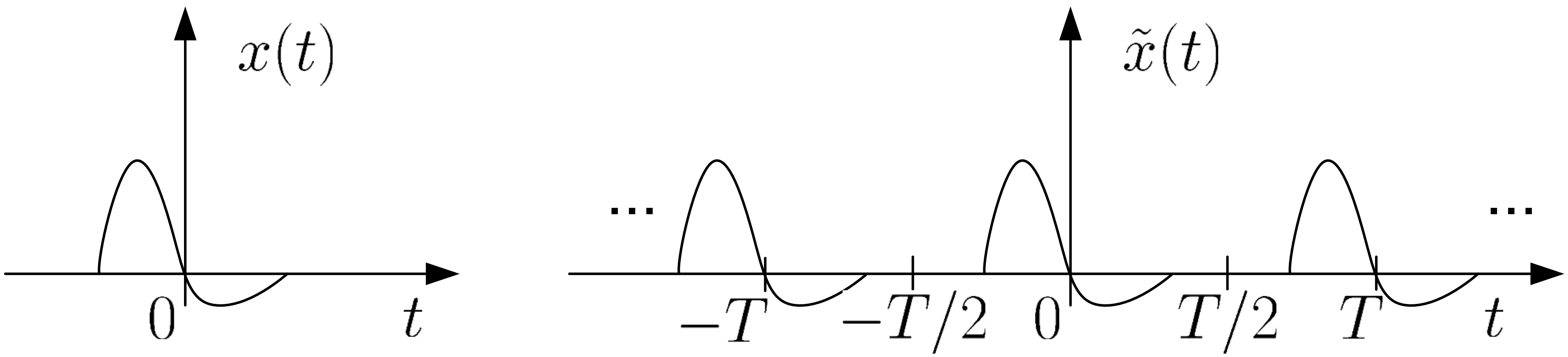


Fig.5.2: Constructing $\tilde{x}(t)$ from $x(t)$

According to (4.4), the Fourier series coefficients of $\tilde{x}(t)$ are:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Omega_0 t} dt \qquad (5.4)$$

where $\Omega_0 = 2\pi/T$.

Noting that $x(t) = \tilde{x}(t)$ for $|t| < T/2$ and $x(t) = 0$ for $|t| > T/2$, (5.4) can be expressed as:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t} dt \qquad (5.5)$$

As $X(j\Omega)$ is function of Ω, substituting $\Omega = k\Omega_0$ in (5.1) gives

$$X(jk\Omega_0) = \int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t} dt \qquad (5.6)$$

We can express a_k as:

$$a_k = \frac{1}{T} X(jk\Omega_0) \tag{5.7}$$

According to (4.3) and using (5.7), we get the Fourier series expansion for $\tilde{x}(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\Omega_0) e^{jk\Omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t} \tag{5.8}$$

As $T \to \infty$ or $\Omega_0 \to 0$, $\tilde{x}(t) \to x(t)$.

Considering $\Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}$ as the area of a rectangle whose height is $X(jk\Omega_0) e^{jk\Omega_0 t}$ and width corresponds to the interval of $[k\Omega_0, (k+1)\Omega_0]$, we obtain:

$$x(t) = \lim_{\Omega_0 \to 0} \tilde{x}(t) = \lim_{\Omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \tag{5.9}$$

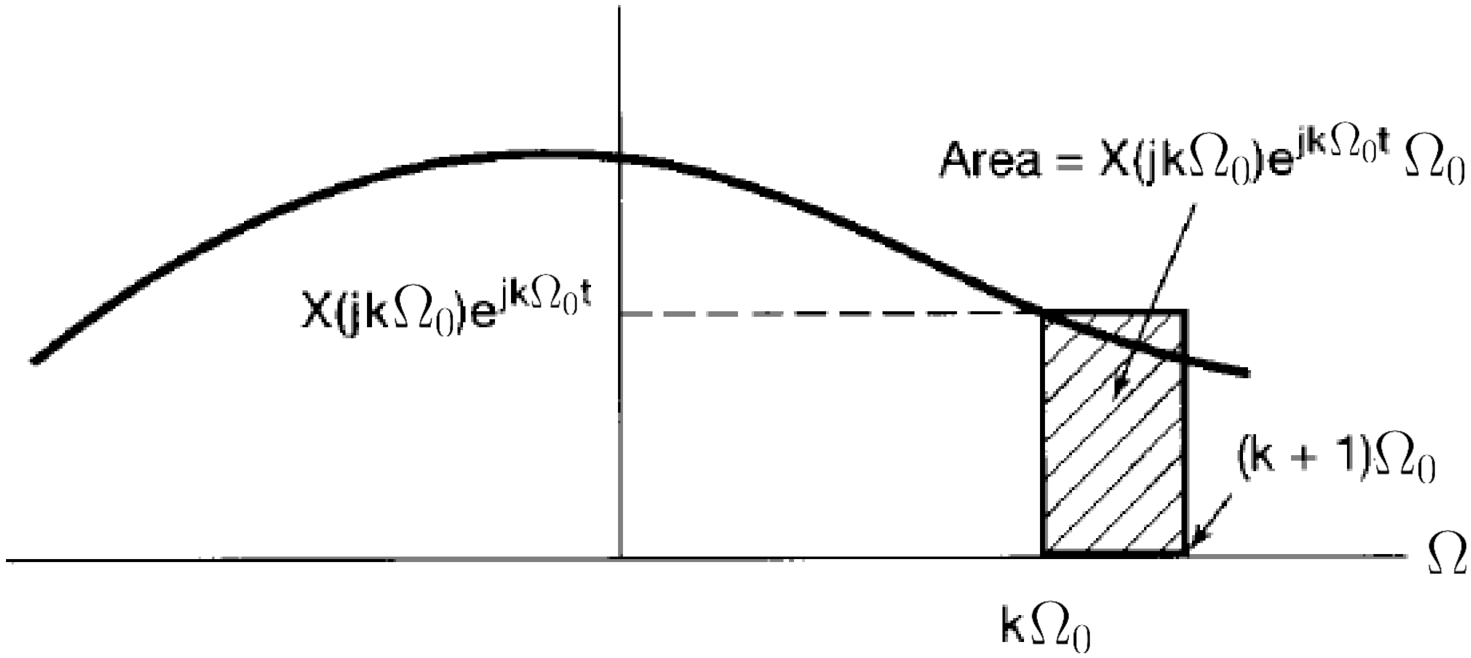


Fig. 5.3: Fourier transform from Fourier series

Example 5.1

Find the Fourier transform of $x(t)$ which is a rectangular pulse of the form:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

Note that the signal is of finite length and corresponds to one period of the periodic function in Example 4.3. Applying (5.1) on $x(t)$ yields:

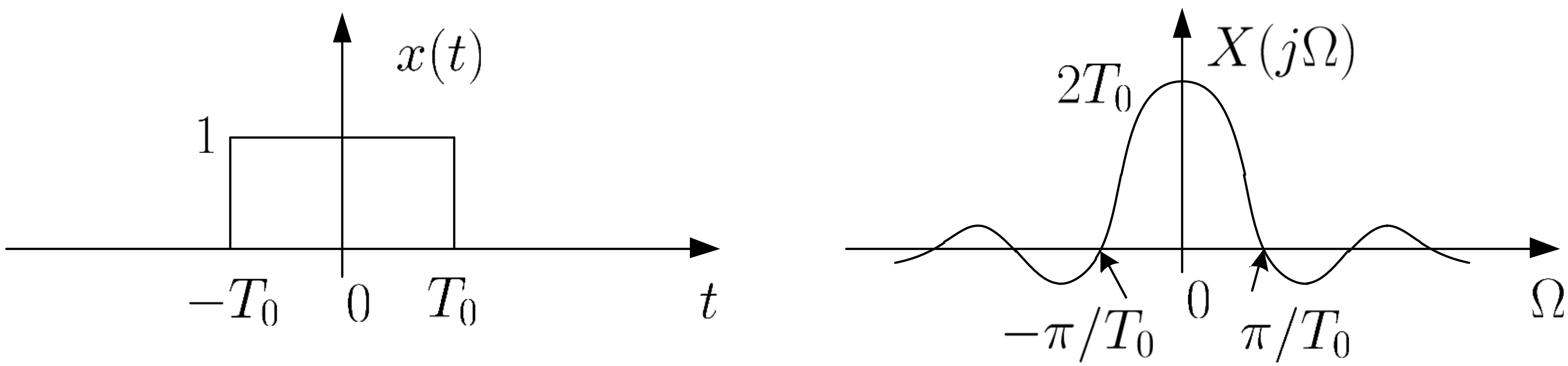
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \int_{-T_0}^{T_0} e^{-j\Omega t}dt = \frac{2\sin(\Omega T_0)}{\Omega}$$

Define the sinc function:

$$\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

It is seen that $X(j\Omega)$ is a scaled sinc function:

$$X(j\Omega) = \frac{2\sin(\Omega T_0)}{\Omega} = 2T_0 \mathrm{sinc}\left(\frac{\Omega T_0}{\pi}\right)$$



We can see that $X(j\Omega)$ is continuous in frequency. When the pulse width decreases, it covers more frequencies and vice versa.

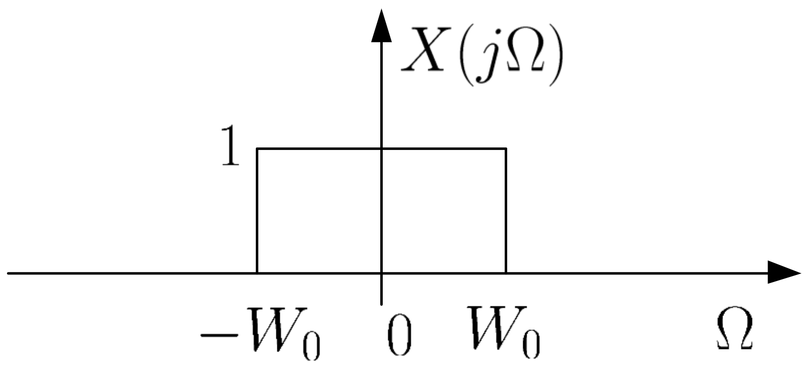
<u>Example 5.2</u>

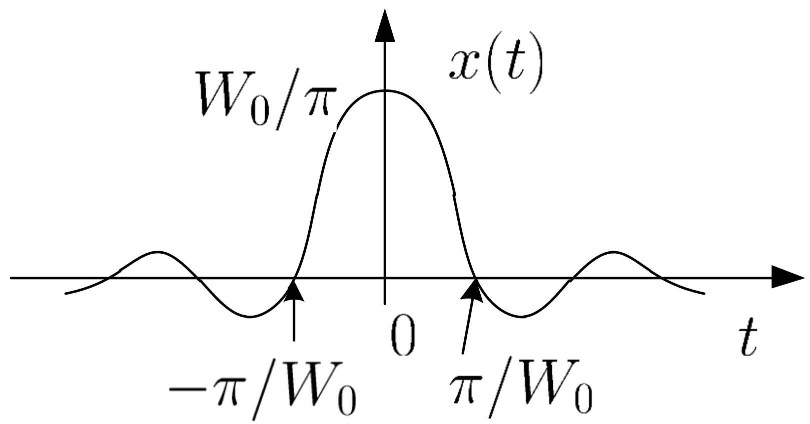
Find the inverse Fourier transform of $X(j\Omega)$ which is a rectangular pulse of the form:

$$X(j\Omega) = \begin{cases} 1, & -W_0 < \Omega < W_0 \\ 0, & \text{otherwise} \end{cases}$$

Using (5.2), we get:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-W_0}^{W_0} e^{j\Omega t}d\Omega = \frac{\sin(W_0 t)}{\pi t} = \frac{W_0}{\pi}\text{sinc}\left(\frac{W_0 t}{\pi}\right)$$





Example 5.3

Find the Fourier transform of $x(t)=e^{-at}u(t)$ with $a>0$.

Employing the property of $u(t)$ in (2.22) and (5.1), we get:

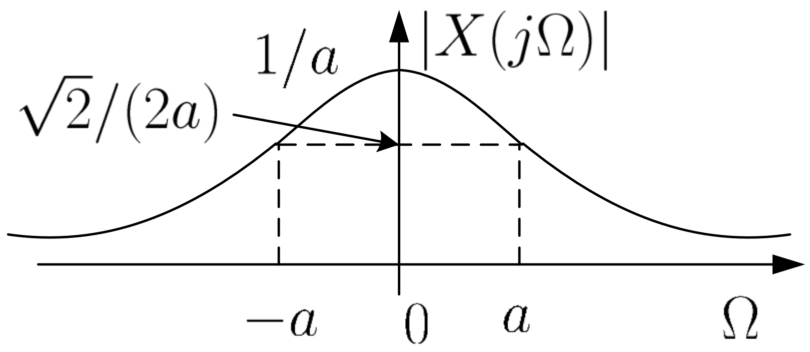
$$X(j\Omega)=\int_0^\infty e^{-at}e^{-j\Omega t}dt=-\frac{1}{a+j\Omega}e^{-(a+j\Omega)t}\Big|_0^\infty=\frac{1}{a+j\Omega}=\frac{a-j\Omega}{a^2+\Omega^2}$$

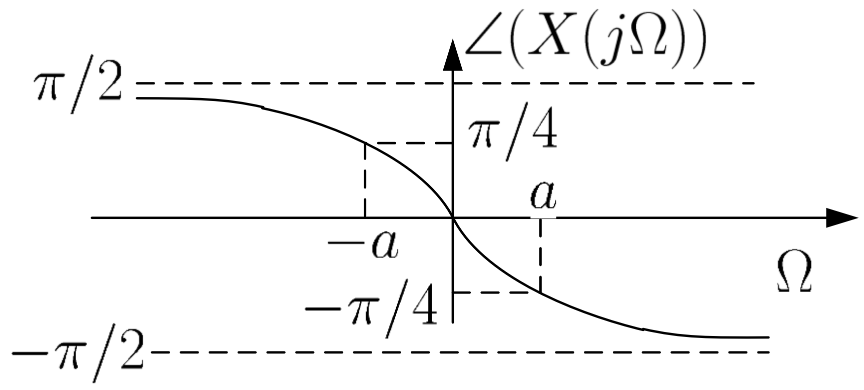
Note that when $t\to\infty$, $e^{-at}\to 0$.

$$|X(j\Omega)|=\frac{1}{\sqrt{a^2+\Omega^2}}$$

and

$$\angle(X(j\Omega))=-\tan^{-1}\left(\frac{\Omega}{a}\right)$$





Example 5.4

Find the Fourier transform of the impulse $x(t) = \delta(t)$.

Using (2.19) and (2.20) with $x(t) = e^{-j\Omega t}$ and $t_0 = 0$, we get:

$$X(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega \cdot 0} dt = e^{-j\Omega \cdot 0} \int_{-\infty}^{\infty} \delta(t) dt = e^{-j\Omega \cdot 0} = 1$$

Spectrum of $\delta(t)$ has unit amplitude at all frequencies. This aligns with Example 5.1 when $T_0 \to 0$. On the other hand, at $T_0 \to \infty$, $x(t)$ will be a DC and only contains frequency 0.

Periodic Signal Representation using Fourier Transform

Fourier transform can be used to represent continuous-time periodic signals with the use of $\delta(t)$.

Instead of time domain, we consider an impulse in the frequency domain:

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0) \quad (5.10)$$

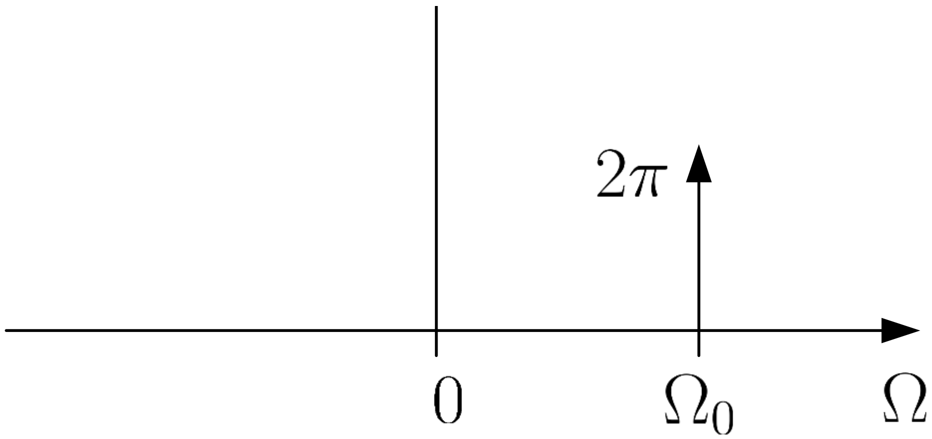


Fig.5.4: Impulse in frequency domain

Taking the inverse Fourier transform of $X(j\Omega)$ and employing the result in Example 5.4, $x(t)$ is computed as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0) e^{j\Omega t} d\Omega = e^{j\Omega_0 t} \qquad (5.11)$$

As a result, the Fourier transform pair is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0) \qquad (5.12)$$

From (4.3) and (5.12), the Fourier transform pair for a continuous-time periodic signal is:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0) \qquad (5.13)$$

Example 5.5

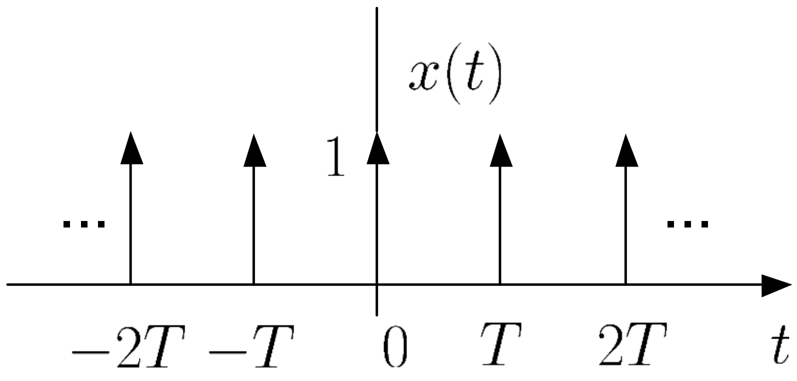
Find the Fourier transform of $x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$ which is called an impulse train.

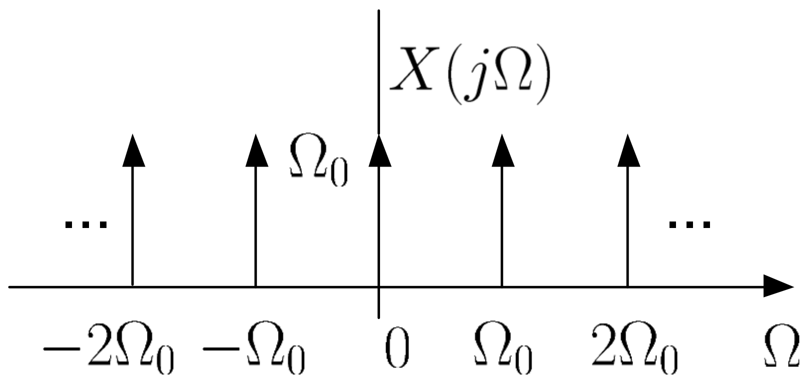
Clearly, $x(t)$ is a periodic signal with a period of T. Using (4.4) and Example 5.4, the Fourier series coefficients are:

$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\Omega_0 t}dt=\frac{1}{T}$$

with $\Omega_0=2\pi/T$. According to (5.13), the Fourier transform is:

$$X(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-\frac{2\pi k}{T}\right)=\Omega_0\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_0)$$





Properties of Fourier Transform

Linearity

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$ax(t) + by(t) \leftrightarrow aX(j\Omega) + bY(j\Omega) \quad (5.14)$$

Time Shifting

A shift of t_0 in $x(t)$ causes a multiplication of $e^{-j\Omega t_0}$ in $X(j\Omega)$:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(t-t_0) \leftrightarrow e^{-j\Omega t_0} X(j\Omega) \quad (5.15)$$

Time Reversal

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(-t) \leftrightarrow X(-j\Omega) \quad (5.16)$$

Time Scaling

For a time-scaled version of $x(t)$, $x(\alpha t)$ where $\alpha \neq 0$ is a real number, we have:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{j\Omega}{\alpha}\right) \quad (5.17)$$

Multiplication

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$x(t) \cdot y(t) \leftrightarrow \frac{1}{2\pi} X(j\Omega) \otimes Y(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\tau) Y(j(\Omega - \tau)) d\tau \quad (5.18)$$

Conjugation

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x^*(t) \leftrightarrow X^*(-j\Omega) \quad (5.19)$$

Parseval's Relation

The Parseval's relation addresses the energy of $x(t)$:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 dt \quad (5.20)$$

Convolution

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$x(t) \otimes y(t) \leftrightarrow X(j\Omega)Y(j\Omega) \tag{5.21}$$

which can be derived as:

$$\begin{aligned}
&\int_{-\infty}^{\infty} x(t) \otimes y(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)y(t-\tau)e^{-j\Omega t} d\tau dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)y(u)e^{-j\Omega\tau}e^{-j\Omega u} d\tau du, \quad u = t-\tau \\
&= \left[\int_{-\infty}^{\infty} x(\tau)e^{-j\Omega\tau} d\tau\right] \cdot \left[\int_{-\infty}^{\infty} y(u)e^{-j\Omega u} du\right] \\
&= X(j\Omega) \cdot Y(j\Omega)
\end{aligned} \tag{5.22}$$

Differentation

Differentiating $x(t)$ with respect to t corresponds to multiplying $X(j\Omega)$ by $j\Omega$ in the frequency domain:

$$\frac{dx(t)}{dt} \leftrightarrow j\Omega X(j\Omega) \Rightarrow \frac{d^k x(t)}{dt^k} \leftrightarrow (j\Omega)^k X(j\Omega) \qquad (5.23)$$

Integration

On the other hand, if we perform integration on $x(t)$, then the frequency domain representation becomes:

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\Omega}X(j\Omega) + \pi X(0)\delta(\Omega) \qquad (5.24)$$

Fourier Transform and Linear Time-Invariant System

Recall in a linear time-invariant (LTI) system, the input-output relationship is characterized by convolution in (3.17):

$$\begin{aligned} y(t) &= x(t) \otimes h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned} \qquad (5.25)$$

Using (5.21), we can consider (5.25) in frequency domain:

$$y(t) = x(t) \otimes h(t) \leftrightarrow Y(j\Omega) = X(j\Omega)H(j\Omega) \qquad (5.26)$$

This suggests apart from computing the output using time-domain approach via convolution, we can convert the input and impulse response to frequency domain, then $y(t)$ is computed from inverse Fourier transform of $X(j\Omega)H(j\Omega)$.

In fact, $H(j\Omega)$ represents the LTI system in the frequency domain, is called the system frequency response.

Recall (3.25) that the input and output of a LTI system satisfy the differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (5.27)$$

Taking the Fourier transform and using the linearity and differentiation properties, we get:

$$Y(j\Omega) \left[\sum_{k=0}^{N} a_k (j\Omega)^k \right] = X(j\Omega) \left[\sum_{k=0}^{M} b_k (j\Omega)^k \right] \qquad (5.28)$$

The system frequency response can also be computed as:

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{\sum_{k=0}^{M} b_k (j\Omega)^k}{\sum_{k=0}^{M} a_k (j\Omega)^k} \qquad (5.29)$$

Example 5.6

Determine the system frequency response for a LTI system described by the following differential equation:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Applying (5.29), we easily obtain:

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{1}{j\Omega + a}$$