ESTIMATION OF THE FREQUENCY AND PHASE OF A SINGLE SINUSOID BASED ON THE DOWNSAMPLING TECHNIQUE

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ABSTRACT

A new phase-based approach for sinusoidal frequency and phase estimation of a single tone is proposed in this paper. The major step is to arrange the observed data into several subsets by downsampling and exploit the new structure for parameter estimation. The maximum likelihood (ML) approach is then developed, and the performance of this estimator is shown to have a uniform performance along the frequency range, which is a major improvement to some conventional phase unwrapping algorithms. Computer simulations also show that the proposed scheme achieves an outstanding performance when the SNR is sufficiently high.

Index Terms— frequency estimation, phase, phase unwrapping, maximum likelihood estimation

1. INTRODUCTION

Estimation of the frequency and phase of a single sinusoid in additive white Gaussian noise is a long studied problem in communications and signal processing. One major type of estimators, either nonparametric or parametric based [1], uses the whole data set at the same time to achieve accurate estimations, while some others, which employ the phase unwrapping technique [2], retrieve the parameters in the sense of time-domain processing when the signal-to-noise ratio (SNR) is sufficiently high.

Phase unwrapping estimators was first suggested by Tretter [3], stating that the phase of a sinusoid is a linear function modulated by 2π . If the phase could be unwrapped correctly, then the parameters can be estimated by linear regression. One common approach, called the Kay's estimator, is to compute the phase difference of adjacent received signals so that the resulting values follow a moving average process, whose parameters can be retrieved by a weighted phase average or weighted linear techniques [4]. This methodology in fact avoids the procedure of phase unwrapping, and has the advantages of performing well when the SNR is large with a limited operations. However, the Kay's estimator will fail at moderate SNRs and also be shown to be inconsistent [6]. Other approaches, such as [5]-[7], which do the unwrapping of the phases in different ways, can be regarded as time-domain processing techniques. A brief introduction of one simple iterative

procedure will be further introduced in Section 2. There are major two drawbacks of employing the phase unwrapping techniques: the high threshold SNR and the fact that the estimation accuracy is depended on the frequency. In this paper, we contribute to derive a subset based maximum likelihood (ML) approach to overcome the latter problem.

The rest of this paper is organized as follows. One general method of the phase unwrapping based algorithms is given in Section 2, then the proposed estimator is derived in Section 3. In Section 4, simulation results are included to evaluate the performance of the developed approach by comparing with the Kay's [4] and phase-based ML [7] algorithms, as well as Cramér-Rao lower bound (CRLB). Finally, conclusions are drawn in Section 5.

2. A GENERAL MODEL AND ALGORITHM

The general signal model used in this paper can be described as follows:

$$y(n) = x(n) + q(n) \quad (1)$$

$$x(n) = Ae^{j(\omega n + \theta)}, \quad n = 0, 1, \dots, N-1 \quad (2)$$

where $\omega, \theta \in [-\pi, \pi)$ are the unknown frequency and phase needed to be estimated. A is the signal amplitude and $q(n)$ is a zero mean complex white Gaussian noise with variance σ^2 . We define $\angle y(n) = (\omega n + \theta + \angle q(n)) \bmod (2\pi) \in [-\pi, \pi)$ as the phase of $y(n)$ before phase unwrapping while $\angle \bar{y}(n) = (\omega n + \theta + \angle q(n))$ being the phase after phase unwrapping. In many articles, such as [6] [7], the authors propose estimation functions

$$[\hat{\omega}^{(n)} \quad \hat{\theta}^{(n)}] = f(y(n), \angle \bar{y}(n)) \quad (3)$$

to get the estimate of frequency and phase in the n -th order. As long as $\hat{\omega}^{(n)}$ and $\hat{\theta}^{(n)}$ are obtained, when the SNR is sufficiently high and ω is not close to $\pm\pi$, one easy way to determine $\angle \bar{y}(n+1)$ is to choose an appropriate integer C so that

$$\angle \bar{y}(n+1) = \angle y(n+1) + 2\pi C \quad (4)$$

lies in the interval $[\hat{\omega}^{(n)}(n+1) + \hat{\theta}^{(n)} - \pi, \hat{\omega}^{(n)}(n+1) + \hat{\theta}^{(n)} + \pi)$, and then an updated estimate of $\hat{\omega}^{(n+1)}$ and $\hat{\theta}^{(n+1)}$ can be obtained using (3). By an initial choice of $\hat{\omega}^{(0)} = 0$ and $\hat{\theta}^{(0)} = \angle \bar{y}(0) = \angle y(0)$, this type of algorithms can go on

when a new sample comes in, therefore is sufficient for real time computation.

One of the major problems of most of these algorithms is that their performances are expected to depend on ω [7]. These approaches perform well when ω is close to 0, but might fail at the areas close to $\pm\pi$, as a small noise will make the actual value of $\omega(n+1) + \theta$ falling outside of the boundaries of the interval $[\hat{\omega}^{(n)}(n+1) + \hat{\theta}^{(n)} - \pi, \hat{\omega}^{(n)}(n+1) + \hat{\theta}^{(n)} + \pi]$. This further leads to an inaccurate phase unwrapping of $\angle \bar{y}(n+1)$ in (4), then a degradation of performance. In next section, we will first propose a general idea to overcome this problem when N is sufficiently large and then apply this idea to derive an approximate ML estimator.

3. PROPOSED ALGORITHM

3.1. A General Solution

Consider the downsampling technique:

$$y_m(n_m) = x_m(n_m) + q_m(n_m), m = 0, 1, \dots, M-1 \quad (5)$$

$$\begin{aligned} x_m(n_m) &= x(Mn_m + m) \\ &= Ae^{j(M\omega n_m + m\omega + \theta)} = Ae^{j(\mu n_m + \phi_m)} \end{aligned} \quad (6)$$

i.e., we separate the whole data set into M subsets where

$$\mu = (M\omega) \bmod (2\pi) = M\omega + 2\pi B \in [-\pi, \pi) \quad (7)$$

$$\phi_m = (\theta + m\omega) \bmod (2\pi) \quad (8)$$

are the new common frequency and phase for the m th subset and B is an integer. By doing this segmentation, although the original frequency is fixed, we can choose an appropriate M to make μ not close to $\pm\pi$. Furthermore, instead of estimating $\hat{\omega}$ and $\hat{\theta}$ from $y(n)$, we retrieve $\hat{\mu}$ and the new phase vector $\hat{\Phi} = [\hat{\phi}_0 \ \hat{\phi}_1 \ \dots \ \hat{\phi}_{M-1}]$ from $y_m(n_m)$:

$$\begin{bmatrix} \hat{\mu} & \hat{\Phi} \end{bmatrix} = f(y_0(n_0), y_1(n_1), \dots, y_{M-1}(n_{M-1})) \quad (9)$$

says, from all signals but under different structure, then the estimator will not degrades, and the final estimate $\hat{\omega}$ can be determined as

$$\hat{\omega} = \frac{\hat{\mu} - 2\pi B}{M} \quad (10)$$

Defining $\hat{\theta}_m = \hat{\phi}_m - m\hat{\omega} + 2\pi B_m \in [-\pi, \pi)$ where B_m , $m = 0, 1, \dots, M-1$, are integers, according to (8), the final estimate of the phase becomes

$$\hat{\theta} = \frac{\sum_{m=0}^{M-1} \hat{\theta}_m}{M} \quad (11)$$

Now the problem is to select an M to make (5)-(11) work. Here we propose to apply the Kay's estimator [4] to the first R samples to get $\hat{\omega}^{\text{rough}}$ and set

$$\omega_r = (r\hat{\omega}^{\text{rough}}) \bmod (2\pi), \quad r = 1, 2, \dots, R \quad (12)$$

We further define

$$M = \arg \min_{r=1,2,\dots,R} |\omega_r| \quad (13)$$

which is, we let μ being the frequency closest to 0 among all the ω_r s. In section 4, we show that choosing $R = 5$ in the following approximate ML methodology can achieve a very outstanding performance, which is acceptable.

3.2. The Approximate ML Estimator

In this part, we consider the ML criterion, one basic criterion that is widely used in parameter estimation, to derive the estimation function f in (9).

At each time point $n = N-1$, we want to estimate ω and θ based on all the N received samples $y(n)$ up to time $N-1$, which is, from all the subsets $y_m(n_m)$ defined in (5). Decomposing $N = ML + m + 1$ where $m \in [0, M-1]$ and L are integers. We notice that each subset $y_m(n_m)$ has a length L_m where $L_0 = L_1 = \dots = L_m = L+1$, $L_{m+1} = L_{m+2} = \dots = L_{M-1} = L$ and $\sum_{m=0}^{M-1} L_m = N$. Writing

$$\mathbf{y} = [\mathbf{y}_0^T \ \mathbf{y}_1^T \ \dots \ \mathbf{y}_{M-1}^T] \quad (14)$$

$$\mathbf{y}_m = [y_m(0) \ y_m(1) \ \dots \ y_m(L_m-1)]^T \quad (15)$$

and the signal part and noise part being $\mathbf{x}$ and $\mathbf{q}$, correspondingly. As long as the noises are identical independent distributed, the ML estimates of $\hat{\mu}$ and $\hat{\Phi}$ correspond to maximize

$$\begin{aligned} p(\mathbf{x}|\mu, \Phi) &= \frac{1}{(\pi\sigma^2)^N} \exp\left(-\frac{\sum_{m=0}^{M-1} \sum_{n_m=0}^{L_m-1} |y_m(n_m) - Ae^{j(\mu n_m + \phi_m)}|^2}{\sigma^2}\right) \\ &= D \exp\left(\frac{2A}{\sigma^2} \sum_{m=0}^{M-1} \sum_{n_m=0}^{L_m-1} \cos(\angle \bar{y}_m(n_m) - (\mu n_m + \phi_m))\right) \end{aligned} \quad (16)$$

where $D = \frac{1}{(\pi\sigma^2)^N} e^{-\sum_{m=0}^{M-1} \sum_{n_m=0}^{L_m-1} (|y_m(n_m)|^2 + A^2)/\sigma^2}$ is a constant and $\mathbf{I}_N$ is $N \times N$ identity matrix. Taking the nature logarithm of both sides of (16) yields

$$\begin{aligned} \Lambda(\mu, \Phi) &= \ln p(\mathbf{x}|\mu, \Phi) \\ &= \ln D + \frac{2A}{\sigma^2} \sum_{m=0}^{M-1} \sum_{n_m=0}^{L_m-1} |y_m(n_m)| \cos(\angle \bar{y}_m(n_m) - (\mu n_m + \phi_m)) \end{aligned} \quad (17)$$

differentiating $\Lambda(\mu, \Phi)$ with respect to μ and ϕ_m and equating the resultant expressions to zero result in

$$\begin{aligned} \frac{\partial \Lambda(\mu, \Phi)}{\partial \mu} &= 2\text{SNR} \sum_{m=0}^{M-1} \sum_{n_m=0}^{L_m-1} n_m |y_m(n_m)| \sin(\angle \bar{y}_m(n_m) - (\mu n_m + \phi_m)) = 0 \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{\partial \Lambda(\mu, \Phi)}{\partial \phi_m} &= 2\text{SNR} \sum_{n_m=0}^{L_m-1} |y_m(n_m)| \sin(\angle \bar{y}_m(n_m) - (\mu n_m + \phi_m)) = 0 \end{aligned} \quad (19)$$

where $\text{SNR} = \frac{A}{\sigma^2}$ for $m = 0, 1, \dots, M-1$. The sine function in (18) and (19) makes the equations highly nonlinear therefore a closed form solution might not exist. To solve this problem, we assume that the SNR is sufficiently high, then the phase error $\angle \bar{y}_m(n_m) - (\mu n_m + \phi_m) \rightarrow 0$. By approximating $\sin x \approx x$ for small x , (18) and (19) become

$$2\text{SNR} \sum_{m=0}^{M-1} (P_{m,1} - P_{m,2}\mu - P_{m,3}\phi_m) = 0 \quad (20)$$

$$2\text{SNR}(P_{m,4} - P_{m,3}\mu - P_{m,5}\phi_m) = 0 \quad (21)$$

$$P_{m,1} = \sum_{n_m=0}^{L_m-1} n_m |y_m(n_m)| \angle \bar{y}_m(n_m) \quad (22)$$

$$P_{m,2} = \sum_{n_m=0}^{L_m-1} n_m^2 |y_m(n_m)| \quad (23)$$

$$P_{m,3} = \sum_{n_m=0}^{L_m-1} n_m |y_m(n_m)| \quad (24)$$

$$P_{m,4} = \sum_{n_m=0}^{L_m-1} |y_m(n_m)| \angle \bar{y}_m(n_m) \quad (25)$$

$$P_{m,5} = \sum_{n_m=0}^{L_m-1} |y_m(n_m)| \quad (26)$$

for $m = 0, 1, \dots, M-1$. Solving all the equations yields

$$\hat{\mu} = \frac{\sum_{m=0}^{M-1} Q_{m,1}}{\sum_{m=0}^{M-1} Q_{m,2}} \quad (27)$$

$$\hat{\phi}_m = \frac{P_{m,4} - P_{m,3}\hat{\mu}}{P_{m,5}} \quad (28)$$

where

$$Q_{m,1} = P_{m,1} - \frac{P_{m,3}P_{m,4}}{P_{m,5}} \quad (29)$$

$$Q_{m,2} = P_{m,2} - \frac{P_{m,3}^2}{P_{m,5}} \quad (30)$$

indicating that when a new sample at time $n = N-1 = ML + m$ comes in, only the corresponding subset $y_m(n_m)$ is changed, and we only need to update P_{m,i_1} and Q_{m,i_2} , $i_1 = 1, 2, \dots, 5$ and $i_2 = 1, 2$, to get the new estimates $\hat{\mu}$, $\hat{\theta}_m$, then the $\hat{\omega}$, $\hat{\theta}$ could be solved using (10) and (11). It is also worth notice that this approach is a generalization of [7], when $M = 1$, it is reduced to [7]. Finally, the proposed estimator is concluded in Table 1.

4. SIMULATION RESULTS

Computer simulations have been carried out to evaluate the performance of the proposed algorithm in the presence of white Gaussian noise. The average mean square frequency error (MSFE) is assigned to evaluate the algorithm performance. All results provided are averages of 2000 independent runs.

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- (i) Use the first L samples to determine M by (12) and (13). Set the initial estimates $\hat{\omega} = \hat{\mu} = 0$ and $\hat{\theta}_m = \angle y_m(n_m)$ for $m = 0, 1, \dots, M-1$.
 - (ii) for time point $n = N-1 > M-1$, stack the new coming sample $y(n)$ in the end of the corresponding subset $y_m(n_m)$ according to (6), get the $\angle \bar{y}_m(n_m)$ from (4) by substituting $\hat{\omega}$ and $\hat{\theta}$ with $\hat{\mu}$ and $\hat{\theta}_m$.
 - (iii) calculate the new P_{m,i_1} and Q_{m,i_2} , then get an updated $\hat{\mu}$ and $\hat{\theta}_m$ using (22) - (30).
 - (iv) update $\hat{\omega}$ and $\hat{\theta}$ by (10) and (11).
 - (v) repeat (ii) - (iv) every time a new sample comes in.
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Table 1. ML Estimation Algorithm

In the first test, we study the performance of the proposed algorithm comparing with the Kay's weighted phase average estimator [4] and the phase-based ML estimator [7]. The signal amplitude and frequency are $A = 1$ and $\omega = 0.6\pi$ while the phase varies from $-\pi$ to π in each independent run. We choose $R = 5$ in the proposed algorithm and three cases $N = 10$, $N = 30$ and $N = 100$ are tested, while the MSFEs versus SNR is plot in Figure 1. The proposed approach performs the best in the sense of achieving the threshold SNR of 12dB in all the three cases. It is also shown that the threshold performance of the proposed estimator is independent of the length of the data, indicating that it in fact only depends on selection of M and the accuracy of the phase unwrapping procedure (4) under the new frequency μ , which in turns only depends on the SNR. This nature also shows that the idea of dividing the whole data set into several subsets will not degrades the performance as long as the number of subsets, M , is correctly chosen to make μ not close to $\pm\pi$.

Figures 2 to 4 plot the frequency versus SNR contours of MSFE for the Kay's, phase-based ML and the proposed methods under $N = 30$. It is observed that proposed scheme has a desirable merit of uniform performance. It also has the best threshold performance, between 8-14dB when the real ω varies from -0.8π to 0.8π , among all the methods, which agrees with the first test. Notice that when the true frequency is close to 0, the proposed method performs the same as the phase-based ML algorithm [7], as in that case, the system will choose $M = 1$ and it is reduced to the phase-based ML estimator.

5. CONCLUSION

In this paper, we first discuss a general procedure of the so call phase-based algorithms, and then an idea of dividing the whole data set into several subsets is introduced. The ML criterion is also applied to derived a new estimator bases on this downsampling technique. Computer simulation shows that the segmentation of the data does not degrade the perfor-

mance of the estimator, and the proposed scheme performs outstandingly in terms of estimation accuracy under a majority range of frequency.

6. REFERENCES

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Fig. 1. Average mean square frequency error versus SNR.

Fig. 2. Contour plot of Kay's weighted phase average estimator.

Fig. 3. Contour plot of phase-based ML method.

Fig. 4. Contour plot of the proposed estimator.