Chapter 1

- **Brief Review of Discrete-Time Signal Processing**

- **Brief Review of Random Processes**

References:

- A.V.Oppenheim and A.S.Willsky, *Signals and Systems*, Prentice Hall, 1996
- J.G.Proakis and D.G.Manolakis, *Introduction to Digital Signal Processing*, Macmillan, 1988
- A.V.Oppenheim and R.W.Schafer, *Discrete-Time Signal Processing*, Prentice Hall, 1998
- P.M.Clarkson, *Optimal and Adaptive Signal Processing*, CRC, 1993
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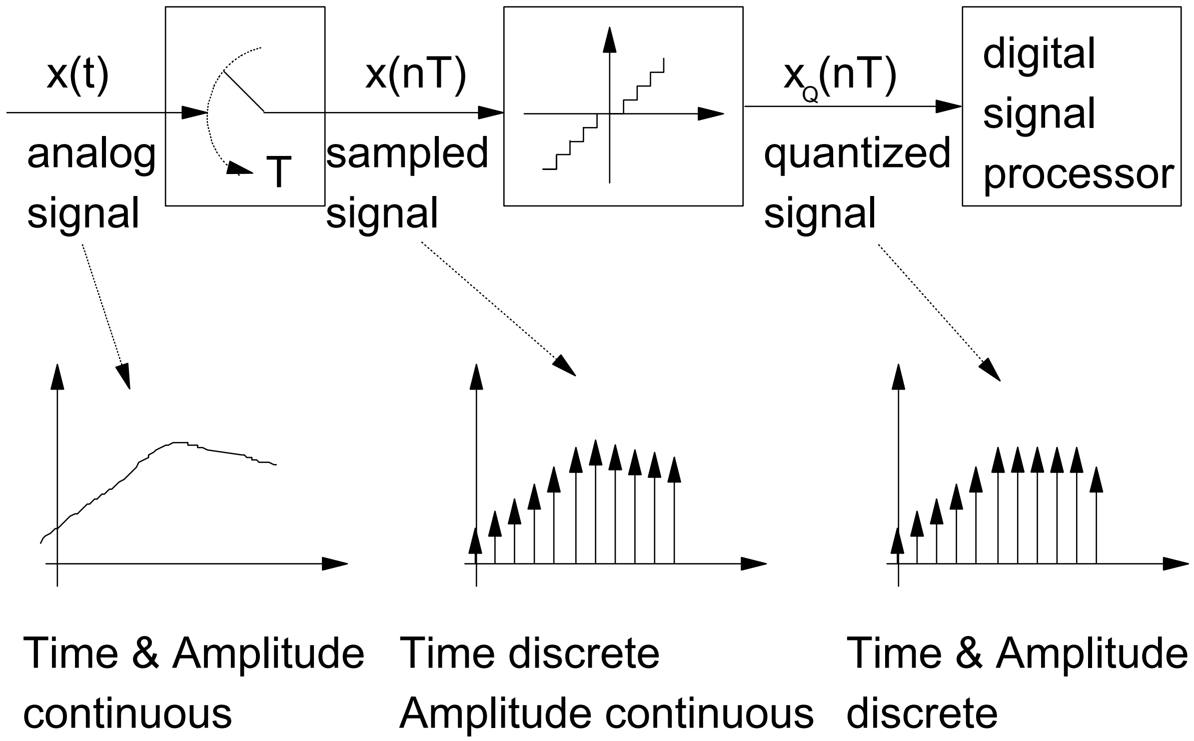
Brief Review of Discrete-Time Signal Processing

There are 3 types of signals that are functions of time:

- continuous-time (analog) : defined on a continuous range of time
- discrete-time : defined only at discrete instants of time (…,(n-1)T,nT,(n+1)T,…)
- digital (quantized) : both time and amplitude are discrete



Digital Signal Processing Applications

- Speech
  - Coding (compression)
  - Synthesis (production of speech signals, e.g., speech development kit by Microsoft)
  - Recognition (e.g., PCCW's 1083 telephone number enquiry system and many applications for disabled persons as well as security)
  - Animal sound analysis
- Music
  - Generation of music by different musical instruments such as piano, cello, guitar and flute using computer
  - Song with low-cost electronic piano keyboard quality

- Image
  - Compression
  - Recognition such as face, palm and fingerprint
  - Construction of 3D objects from 2D images
  - Animation, e.g., “Toy Story (反斗奇兵)”
  - Special effects such as adding Forrest Gump to a film of President Nixon in “阿甘正傳” and removing some objects in a photograph or movie
- Digital Communications
  - Encryption
  - Transmission and Reception (coding / decoding, modulation / demodulation, equalization)
- Biometrics and Bioinformatics
- Digital Control

Transform from Time to Frequency

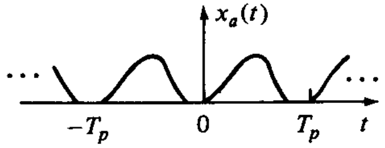
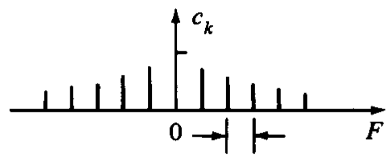
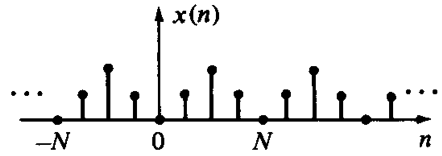
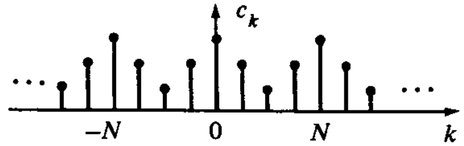
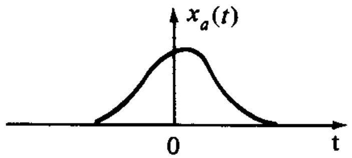
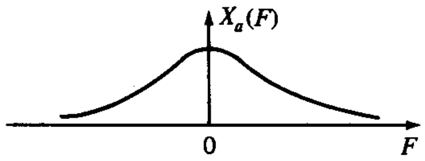
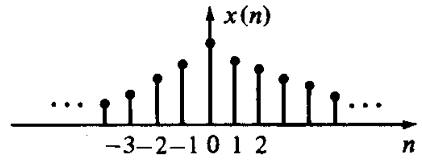
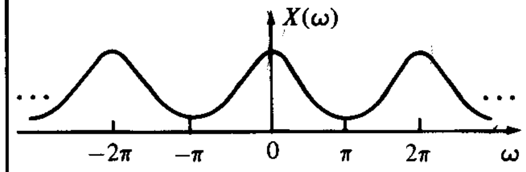
$$x(t) \underset{\text{inverse transform}}{\overset{\text{transform}}{\rightleftarrows}} X(\omega)$$

Fourier Series

- express periodic signals using *harmonically related sinusoids*
- different definitions for continuous-time & discrete-time signals
- frequency ω takes discrete values: $\omega_0, 2\omega_0, 3\omega_0, \ldots$

Fourier Transform

- frequency analysis tool for aperiodic signals
- defined on a continuous range of ω
- different definitions for continuous-time & discrete-time signals
- Fast Fourier transform (FFT) – an computationally efficient method for computing Fourier transform of discrete signals

	CONTINUOUS–TIME SIGNALS		DISCRETE–TIME SIGNALS	
	TIME–DOMAIN	FREQUENCY–DOMAIN	TIME–DOMAIN	FREQUENCY–DOMAIN
PERIODIC SIGNALS / FOURIER SERIES		 $F_0 = \frac{1}{T_p}$		
	$c_k = \frac{1}{T_p}\int_{T_p} x_a(t)e^{-j2\pi kF_0 t}\,dt$ ⟹	⟸ $x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$ ⟹	⟸ $x(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$
	CONTINUOUS AND PERIODIC	DISCRETE AND APERIODIC	DISCRETE AND PERIODIC	DISCRETE AND PERIODIC
APERIODIC SIGNALS / FOURIER TRANSFORMS				
	$X_a(F) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi Ft}\,dt$ ⟹	⟸ $x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}\,dF$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$ ⟹	⟸ $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$
	CONTINUOUS AND APERIODIC	CONTINUOUS AND APERIODIC	DISCRETE AND APERIODIC	CONTINUOUS AND PERIODIC

Transform	Time Domain	Frequency Domain
Fourier Series	periodic & continuous $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, $\omega_0 = 2\pi / T_P$	aperiodic & discrete $c_k = \frac{\omega_0}{2\pi} \int_{-T_P/2}^{T_P/2} x(t) e^{-j\omega_0 kt} dt$, T_P is the period
Fourier Transform	aperiodic & continuous $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$	aperiodic & continuous $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
Discrete-Time Fourier Transform	aperiodic & discrete $x(nT) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} X(\omega) e^{j\omega nT} d\omega$, T is the sampling interval	periodic & continuous $X(\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}$
Discrete(-Time) Fourier Series	periodic & discrete $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$, $T_P = N$ and $T = 1$	periodic & discrete $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

Fourier Series

Fourier series are used to represent the frequency contents of a periodic and continuous-time signal. A continuous-time function $x(t)$ is said to be periodic if there exists $T_P > 0$ such that

$$x(t) = x(t + T_P), \qquad t \in (-\infty, \infty) \tag{I.1}$$

The smallest T_P for which (I.1) holds is called the fundamental period. Every periodic function can be expanded into a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \qquad t \in (-\infty, \infty) \tag{I.2}$$

where

$$c_k = \frac{\omega_0}{2\pi} \int_{-T_P/2}^{T_P/2} x(t) e^{-j\omega_0 kt} dt \tag{I.3}$$

and $\omega_0 = 2\pi / T_P$ is called the fundamental frequency.

Example 1.1

The signal $x(t) = \cos(100\pi t) + \cos(200\pi t)$ is a periodic and continuous-time signal.

The fundamental frequency is $\omega_0 = 100\pi$. The fundamental period is then $T_P = 2\pi/(100\pi) = 1/50$:

$$x\left(t + \frac{1}{50}\right) = \cos\left(100\pi\left(t + \frac{1}{50}\right)\right) + \cos\left(200\pi\left(t + \frac{1}{50}\right)\right)$$
$$= \cos(100\pi t + 2\pi) + \cos(200\pi t + 4\pi)$$
$$= \cos(100\pi t) + \cos(200\pi t) = x(t)$$

Since $x(t) = \cos(100\pi t) + \cos(200\pi t) = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \dfrac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2}$

By inspection and using (I.2), we have $c_1 = 1/2$, $c_{-1} = 1/2$, $c_2 = 1/2$, $c_{-2} = 1/2$ while all other Fourier series coefficients are equal to zero.

Fourier Transform

Fourier transform is used to represent the frequency contents of an aperiodic and continuous-time signal $x(t)$:

Forward transform:

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt \tag{I.4}$$

and

Inverse transform:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega \tag{I.5}$$

Some points to note:

- Fourier spectrum (both magnitude and phase) are continuous in frequency and aperiodic
- Convolution in time domain corresponds to multiplication in Fourier transform domain, i.e., $x(t)\otimes y(t)\leftrightarrow X(\omega)\cdot Y(\omega)$

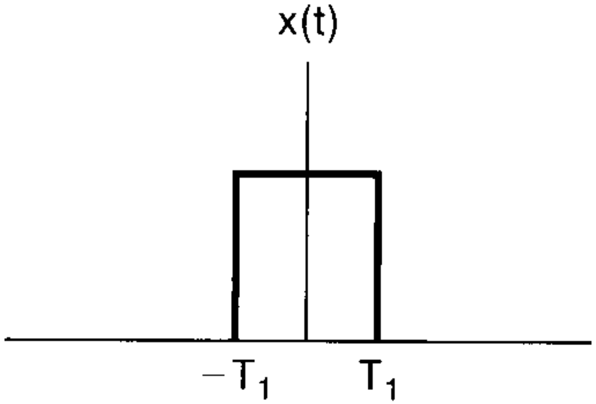
Example 1.2

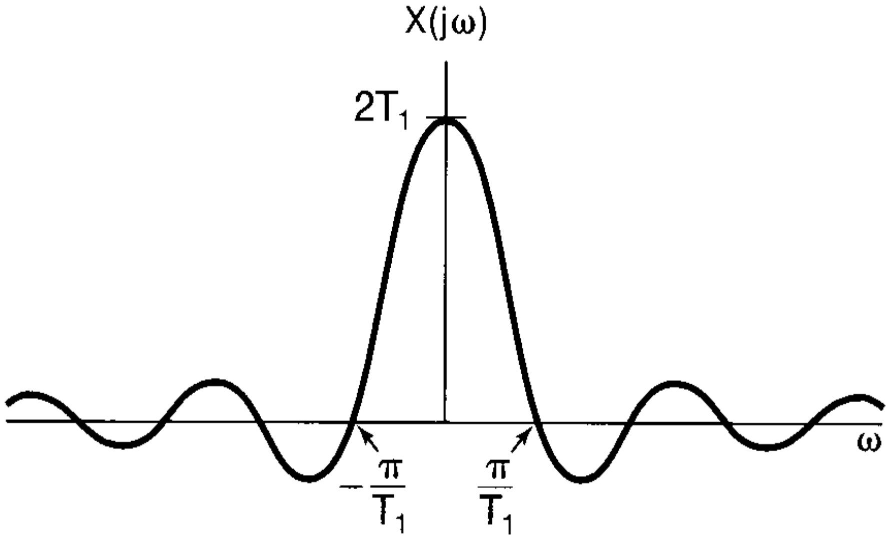
Find the Fourier transform of the following rectangular pulse:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

Using (I.4),

$$X(\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin(\omega T_1)}{\omega}$$





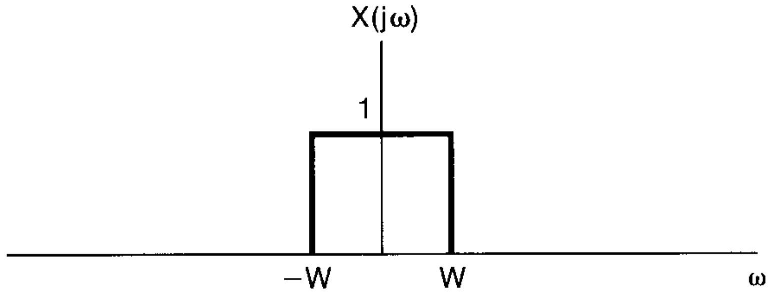
Example 1.3

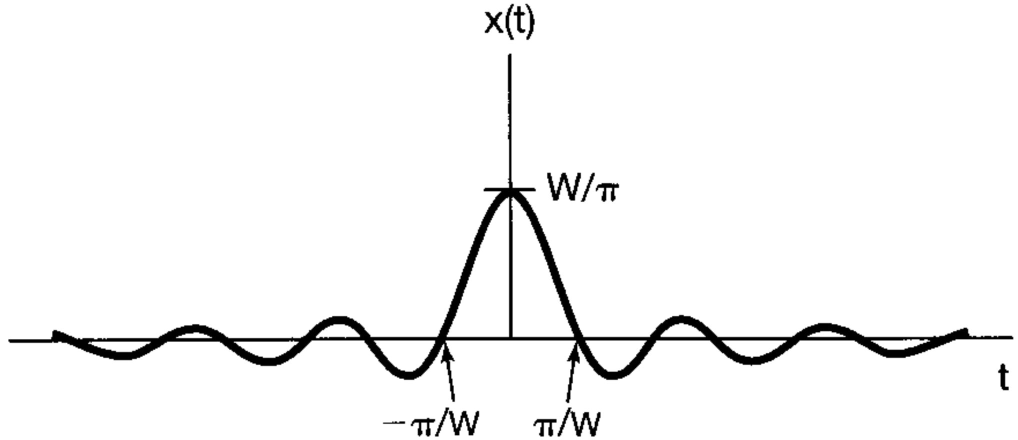
Find the inverse Fourier transform of

$$X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

Using (I.5),

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{\sin(Wt)}{\pi t}$$

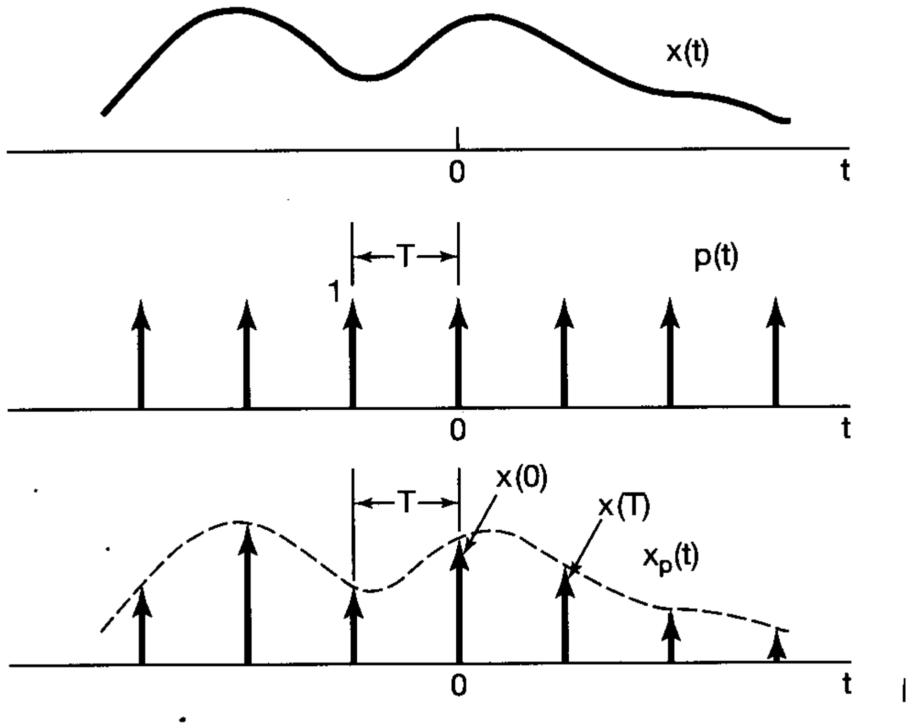




Discrete-Time Fourier Transform (DTFT)

DTFT is a frequency analysis tool for aperiodic and discrete-time signals. If we sample an aperiodic and continuous-time function $x(t)$ with a sampling interval T, the sampled output $x_s(t)$ is expressed as

$$x_s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad (I.6)$$



The DTFT can be obtained by substituting $x_s(t)$ into the Fourier transform equation of (1.4):

$$X(\omega) = \int_{-\infty}^{\infty} x_s(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) \delta(t - nT) e^{-j\omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT} \tag{I.7}$$

where sifting property of unit-impulse function is employed to obtain (1.7):

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

Some points to note:

- DTFT spectrum (both magnitude and phase) is continuous in frequency and periodic with period $2\pi / T$
- When the sampling interval is normalized to 1, we have

Forward Transform: $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \text{(I.8)}$$

and

Inverse Transform: $$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \quad \text{(I.9)}$$

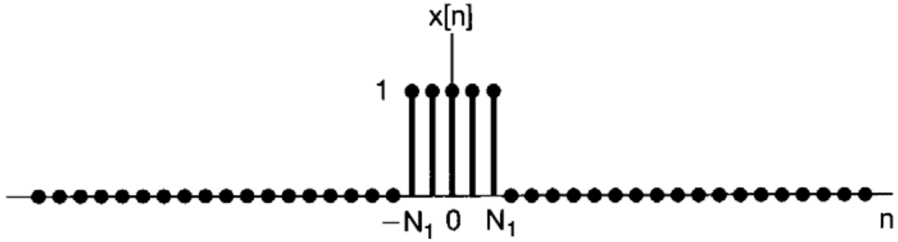
Discrete-Time Fourier Series (DTFS)

DTFS is used for analyzing discrete-time periodic signals. It can be derived from the Fourier series.

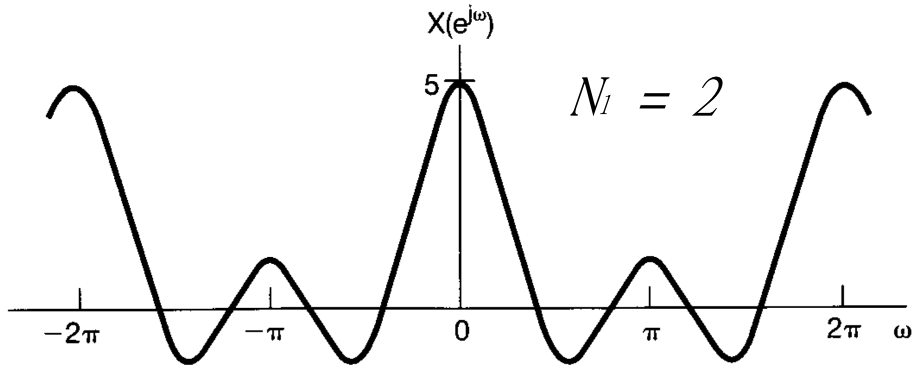
Example 1.4

Find the DTFT of the following discrete-time signal:

$$x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases}$$



Using (I.8),



$$X(\omega)=\sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$=e^{j\omega N_1}(1+e^{-j\omega}+e^{-j2\omega}+\cdots+e^{-j2N_1\omega})=\frac{\sin((N_1+1/2)\omega)}{\sin(\omega/2)}$$

z-Transform

It is a useful transform of processing discrete-time signal. In fact, it is a generalization of DTFT for discrete-time signals

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad \text{(I.10)}$$

where z is a complex variable. Substituting $z = e^{j\omega}$ yields DTFT.

Moreover, substituting $z = re^{j\omega}$ gives

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-jn\omega} = F\{x[n] r^{-n}\} \qquad \text{(I.11)}$$

Advantages of using z-transform over DTFT:

- can encompass a broader class of signal since Fourier transform does not converge for all sequences:

  A sufficient condition for convergence of the DTFT is

$$|X(\omega)| \leq \sum_{n=-\infty}^{\infty} |x(n)| \cdot |e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x(n)| < \infty \tag{I.12}$$

  Therefore, if $x(n)$ is absolutely summable, then $X(\omega)$ exists.

  On the other hand, by representing $z = re^{j\omega}$, the z-transform exists if

$$|X(z)| = |X(re^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| \cdot |e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \tag{I.13}$$

$\Rightarrow$ we can choose a region of convergence (ROC) for z such that the z-transform converges

- notation convenience : $z \leftrightarrow e^{j\omega}$

- can solve problems in discrete-time signals and systems, e.g. difference equations

Example 1.5

Determine the z-transform of $x[n] = a^n u[n]$.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$X(z)$ converges if $\sum_{n=0}^{\infty} \left| az^{-1} \right|^n < \infty$. This requires $\left| az^{-1} \right| < 1$ or $|z| > |a|$, and

$$X(z) = \frac{1}{1 - az^{-1}}$$

Notice that for another signal $x[n] = -a^n u[-n-1]$,

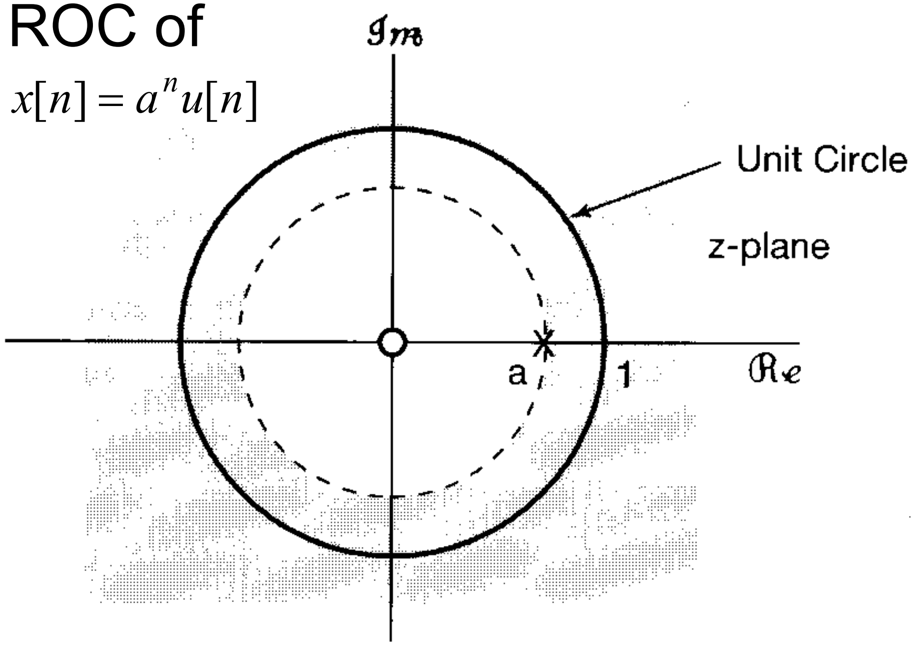
$$X(z) = \sum_{n=-\infty}^{-1} (-a^n) z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} \left(a^{-1} z \right)^m$$

In this case, $X(z)$ converges if $\left|a^{-1}z\right| < 1$ or $|z| < |a|$, and

$$X(z) = \frac{1}{1 - az^{-1}}$$

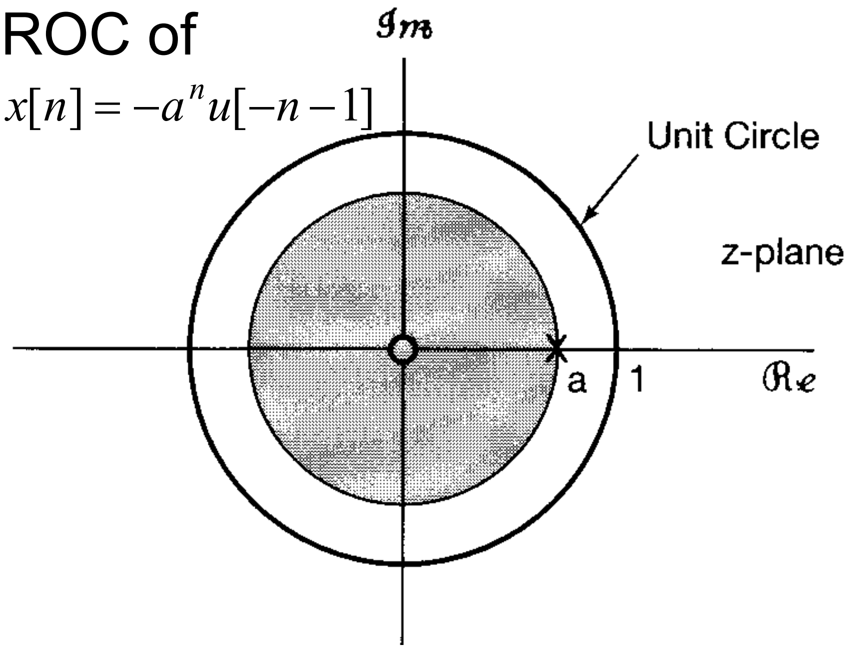
ROC of

$x[n] = a^n u[n]$



ROC of

$x[n] = -a^n u[-n-1]$



Some points to note:

- Different signals can give same z-transform, although the ROCs differ
- When $x[n] = a^n u[n]$ with $|a| > 1$, its DTFT does not exist

Property	Signal	z-Transform	ROC
	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Initial Value Theorem

If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Transfer Function and Difference Equation

A linear time-invariant (LTI) system with input sequence $x(n)$ and output sequence $y(n)$ are related via an *Nth-order* *linear constant coefficient difference equation* of the form:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k), \qquad a_0 \neq 0, b_0 \neq 0 \tag{I.14}$$

Applying z-transform to both sides with the use of the linearity property and time-shifting property, we have

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \tag{I.15}$$

The system (or filter) transfer function is expressed as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \tag{I.16}$$

where each $(1-c_k z^{-1})$ contributes a zero at $z = c_k$ and a pole at $z = 0$ while each $(1-d_k z^{-1})$ contributes a pole at $z = d_k$ and a zero at $z = 0$.

The frequency response of the system or filter can be computed as

$$H(\omega) = H(z)\big|_{z=\exp(j\omega)} \tag{I.17}$$

From (1.14), the output $y(n)$ is expressed as

$$y(n) = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)\right) \tag{I.18}$$

When at least one of the $\{a_1, a_2, \cdots, a_N\}$ is non-zero, then $y(n)$ depends on its past samples as well as the input signal $x(n)$. The system or filter in this case is known as an infinite impulse response (IIR) system. Applying inverse DTFT or z transform to the transfer function, it can be shown that the system impulse response is of infinite duration.

When all $\{a_1, a_2, \cdots, a_N\}$ are equal to zero, $y(n)$ depends on $x(n)$ only. It is known as a finite impulse response (FIR) system because the impulse response is of finite duration.

Example 1.6

Consider a LTI system with the input $x[n]$ and output $y[n]$ satisfy the following linear constant-coefficient difference equation,

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

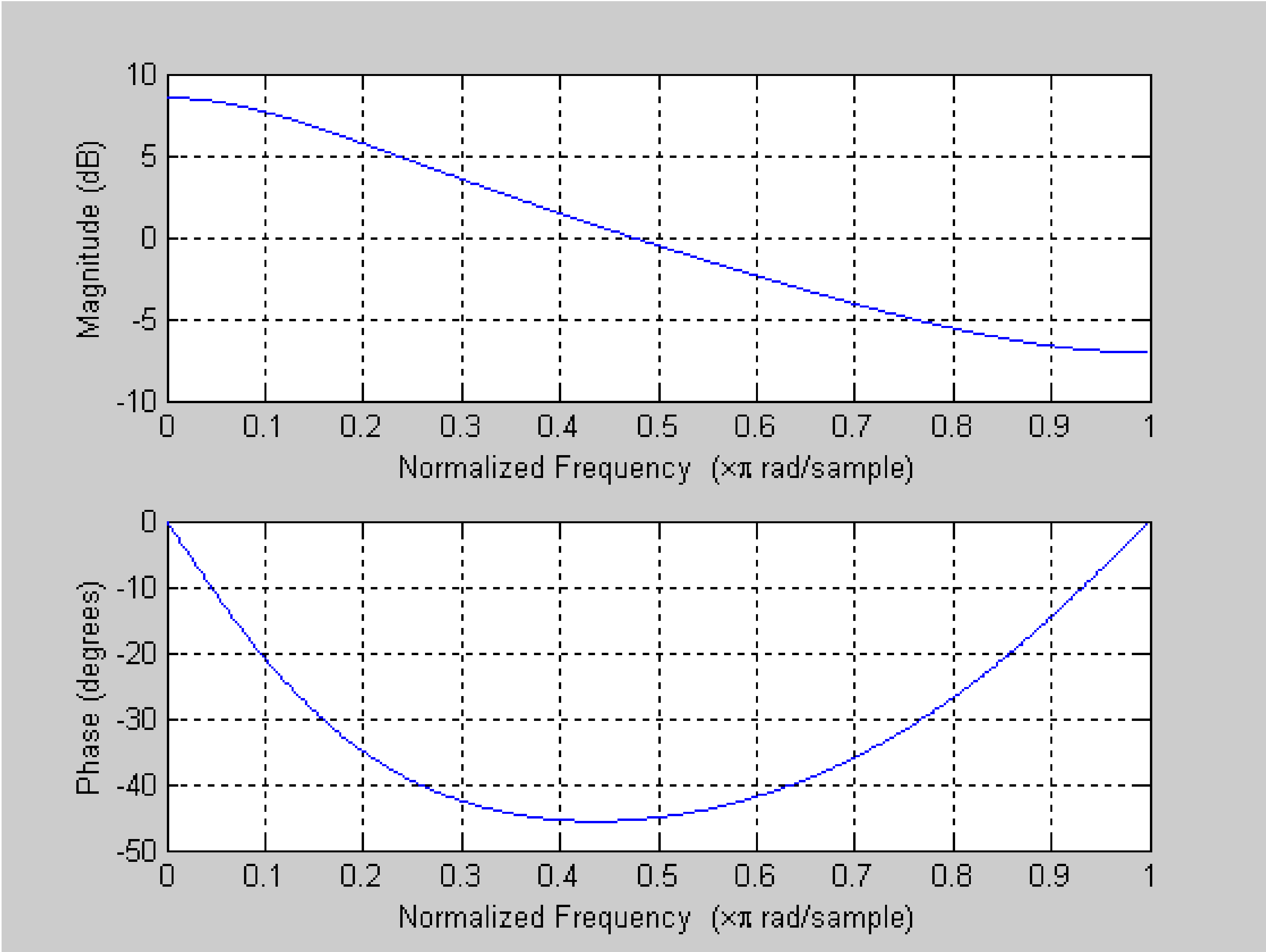
Find the system function and frequency response.

Taking z-transform on both sides,

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z)$$

Thus,

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} \quad \text{and} \quad H(\omega)=H(z)\big|_{z=\exp(j\omega)}=\frac{1+\frac{1}{3}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$



10
5
0
-5
-10
Magnitude (dB)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)
0
-10
-20
-30
-40
-50
Phase (degrees)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)

Example 1.7

Suppose you need to high-pass the signal $x[n]$ by the high-pass filter with the following transfer function

$$H(z) = \frac{1}{1 + 0.99z^{-1}}$$

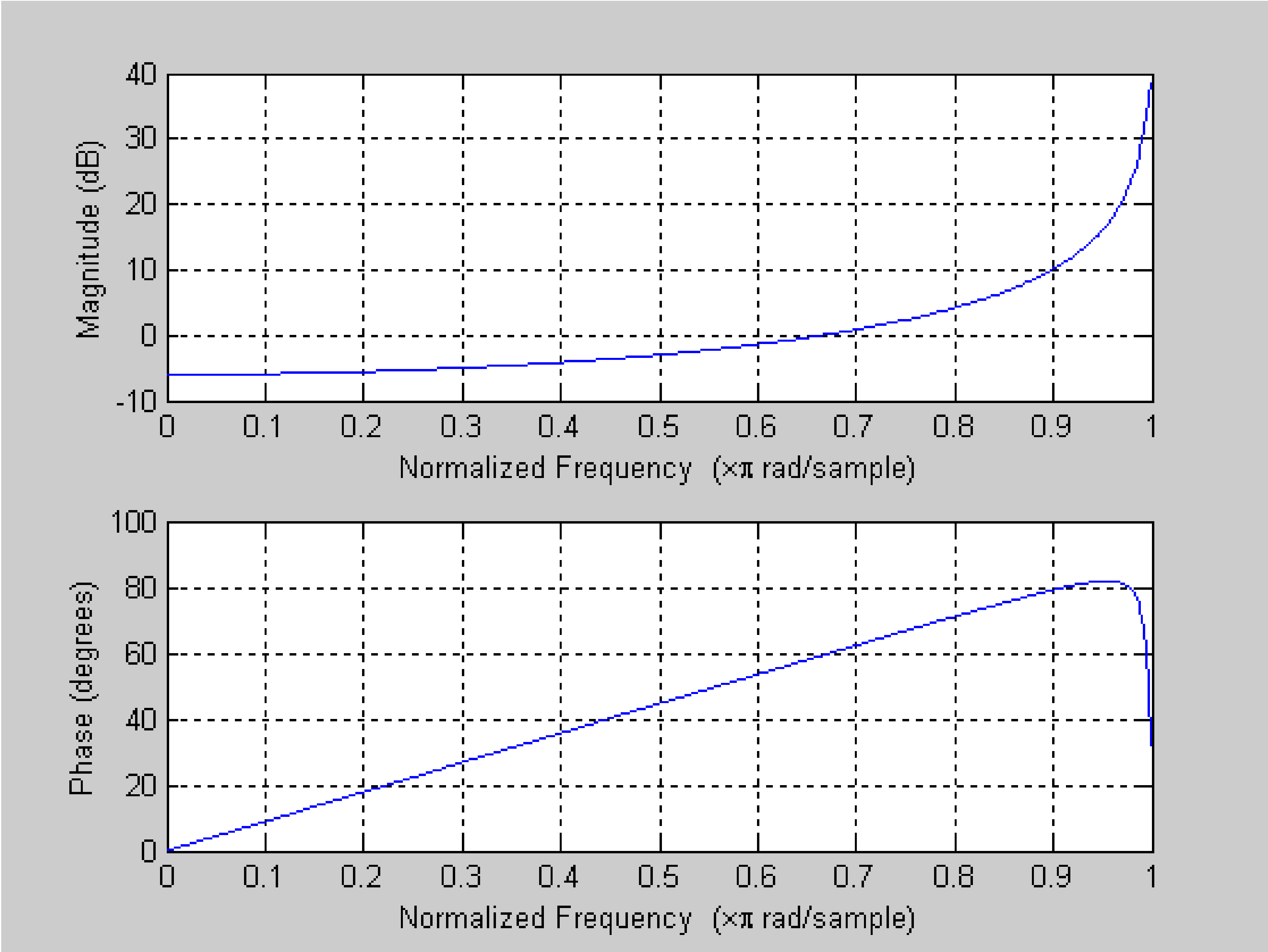
How to obtain the filtered signal $y[n]$?

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + 0.99z^{-1}} \Rightarrow Y(z) + 0.99z^{-1}Y(z) = X(z)$$

Taking the inverse z-transform

$$y[n] + 0.99y[n-1] = x[n]$$

$\Rightarrow y[n] = -0.99y[n-1] + x[n]$ （$y[-1] = 0$ for initialization）


40
30
20
10
0
-10
Magnitude (dB)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)
100
80
60
40
20
0
Phase (degrees)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)

Causality, Stability and ROC:

Causality condition: $h[n]=0$ for all $n<0$,

$\downarrow$

$h[n]$ is right-sided

$\downarrow$

The ROC for $H(z)$ is

the exterior of an origin-centered circle (including $z=\infty$)

$\downarrow$

If $H(z)$ is **rational**, the ROC for $H(z)$ is

the exterior outside the outermost pole.

Stability condition: $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$

$\downarrow$

$H(e^{j\omega})$, i.e., the Fourier transform of $h[n]$, converges

$\downarrow$

The ROC for $H(z)$ **includes the unit circle** $|z|=1$

Example 1.8

Verify if the system impulse response $h[n] = 0.5^n u[n]$ is causal and stable.

It is obvious that $h[n]$ is causal because $h[n] = 0$ for all $n < 0$. On the other hand,

$$H(z) = \sum_{n=-\infty}^{\infty} 0.5^n u[n] z^{-n} = \sum_{n=0}^{\infty} (0.5z^{-1})^n = \frac{1}{1-0.5z^{-1}}$$

$H(z)$ converges if $\sum_{n=0}^{\infty} \left|0.5z^{-1}\right|^n < \infty$. This requires $\left|0.5z^{-1}\right| < 1$ or $|z| > 0.5$,

i.e., ROC for $H(z)$ is the exterior outside the pole of 0.5

(Notice that for another impulse response $h[n] = -0.5^n u[-n-1]$, and it corresponds to an unstable system because the ROC for $H(z)$ is $|z| < 0.5$)

The *z*-transform for $h[n]$ is

$$H(z) = \frac{1}{1-0.5z^{-1}}, \qquad |z| > 0.5$$

Hence it is stable because the ROC for $H(z)$ includes the unit circle $|z| = 1$

On the other hand, its stability can also be shown using:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 0.5^n = 1 + 0.5^2 + 0.5^3 + \cdots$$

$$= \frac{1}{1-0.5} = 2$$

$$< \infty$$

Brief Review of Random Processes

Basically there are two types of signals:

- Deterministic Signals
  - exactly specified according to some mathematical formulae
  - characterized by finite parameters
  - e.g., exponential signal, sinusoidal signal, ramp signal, etc.
  - a simple mathematical model of a musical signal is

$$x(t) = a(t) \sum_{m=1}^{\infty} c_m \cos(2\pi m f_0 t + \phi_m)$$

where:

f_0 is the fundamental frequency or pitch

c_m is the amplitude and ϕ_m is the phase of the *m*th harmonic

$a(t)$ is the envelope

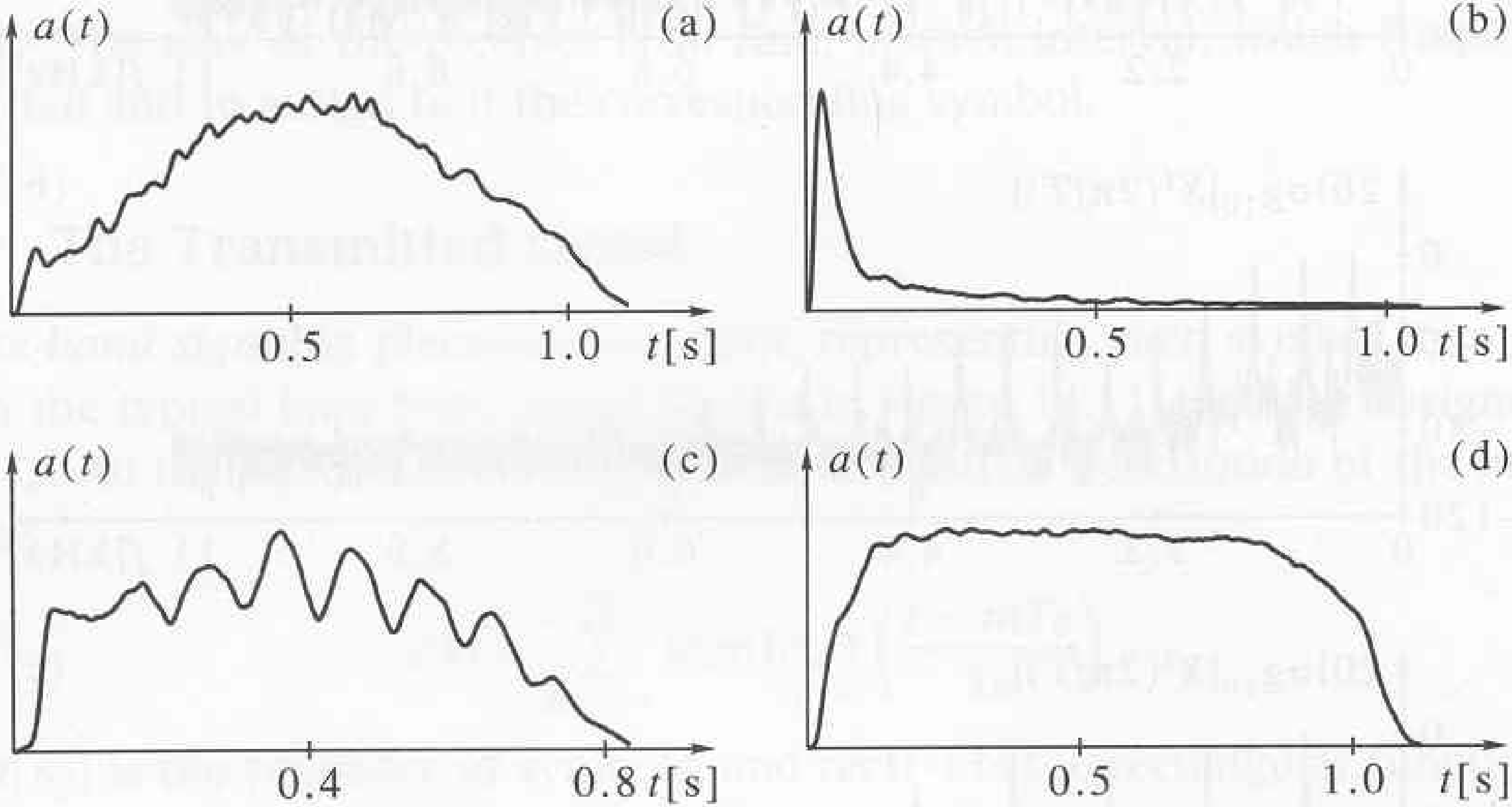


Figure 14.8 Envelope waveforms of musical instruments: (a) cello; (b) classical guitar; (c) flute; (d) French horn.

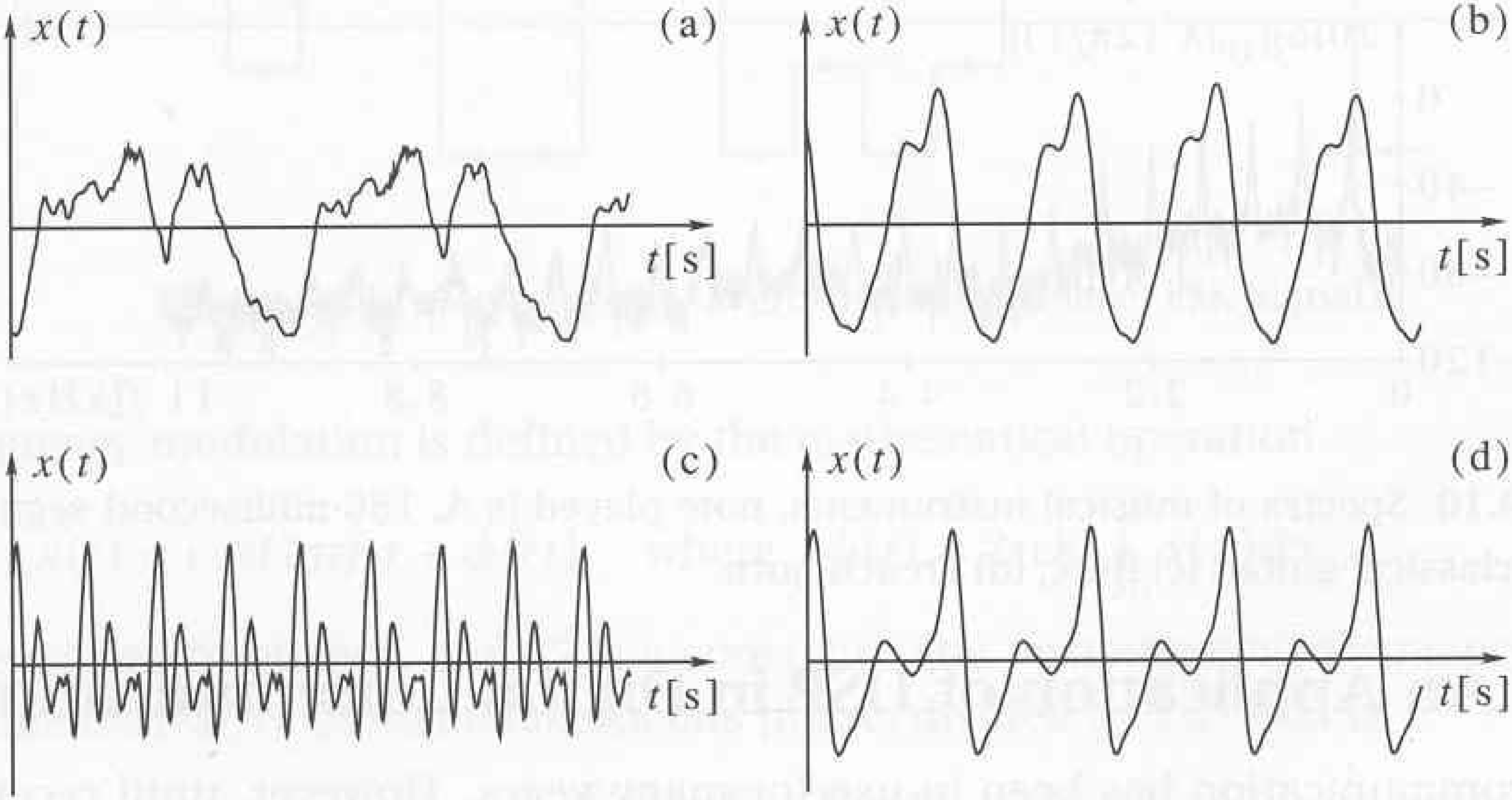


Figure 14.9 Waveforms of musical instruments, note played is A, 10-millisecond segments: (a) cello; (b) classical guitar; (c) flute; (d) French horn.

- Random Signals
  - cannot be directly generated by any formulae and their values cannot be predicted
  - characterized by probability density function (PDF), mean, variance, power spectrum, etc.
  - e.g., thermal noise, stock values, autoregressive (AR) process, moving average (MA) process, etc.
  - a simple voiced discrete-time speech model is

$$x[n] = \sum_{i=1}^{P} a_i x[n-i] + w[n]$$

where

$\{a_i\}$ are called the AR parameters

$w[n]$ is a noise-like process

P is the order of the AR process

Definitions and Notations

1. **Mean Value**

The mean value of a real random variable $x(n)$ at time n is defined as

$$\mu(n) = E\{x(n)\} = \int_{-\infty}^{\infty} x(n) f(x(n)) d\big(x(n)\big) \qquad \text{(I.19)}$$

where $f(x(n))$ is the PDF of $x(n)$ such that

$$\int_{-\infty}^{\infty} f(x(n)) d\big(x(n)\big) = 1 \qquad \text{and} \qquad f(x(n)) \geq 0$$

Note that, in general,

$$\mu(m) \neq \mu(n), \qquad m \neq n \qquad \text{(I.20)}$$

and

$$\mu(m) \neq \frac{1}{N} \sum_{n=0}^{N-1} x(n) \qquad \text{(I.21)}$$

The mean value is also called *expected value* and *ensemble mean*.

2. **Moment**

Moment is the generalization of the mean value:

$$E\{(x(n))^m\} = \int_{-\infty}^{\infty} (x(n))^m f(x(n))d(x(n)) \tag{I.22}$$

When $m = 1$, it is the mean while when $m = 2$, it is called the mean square value of $x(n)$.

3. **Variance**

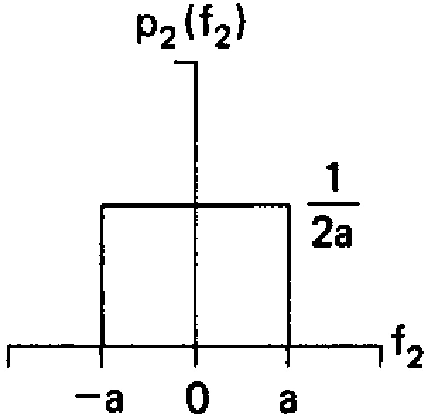
The variance of a real random variable $x(n)$ at time n is defined as

$$\sigma^2(n) = E\{(x(n) - \mu(n))^2\} = \int_{-\infty}^{\infty} (x(n) - \mu(n))^2 f(x(n))d(x(n)) \tag{I.23}$$

It is also called *second central moment*.

Example 1.9

Determine the mean, second-order moment, variance of a quantization error, x, with the following PDF:



$$\mu = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_{-a}^{a} x \cdot \frac{1}{2a} dx = \frac{1}{2a} \cdot \frac{1}{2} x^2 \Big|_{-a}^{a} = 0$$

$$E\{x^2\} = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_{-a}^{a} x^2 \cdot \frac{1}{2a} dx = \frac{1}{2a} \cdot \frac{1}{3} x^3 \Big|_{-a}^{a} = \frac{a^2}{3}$$

$$\sigma^2 = E\{(x-\mu)^2\} = E\{x^2\} = \frac{a^2}{3}$$

4. **Autocorrelation**

The autocorrelation of a real random signal $x(n)$ is defined as

$$R_{xx}(m,n) = E\{x(m)x(n)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(m)x(n) f\big(x(m),x(n)\big) d\big(x(m)\big) d\big(x(n)\big) \qquad \text{(I.24)}$$

where $f\big(x(m),x(n)\big)$ is the joint PDF of $x(m)$ and $x(n)$. It measures the degree of association or dependence between x at time index n and at index m.

In particular,

$$R_{xx}(n,n) = E\{x^2(n)\} \qquad \text{(I.25)}$$

is the mean square value or average power of $x(n)$. Moreover, when $x(n)$ has zero-mean, then

$$\sigma^2(n) = R_{xx}(n,n) = E\{x^2(n)\} \qquad \text{(I.26)}$$

That is, the power of $x(n)$ is equal to the variance of $x(n)$.

5. **Covariance**

The covariance of a real random signal $x(n)$ is defined as

$$C_{xx}(m,n) = E\{(x(m) - \mu(m))(x(n) - \mu(n))\} \tag{I.27}$$

Expanding (I.27) gives

$$C_{xx}(m,n) = E\{x(m)x(n)\} - \mu(m)\mu(n)$$

In particular,

$$C_{xx}(n,n) = E\{(x(n) - \mu(n))^2\} = \sigma^2(n)$$

is the variance, and for zero-mean $x(n)$, we have

$$C_{xx}(m,n) = R_{xx}(m,n)$$

6. **Crosscorrelation**

The crosscorrelation of two real random signals $x(n)$ and $y(n)$ is defined as

$$R_{xy}(m,n) = E\{x(m)y(n)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(m)y(n)f\big(x(m),y(n)\big)d\big(x(m)\big)d\big(y(n)\big) \qquad \text{(I.28)}$$

where $f\big(x(m),y(n)\big)$ is the joint PDF of $x(m)$ and $y(n)$. It measures the correlation of $x(n)$ and $y(n)$. The signals $x(m)$ and $y(n)$ are *uncorrelated* if $R_{xy}(m,n) = E\{x(m)\}\cdot E\{x(n)\}$.

7. **Independence**

Two real random variables $x(n)$ and $y(n)$ are said to be independent if

$$f\big(x(n),y(n)\big) = f\big(x(n)\big)\cdot f\big(y(n)\big) \Rightarrow E\{x(n)y(n)\} = E\{x(n)\}\cdot E\{y(n)\} \qquad \text{(I.29)}$$

Q.: Does “uncorrelated” implies “independent” or vice versa?

8. **Stationarity**

A discrete random signal is said to be *strictly stationary* if its k-th order PDF $f(x(n_1), x(n_2), \cdots, x(n_k))$ is shift-invariant for any set of $n_1, n_{2,} \cdots, n_k$ and for any k. That is

$$f(x(n_1), x(n_2), \cdots, x(n_k)) = f(x(n_1 + n_0), x(n_2 + n_0), \cdots, x(n_k + n_0)) \qquad \text{(I.30)}$$

where n_0 is an arbitrary shift and for all k. In particular, a real random signal is said to be *wide-sense stationary* (WSS) if the first and second order moments, viz., its mean and autocorrelation, are shift-invariant.

This means

$$\mu = E\{x(n)\} = E\{x(m)\}, \qquad m \neq n \qquad \text{(I.31)}$$

and

$$R_{xx}(i) = R_{xx}(m - n) = R_{xx}(m, n) = E\{x(m)x(n)\} \qquad \text{(I.32)}$$

where $i = m - n$ is called the correlation lag.

Three important properties of $R_{xx}(i)$:

(i) $R_{xx}(i)$ is an even sequence, i.e.,

$$R_{xx}(i) = R_{xx}(-i) \qquad \text{(I.33)}$$

and hence is symmetric about the origin.

Q.: Why is it an even sequence?

(ii) The mean square value or power is greater than or equal the magnitude of the correlation for any other lag, i.e.,

$$E\{x^2(n)\} = R_{xx}(0) \geq | R_{xx}(i) |, \qquad i \neq 0 \qquad \text{(I.34)}$$

which can be proved by the Cauchy-Schwarz inequality:

$$| E\{a \cdot b\} | \leq \sqrt{E\{a^2\}} \cdot \sqrt{E\{b^2\}}$$

(iii) When $x(n)$ has zero-mean, then

$$\sigma^2 = E\{x^2(n)\} = R_{xx}(0) \qquad \text{(I.35)}$$

9. **Ergodicity**

A stationary process is said to be ergodic if its time average using infinite samples equals its ensemble average. That is, the statistical properties of the process can be determined by time averaging over a single sample function of the process. For example,

- Ergodic in the mean if

$$\mu = E\{x(n)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-N/2}^{N/2-1} x(n)$$

- Ergodic in the autocorrelation function if

$$R_{xx}(i) = E\{x(n)x(n-i)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-N/2}^{N/2-1} x(n)x(n-i)$$

Unless stated otherwise, we assume that random signals are ergodic (and thus stationary) in this course.

Example 1.10

Consider an ergodic stationary process $\{x[n]\}, = \cdots,-1,0,1,\cdots$ which is uniformly distributed between 0 and 1.

The ensemble average or mean of $x[n]$ at time m is

$$\mu[m] = \int_{-\infty}^{\infty} x[m] \cdot f(x[m]) dx[m] = \int_{0}^{1} x[m] dx[m] = \frac{1}{2} x^2[m] \Big|_0^1 = \frac{1}{2}$$

It is clear that the mean of $x[n]$ is also $\mu = 0.5$ for all n

Because of ergodicity, the time average is

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=-N/2}^{N/2-1} x[n] = \mu = \frac{1}{2}$$

10. **Power Spectrum**

For random signals, power spectrum or power spectral density (PSD) is used to describe the frequency spectrum.

Q.: Can we use DTFT to analyze the spectrum of random signal? Why?

The PSD is defined as:

$$\Phi_{xx}(\omega) = \sum_{i=-\infty}^{\infty} R_{xx}(i)e^{-j\omega i} = Z\left[R_{xx}(i)\right]\big|_{z=\exp(j\omega)} \quad \text{(I.36)}$$

Given $\Phi_{xx}(\omega)$, we can get $R_{xx}(i)$ using

$$R_{xx}(i) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(\omega)e^{j\omega i} d\omega \quad \text{(I.37)}$$

Q.: Why?

Under a mild assumption:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=-N}^{N} |k| \cdot |R_{xx}(k)| = 0$$

it can be proved (1.36) is equivalent to

$$\Phi_{xx}(\omega) = \lim_{N\to\infty} E\left\{ \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2 \right\} \tag{1.38}$$

Since $\sum_{n=0}^{N-1} x(n) e^{-j\omega n}$ corresponds to the DTFT of $x(n)$, we can consider the PSD as the time average of $|X(\omega)|^2$ based on infinite samples.

(1.38) also implies that the PSD is a measure of the mean value of the DTFT of $x(n)$.

Common Random Signal Models

1. **White Process**

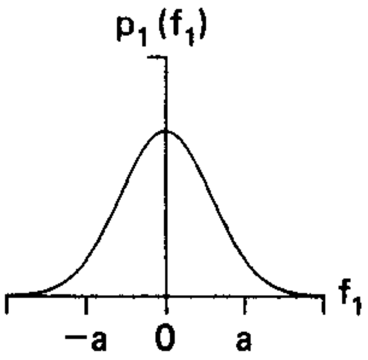
A discrete-time zero-mean signal $w(n)$ is said to be white if

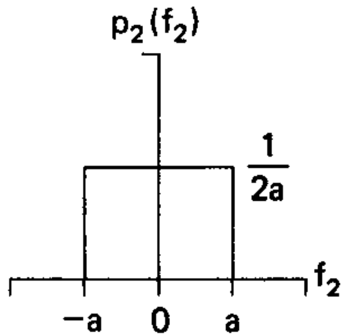
$$R_{ww}(m-n) = E\{w(n)w(m)\} = \begin{cases} \sigma_w^2, & m=n \\ 0, & \text{otherwise} \end{cases} \tag{I.39}$$

Moreover, the PSD of $w(n)$ is flat for all frequencies:

$$\Phi_{ww}(\omega) = \sum_{i=-\infty}^{\infty} R_{ww}(i)e^{-j\omega i} = R_{ww}(0)\cdot e^{-j\omega\cdot 0} = \sigma_w^2$$

Notice that white process does not specify its PDF. They can be of Gaussian-distributed, uniform-distributed, etc.





2. **Autoregressive Process**

An autoregressive (AR) process of order M is defined as

$$x(n) = a_1 x(n-1) + a_2 x(n-2) + \cdots + a_M x(n-M) + w(n) \quad \text{(I.40)}$$

where $w(n)$ is a white process.

Taking the z-transform of (1.40) yields

$$H(z) = \frac{X(z)}{W(z)} = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_M z^{-M}}$$

Let $h(n) = Z^{-1}\{H(z)\}$, we can write

$$x(n) = h(n) \otimes w(n) = \sum_{k=-\infty}^{\infty} h(n-k) w(k) = \sum_{k=-\infty}^{\infty} w(n-k) h(k)$$

Q.: What Is the mean value of $x(n)$?

Input-output relationship of random signals is:

$$R_{xx}(m) = E\{x(n)x(n+m)\}$$

$$= E\left\{ \sum_{k_1=-\infty}^{\infty} h(k_1)w(n-k_1) \cdot \sum_{k_2=-\infty}^{\infty} h(k_2)w(n+m-k_2) \right\}$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1)h(k_2)E\{w(n-k_1) \cdot w(n+m-k_2)\}$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1)h(k_2)R_{ww}(m+k_1-k_2)$$

$$= \sum_{k=-\infty}^{\infty} R_{ww}(m-k) \cdot \sum_{k_1=-\infty}^{\infty} h(k_1)h(k+k_1), \qquad k = k_2 - k_1$$

$$\Rightarrow R_{xx}(m) = R_{ww}(m) \otimes g(m), \qquad g(k) = \sum_{k_1=-\infty}^{\infty} h(k_1)h(k+k_1) = h(k) \otimes h(-k)$$

$$\Rightarrow \Phi_{xx}(\omega) = \Phi_{ww}(\omega) \cdot G(\omega), \qquad G(\omega) = |H(\omega)|^2$$

$$\Rightarrow \qquad \Phi_{xx}(\omega) = \Phi_{ww}(\omega) \cdot |H(\omega)|^2 \qquad \text{(I.41)}$$

Note that (1.41) applies for all stationary input processes and impulse responses.

In particular, for the AR process, we have

$$\Phi_{xx}(\omega) = \frac{\sigma_w^2}{\left|1 - a_1 e^{-j\omega} - a_2 e^{-j2\omega} - \cdots - a_M e^{-jM\omega}\right|^2} \qquad (1.42)$$

3. **Moving Average Process**

A moving average (MA) process of order N is defined as

$$x(n) = b_0 w(n) + b_1 w(n-1) + \cdots + b_N w(n-N) \qquad (I.43)$$

Applying (1.41) gives

$$\Phi_{xx}(\omega) = \left|b_0 + b_1 e^{-j\omega} + \cdots + b_N e^{-j\omega N}\right|^2 \cdot \sigma_w^2 \qquad (I.44)$$

4. **Autoregressive Moving Average Process**

An autoregressive moving average (ARMA) process is defined as

$$\begin{aligned} x(n) = a_1 x(n-1) + a_2 x(n-2) + \cdots + a_M x(n-M) \\ + b_0 w(n) + b_1 w(n-1) + \cdots + b_N w(n-N) \end{aligned} \qquad \text{(I.45)}$$

Applying (1.41) gives

$$\Phi_{xx}(\omega) = \frac{\left| b_0 + b_1 e^{-j\omega} + \cdots + b_N e^{-jN\omega} \right|^2}{\left| 1 - a_1 e^{-j\omega} - a_2 e^{-j2\omega} - \cdots - a_M e^{-jM\omega} \right|^2} \cdot \sigma_w^2 \qquad \text{(1.46)}$$

Questions for Discussion

1. Consider a signal $x(n)$ and a stable system with transfer function $H(z) = B(z)/A(z)$. Let the system output with input $x(n)$ be $y(n)$.

   Can we always recover $x(n)$ from $y(n)$? Why? You may consider the simple cases of $B(z) = 1 + 2z^{-1}$ and $A(z) = 1$ as well as $B(z) = 1 + 0.5z^{-1}$ and $A(z) = 1$.

2. Given a random variable x with mean μ_x and variance σ_x^2. Determine the mean, variance, mean square value of

$$y = ax + b$$

   where a and b are finite constants.

3. Is AR process really stationary? You can answer this question by examining the autocorrelation function of a first-order AR process, say,

$$x(n) = ax(n-1) + w(n)$$