Chaotic Systems with Any Number of Equilibria and Their Hidden Attractors

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Attractor with Multi-Equilibia

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Acknowledgements

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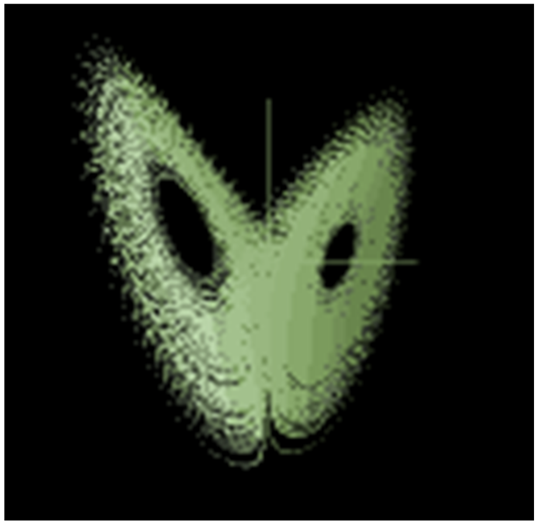
Lorenz System

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz \end{cases}$$

$$a = 10, b = 8/3, c = 28$$



E. N. Lorenz, "Deterministic non-periodic flow," J. Atmos. Sci., 20: 130-141, 1963.

Main Characteristics

"Simple"

3D quadratic autonomous

Multiple equilibria

Smooth

After all: Chaotic

State of the Art

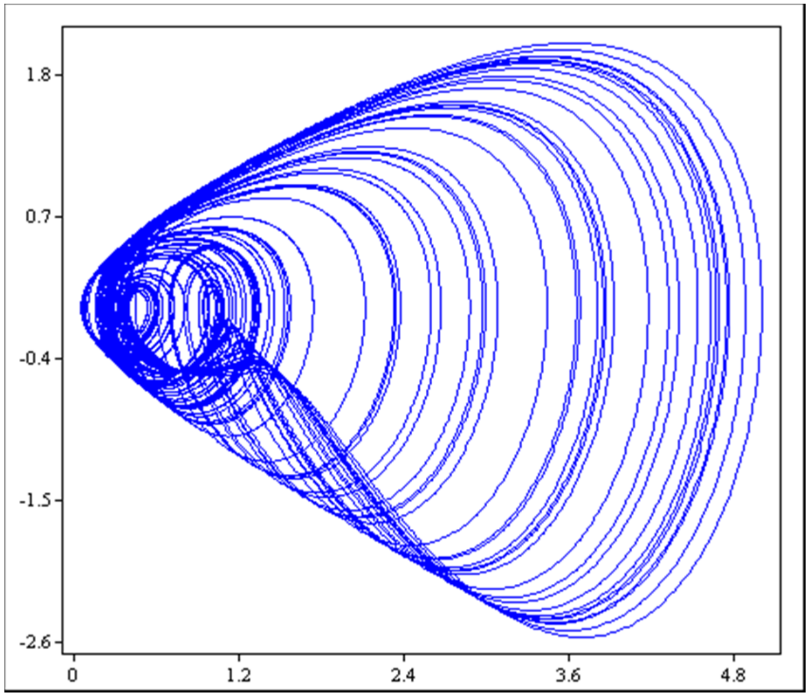
Chaotic systems with

- ✓ No equilibrium
- ✓ One equilibrium
- ✓ Two equilibria
- ✓ Three equilibria
- An arbitrary number of equilibria ?

Chaotic system with no equilibrium

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -y + 3y^2 - x^2 - xz - a. \end{cases}$$

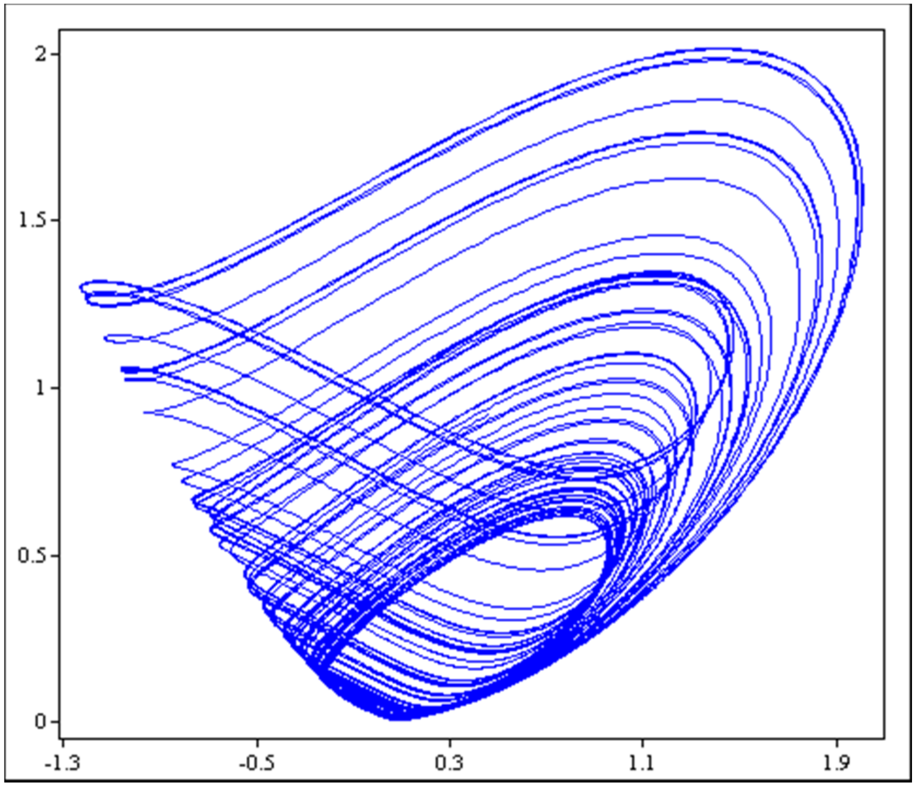
$$a > 0 \text{ no equilibrium}$$



X Wang and G Chen: Constructing a chaotic system with any number of equilibria, Nonl Dynam 2013

Chaotic system with one stable equilibrium

$$\begin{cases} \dot{x} = yz + a \\ \dot{y} = x^2 - y \\ \dot{z} = 1 - 4x \end{cases}$$



X Wang and G Chen: A chaotic system with only one stable equilibrium, Comm in Nonl Sci and Numer Simul, 2012

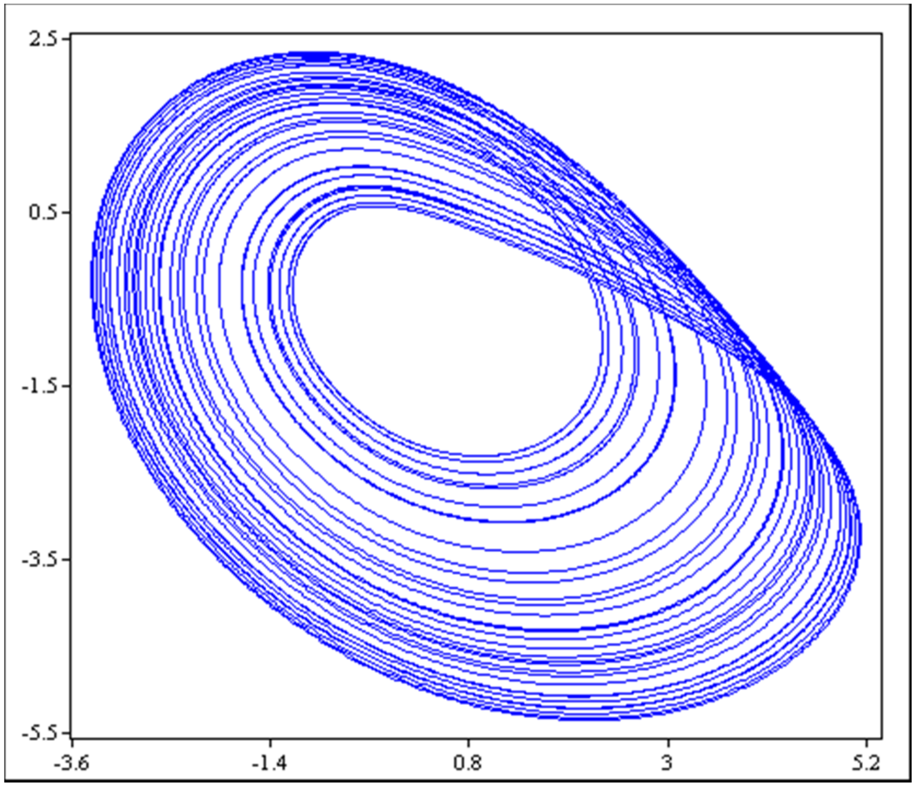
Chaotic systems with two equilibria

Rössler system:

$$\dot{x} = -y - z$$

$$\dot{y} = x + ay$$

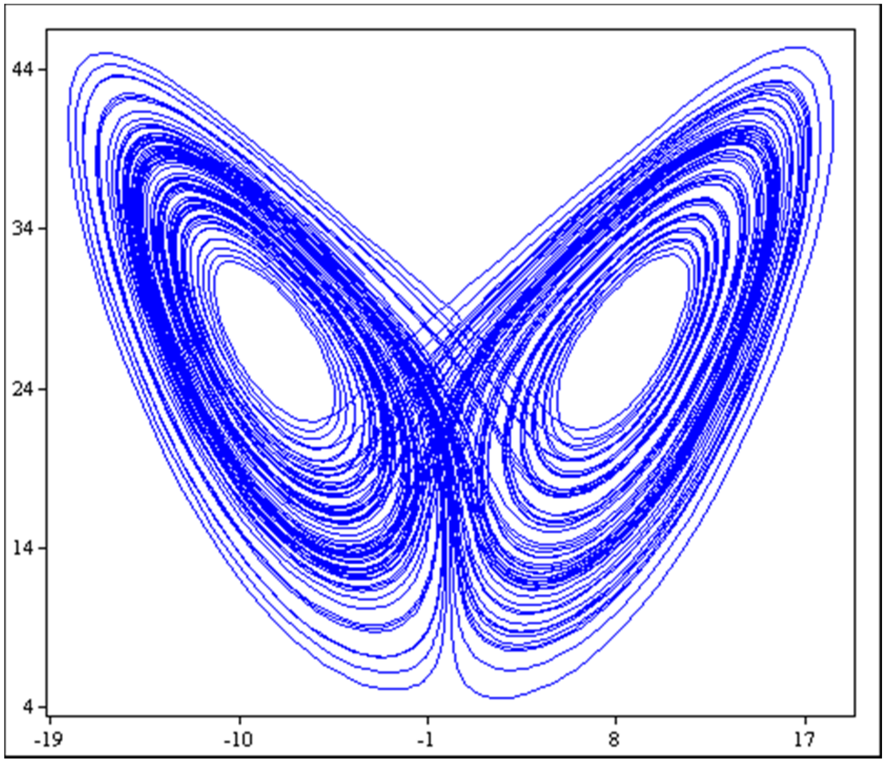
$$\dot{z} = b + z(x - c)$$



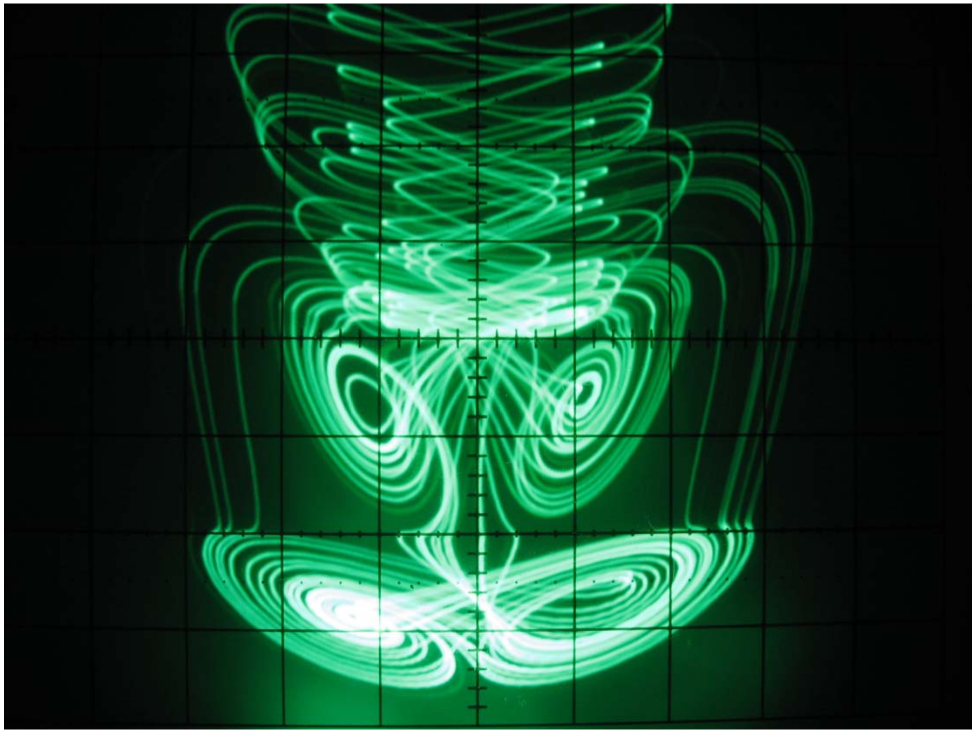
Chaotic systems with three equilibria

Lorenz system:

$$\dot{x} = a(y - x)$$
$$\dot{y} = x(c - z) - y$$
$$\dot{z} = xy - bz$$



How to construct a chaotic system with any number of equilibria ?

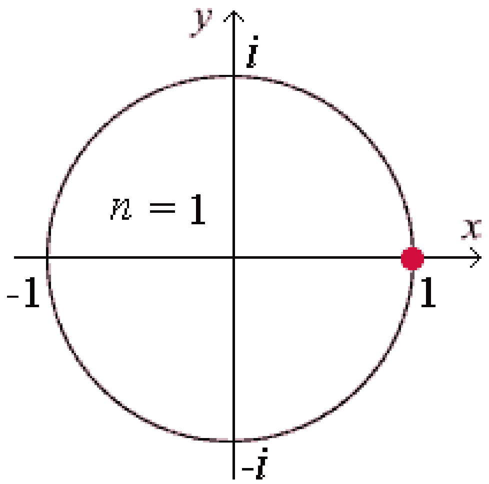


Idea

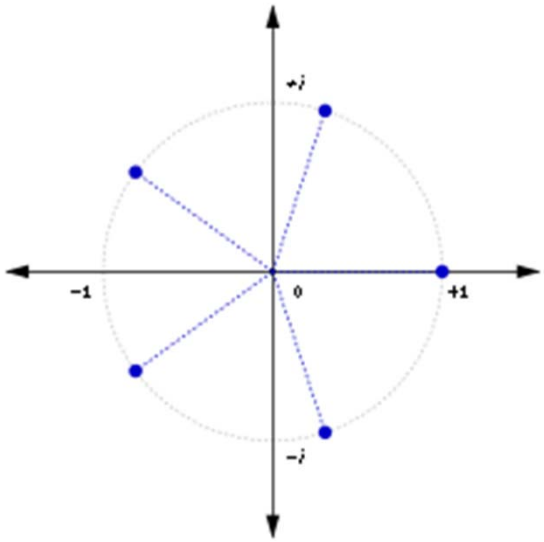
- Start from a chaotic system with one stable equilibrium
- Add symmetry by using a suitable local diffeomorphism

- By-product: stability of the equilibria can be easily adjusted by tuning a single parameter

Symmetry



$$W = Z^n$$



W plane

$$W = (u,v) = u + vi$$

original system:

$$(u,v,w)$$

Z plane

$$Z = (x,y) = x + yi$$

translated system:

$$(x,y,z)$$

Symmetry

$$\mathbb{R}_y(\pi) \quad \begin{cases} u = x^2 - z^2 \\ v = y \\ w = 2xz \end{cases}$$

local diffeomorphism

$$\begin{cases} \dot{u} = vw + a \\ \dot{v} = u^2 - v \\ \dot{w} = 1 - 4u. \end{cases} \longrightarrow \begin{cases} \dot{x} = \frac{1}{2} \frac{z + 2yx^2z + xa - 4x^2z + 4z^3}{x^2 + z^2} \\ \dot{y} = \left(x^2 - z^2\right)^2 - y \\ \dot{z} = -\frac{1}{2} \frac{2yxz^2 + za - 4xz^2 - x + 4x^3}{x^2 + z^2}. \end{cases}$$

One equilibrium

Two equilibria

Stability of the two equilibria

- Two symmetrical equilibria, independent of parameter a

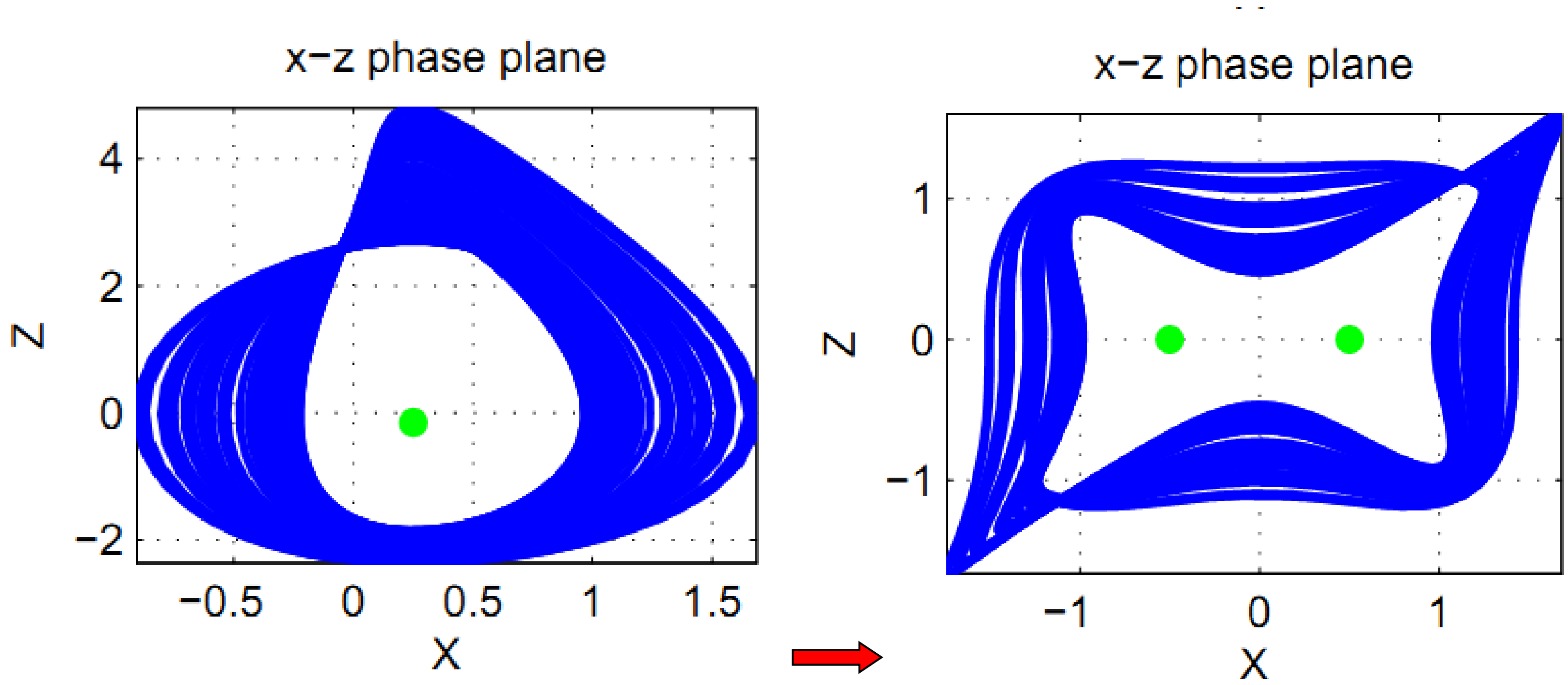
$$P1(\tfrac{1}{2}, \tfrac{1}{16}, 0) \text{ and } P1(-\tfrac{1}{2}, \tfrac{1}{16}, 0)$$

- Eigenvalue of Jacobian:

$$\begin{aligned} \lambda_1 &= -1 < 0, \\ \lambda_2 &= -2a + 0.5i, \\ \lambda_3 &= -2a - 0.5i. \end{aligned}$$

- So, $a > 0$ – stable; $a < 0$ – unstable

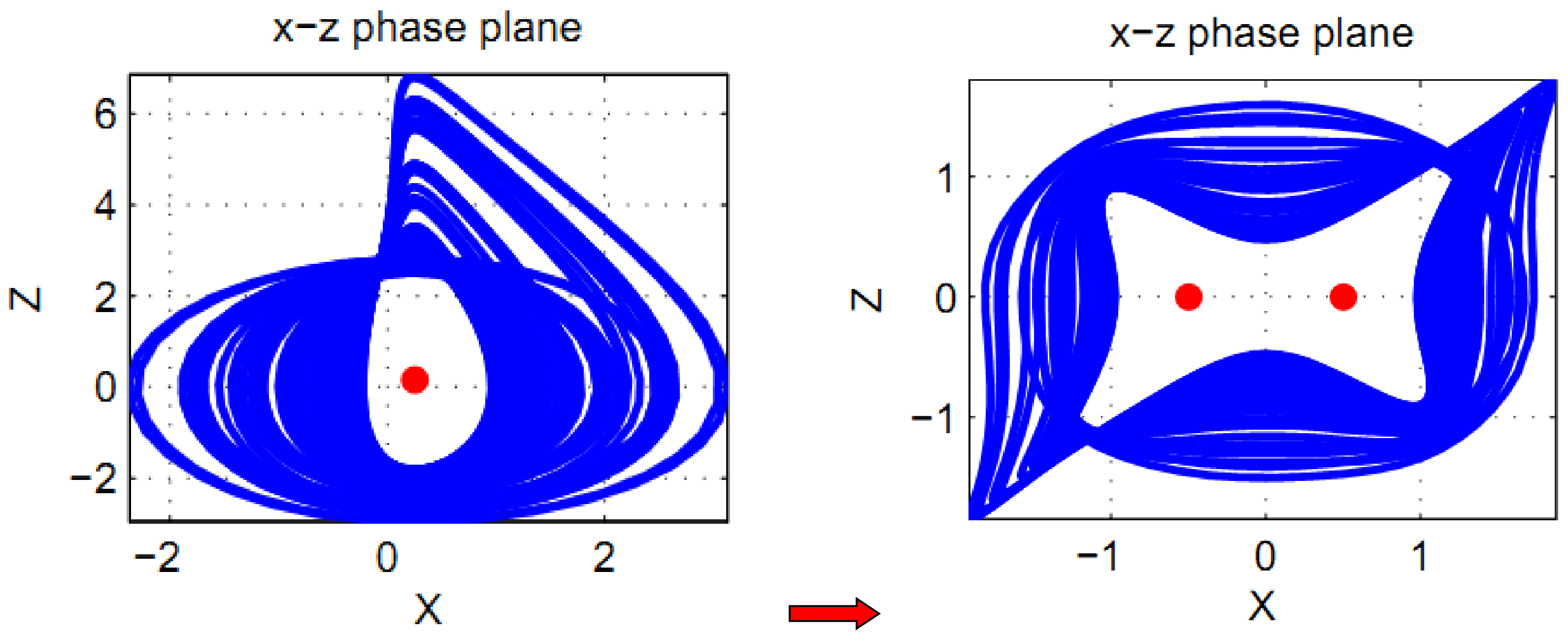
Simulation



$$a = 0.005 > 0$$

stable equilibria

Simulation



$$a = -0.01 < 0$$

unstable equilibria

Symmetry

$$\mathbb{R}_y(\tfrac{2}{3}\pi) \quad \begin{cases} u = x^3 - 3xz^2 \\ v = y \\ w = 3x^2z - z^3 \end{cases}$$

local diffeomorphism

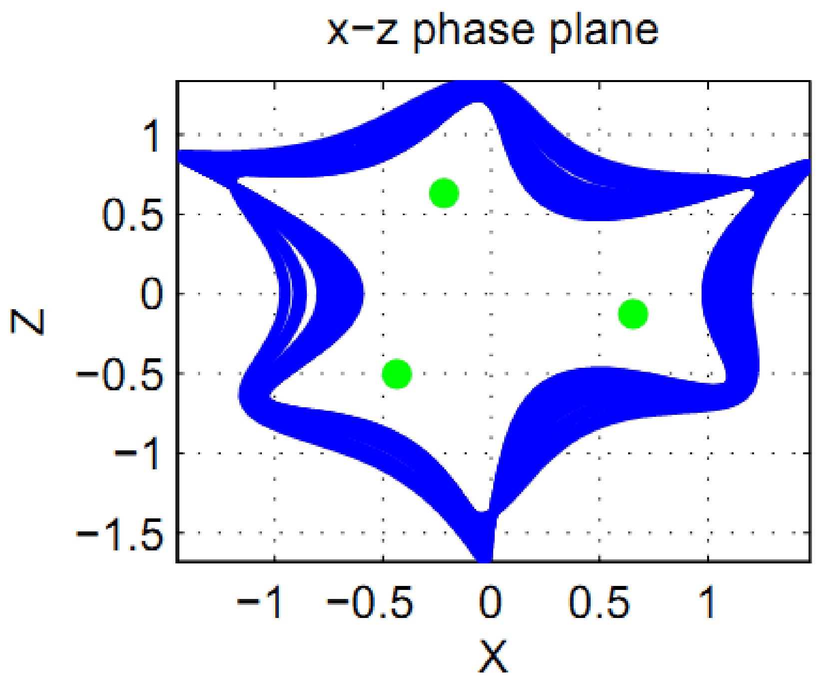
$$\begin{cases} \dot{u} = vw + a \\ \dot{v} = u^2 - v \\ \dot{w} = 1 - 4u. \end{cases}$$

$$\begin{cases} \dot{x} = \frac{1}{3}\frac{3\,x^4zy - 4\,x^2z^3y + x^2a - 8\,x^4z + 2\,zx + 24\,x^2z^3 + z^5y - z^2a}{2\,x^2z^2 + x^4 + z^4} \\ \dot{y} = (x^3 - 3xz^2)^2 - y \\ \dot{z} = -\frac{1}{3}\frac{6\,z^2x^3y - 2\,z^4xy + 2\,zxa + 4\,x^5 - x^2 - 16\,z^2x^3 + z^2 + 12\,z^4x}{2\,x^2z^2 + x^4 + z^4}. \end{cases}$$

Three equilibria are symmetrical with tunable stabilities

	a	Equilibria	Jacobian eigenvalues
Unstable case	-0.01	$P1 = (0.6550, 0.0625, 0.1258)$ $P2 = (-0.2186, 0.0625, -0.6300)$ $P3 = (-0.4365, 0.0625, 0.5044)$	$-1.0617, 0.0308 \pm 0.4843i$
Stable case	0.01	$P1 = (0.6550, 0.0625, -0.1258)$ $P2 = (-0.2186, 0.0625, 0.6300)$ $P3 = (-0.4365, 0.0625, -0.5044)$	$-0.9334, -0.0333 \pm 0.5165i$

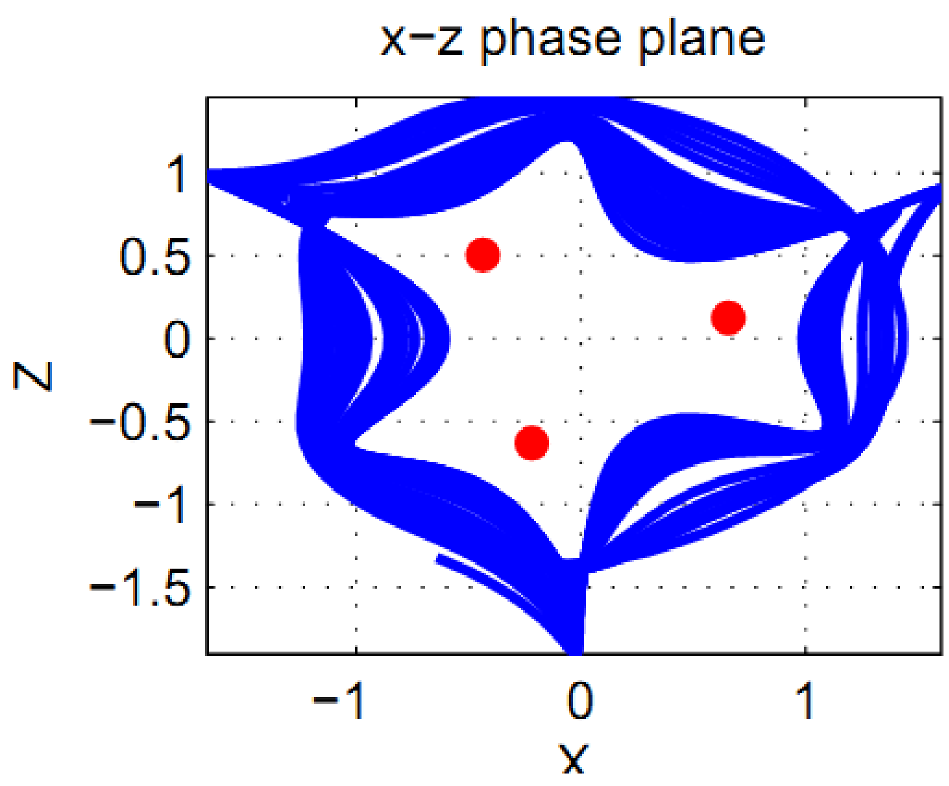
Simulation



$$a = 0.005 > 0$$

stable equilibria

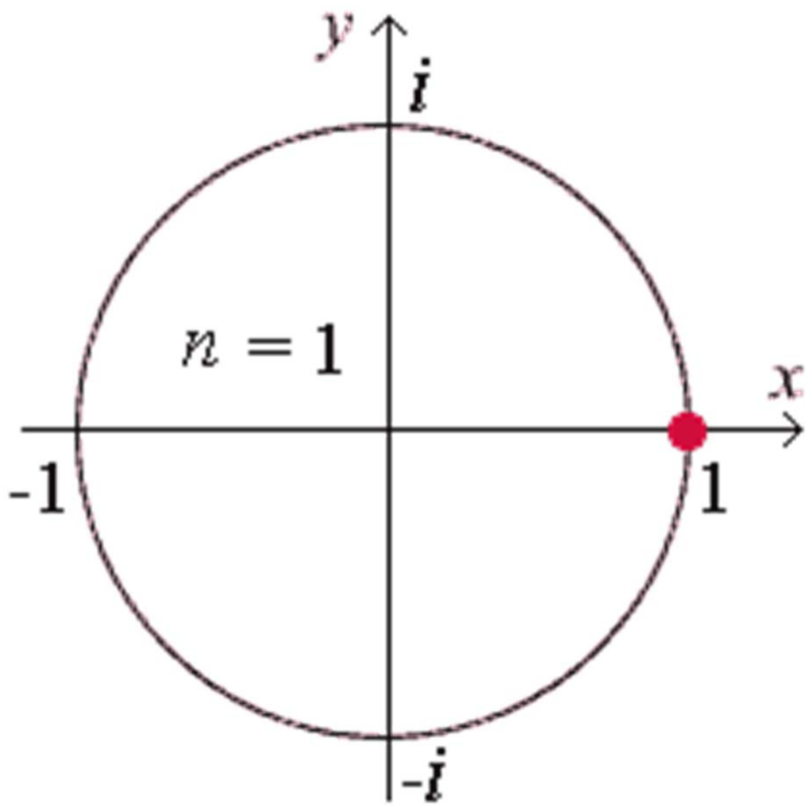
Simulation



$$a = -0.01 < 0$$

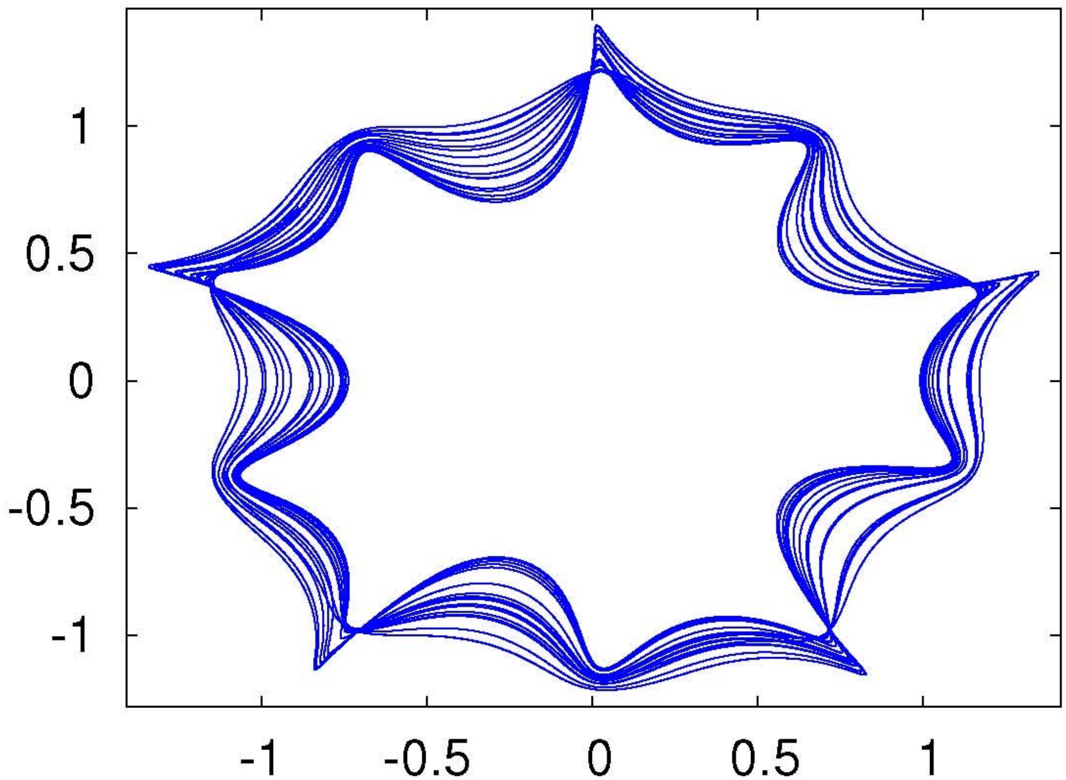
unstable equilibria

Now, theoretically one can create any number of equilibria …



$$(u + wi) = (x + yi)^n$$

For example …



$$R_y\left(\frac{2}{5}\pi\right)$$

Next

Consider multi-equilibrium chaotic systems with hidden attractors

International Journal of Bifurcation and Chaos, Vol. 23, No. 1 (2013) 1330002 (69 pages)

DOI: 10.1142/S0218127413300024

HIDDEN ATTRACTORS IN DYNAMICAL SYSTEMS. FROM HIDDEN OSCILLATIONS IN HILBERT–KOLMOGOROV, AIZERMAN, AND KALMAN PROBLEMS TO HIDDEN CHAOTIC ATTRACTOR IN CHUA CIRCUITS

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Hidden Attractors

Self-excited attractor: if its basin of attraction intersects with a small neighborhood of an equilibrium

Hidden attractor: otherwise

Examples of hidden attractors:

in a system without equilibrium

In a system with only one stable equilibrium

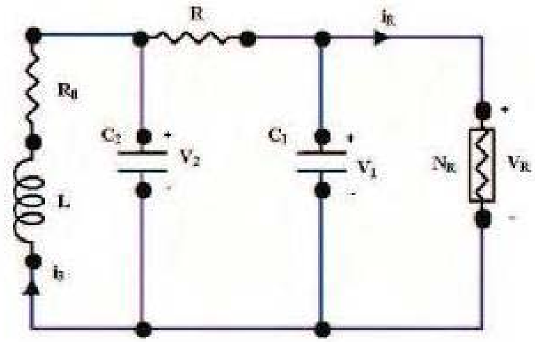
How to predict and compute hidden attractors? See:

Survey 1: G.A. Leonov, N.V. Kuznetsov, IJBC, 23(1): 1330002 (69pp), 2013
Survey 2: G.A. Leonov, N.V. Kuznetsov, Euro. Phys. J, 224: 1421-1458, 2015

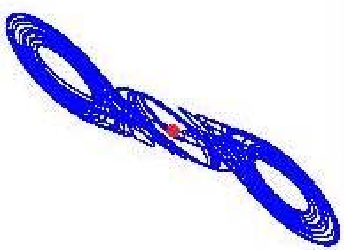
Attractors in Chua's circuits

L.Chua (1983)

$$\dot{x} = \alpha\big(y - x - f(x)\big),$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z),$$



$$f(x) = m_1 x + \mathrm{sat}(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| + |x-1|)$$



1983–now: computations of hundreds Chua self-excited attractors by the standard procedure: a trajectory with initial data from a small neighborhood of unstable zero equilibrium is attracted & visualizes the attractor. [*Bilotta&Pantano, A gallery of Chua attractors, WorldSci. 2008*]

Does Chua attractor exist if the state of equilibrium is stable?

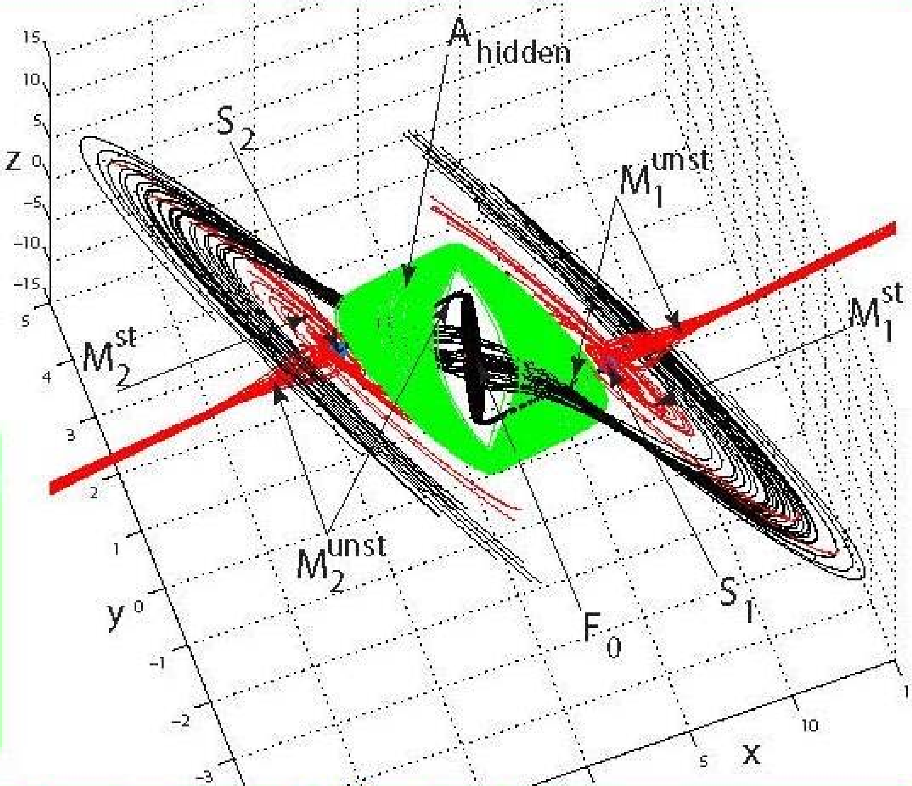
Hidden attractor in Chua's circuit

In 2009-2010 the notion of *hidden attractor* was introduced
and hidden chaotic attractor was found in Chua circuit for the first time:

$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z, \; \dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\mathrm{sat}(x)$$

$\alpha = 8.4562, \beta = 12.0732, \gamma = 0.0052$
$m_0 = -0.1768, m_1 = -1.1468$

Equilibria: stable zero F_0 & 2 saddles $S_{1,2}$
Trajectories: 'from' $S_{1,2}$ tend (black)
to zero F_0 or tend (red) to infinity;
Hidden chaotic attractor (in green)
with positive Lyapunov exponent

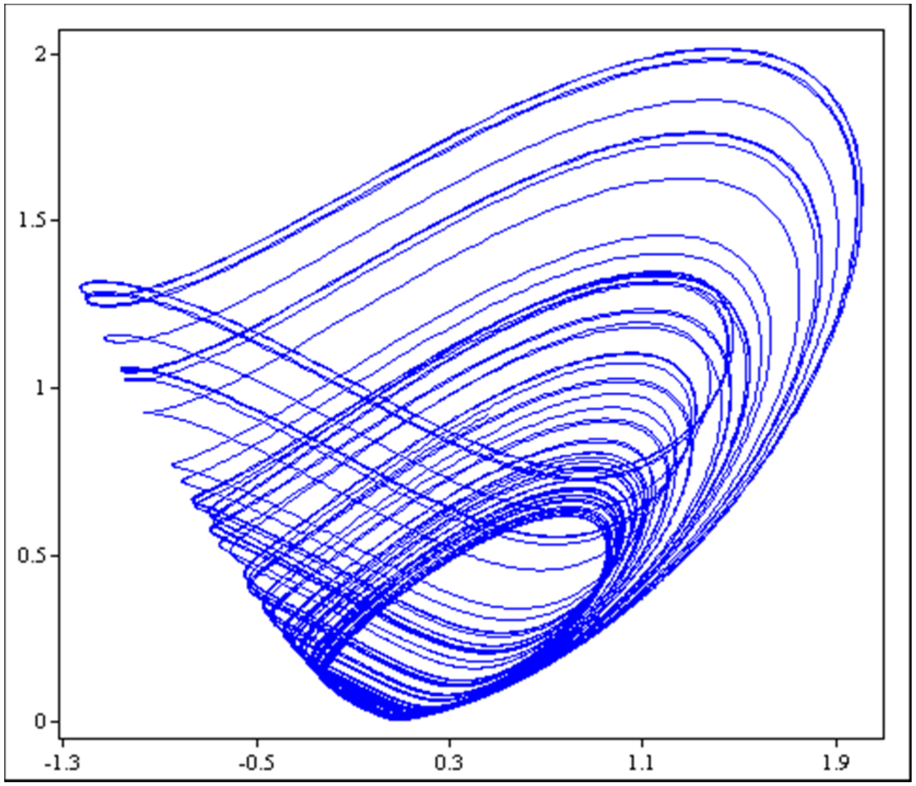


✓ Leonov G.A., Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, Physics Letters A, 375(23), 2011, 2230-2233

Hidden Attractor

of the system with one stable equilibrium

$$\begin{cases} \dot{x} = yz + a \\ \dot{y} = x^2 - y \\ \dot{z} = 1 - 4x \end{cases}$$



X Wang and G Chen: A chaotic system with only one stable equilibrium, Comm in Nonl Sci and Numer Simul, 2012

Following:

International Journal of Bifurcation and Chaos, Vol. 24, No. 11 (2014) 1450146 (6 pages)

DOI: 10.1142/S0218127414501466

Is that Really Hidden? The Presence of Complex Fixed-Points in Chaotic Flows with No Equilibria

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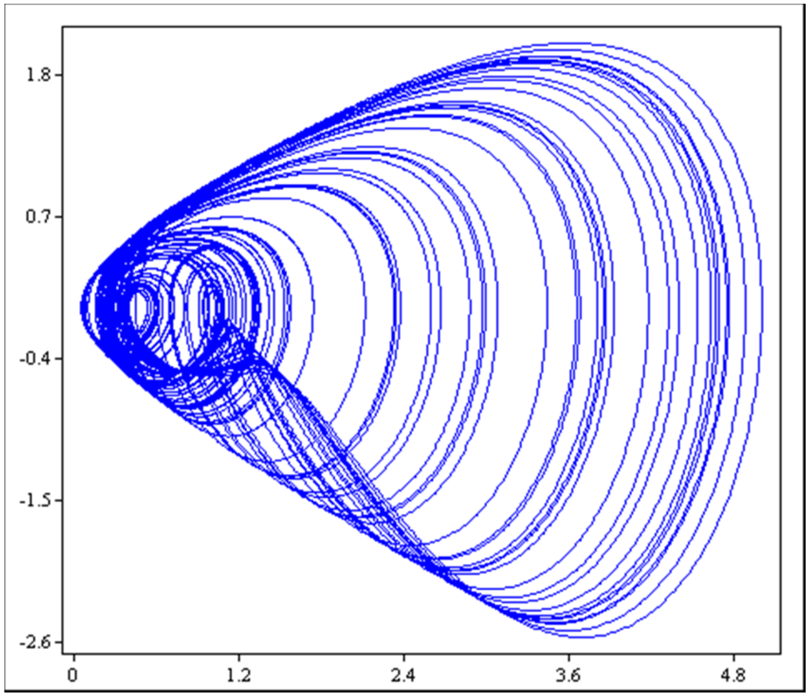
Recall: A chaotic system with no equilibrium

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -y + 3y^2 - x^2 - xz - a. \end{cases}$$

$a > 0$ no equilibrium



X Wang and G Chen: Constructing a chaotic system with any number of equilibria, Nonl Dynam 2013

More: chaotic systems with no equilibrium

Table 1. Fifteen simple chaotic systems with no equilibria and their complex fixed-points (if any).

Model	Equations	a	D_{KY}	Equilibria
NE_1 [Sprott, 1994; Hoover, 1995]	$\dot{x} = y$ $\dot{y} = -x - zy$ $\dot{z} = y^2 - a$	1.0	3.0000	None
NE_2 [Wei, 2011]	$\dot{x} = -y$ $\dot{y} = x + z$ $\dot{z} = 2y^2 + xz - a$	0.35	2.0517	$(\mp\sqrt{a}i, 0, \pm\sqrt{a}i)$
NE_3 [Wang & Chen, 2013]	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y + 0.1x^2 + 1.1xz + a$	1.0	2.0196	$(\pm\sqrt{10a}i, 0, 0)$
NE_4	$\dot{x} = -0.1y + a$ $\dot{y} = x + z$ $\dot{z} = xz - 3y$	1.0	2.0028	$(\mp\sqrt{30a}i, 10a, \pm\sqrt{30a}i)$

… … … … …

V T Pham et al., IJBC, 24(11): 1450146, 2014

Consider Chaotic System NE4

Notice: Complex equilibria always exists

Transformation: 3D (real) → 6D (complex)

$$x \to x_r + x_i \quad y \to y_r + y_i \quad z \to z_r + z_i$$

Take and then slightly vary some initial conditions $(\alpha i, \beta, \gamma)$

(α, β, γ) are real parameters

Some transient chaotic behaviors can be observed, but the original attractor(s) cannot be found.

System NE4

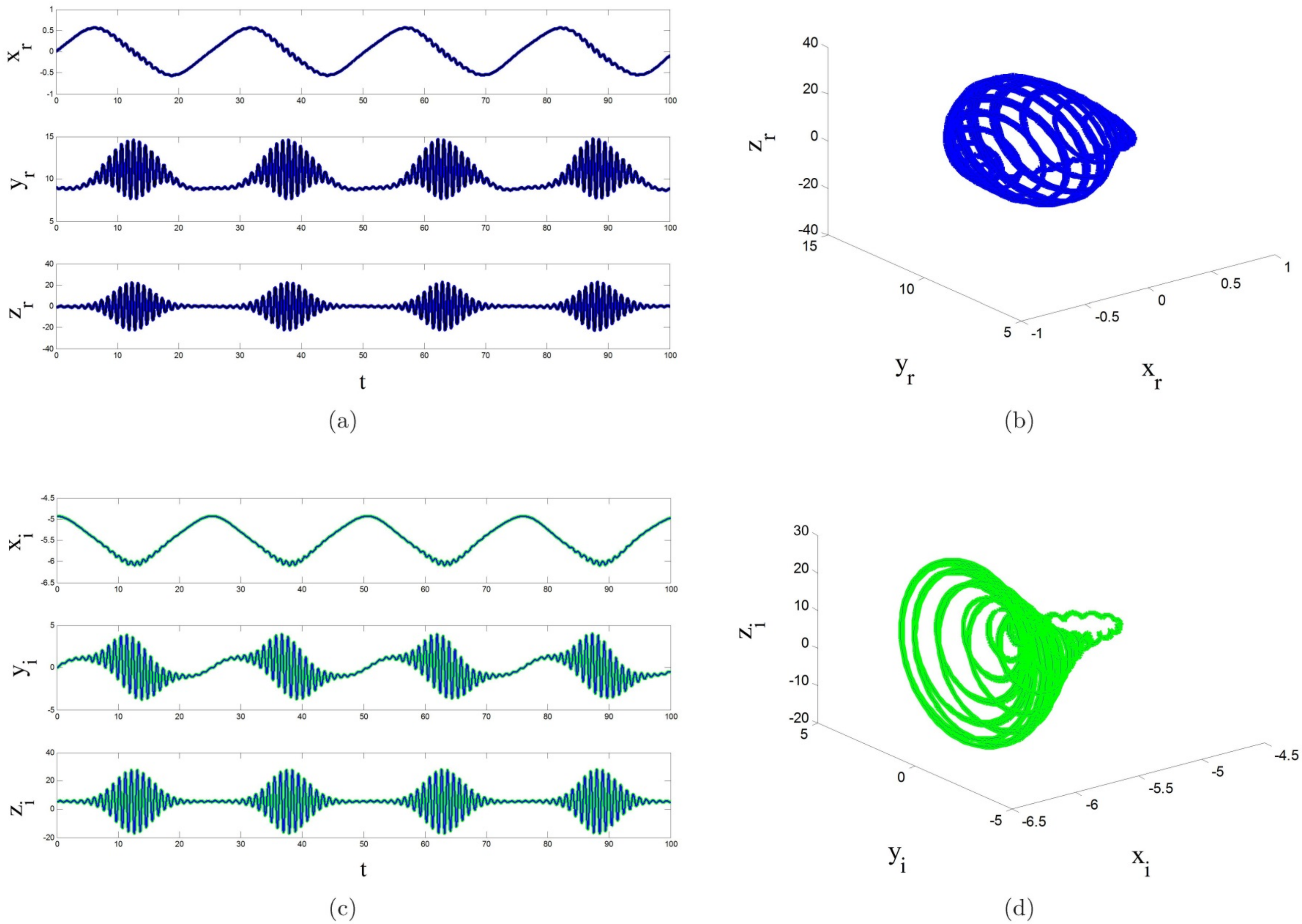
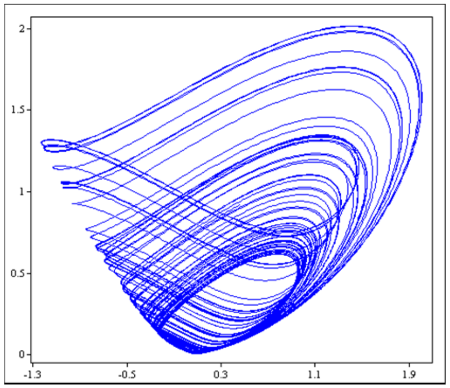


Fig. 1. Torus attractors in NE_4: (a) the real part of the variables in the time domain, (b) the real part of the variables in the state space, (c) the imaginary part of the variables in the time domain and (d) the imaginary part of the variables in the state space. The complex fixed-points in NE_4 are $(\mp\sqrt{30}i, 10, \pm\sqrt{30}i)$. The initial condition we used is $(-0.9\sqrt{30}i, 9, +0.9\sqrt{30}i)$.

The Chaotic System with One Stable Equilibrium



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Take and then slightly vary some initial conditions $(\alpha i, \beta, \gamma)$

(α, β, γ) are real parameters

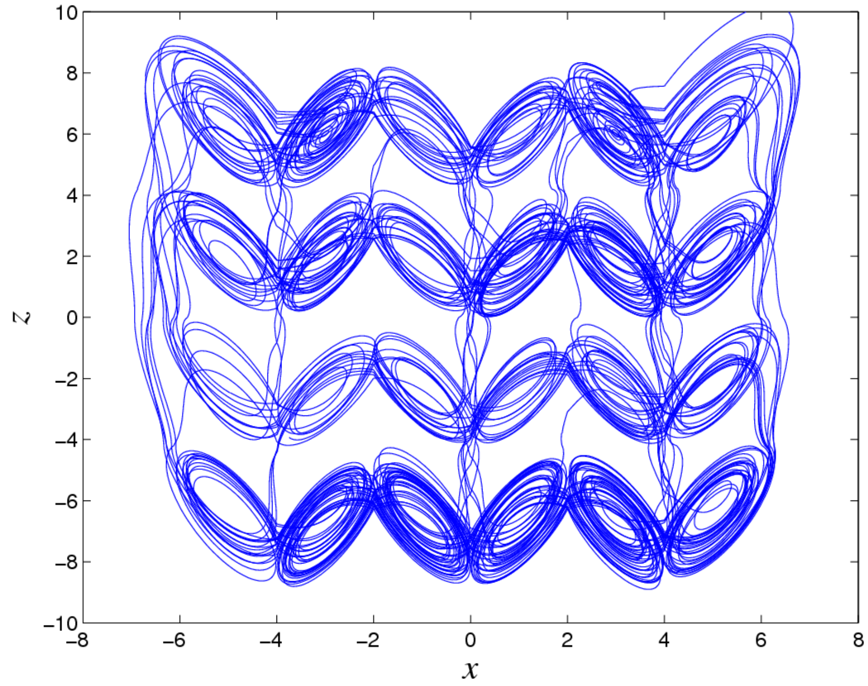
Some transient chaotic behaviors can be observed, but the original attractor(s) cannot be found →

"Yes, that is really hidden; among many other mysterious secrets in the hidden oscillation world."

V T Pham et al., IJBC, 24(11): 1450146, 2014

Chaoticity of Systems with Multi-Equilibia

How to Verify ("Prove") it ?



Existence of Chaotic Attractors

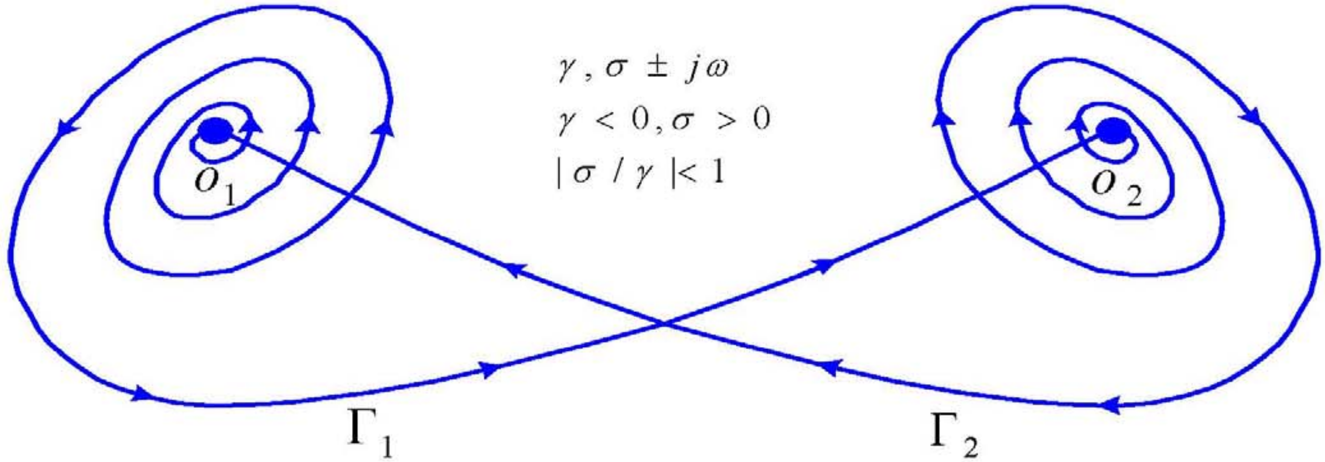
Shilnikov Theorem (1967):

If a 3D autonomous system has two distinct saddle fixed points and there exists a heteroclinic orbit connecting them, and if the eigenvalues of the Jacobin of the system at these fixed points are

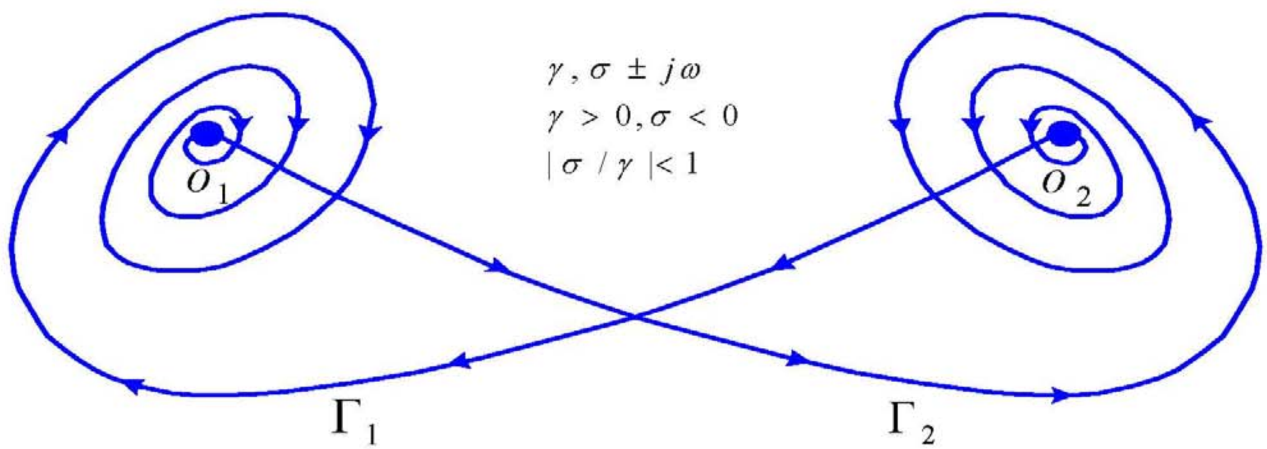
$$\alpha_k, \beta_k \pm j\omega_k (k = 1,2)$$

Satisfying $|\alpha_k| > |\beta_k| > 0 \ (k = 1,2)$ and $\beta_1\beta_2 > 0 \quad or \quad \omega_1\omega_2 > 0$

then the system has infinitely many Smale horseshoes and hence has horseshoe chaos.

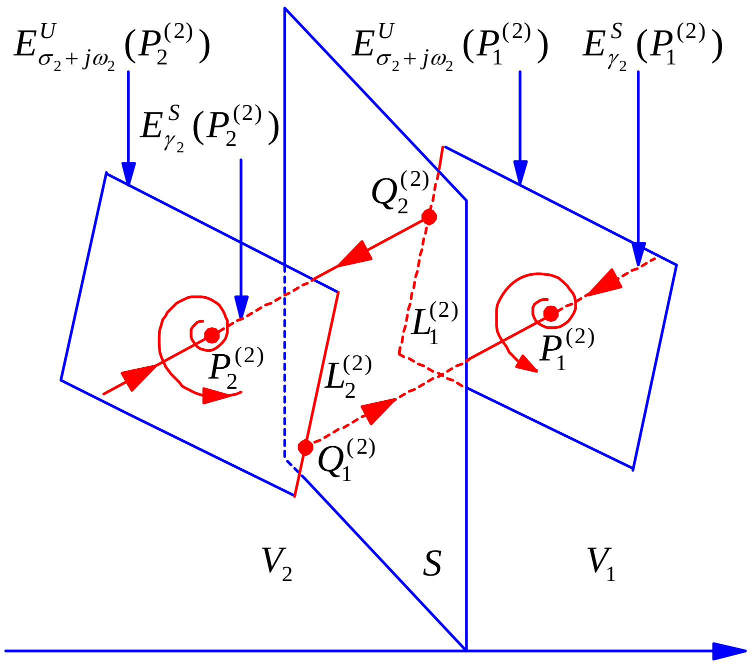


Heteroclinic orbit between two index_2 saddle-focus equilibria

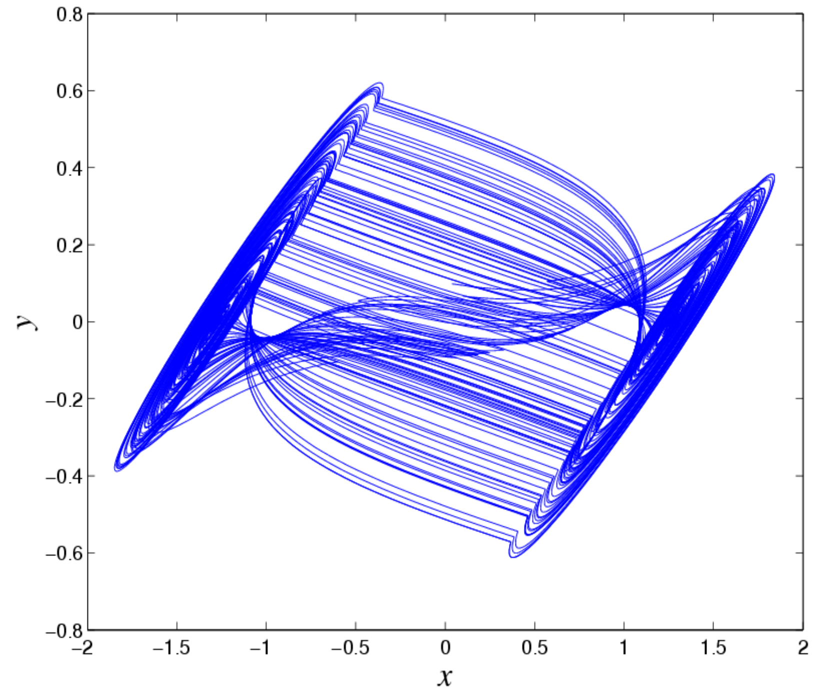


Heteroclinic orbit between two index_1 saddle-focus equilibrium

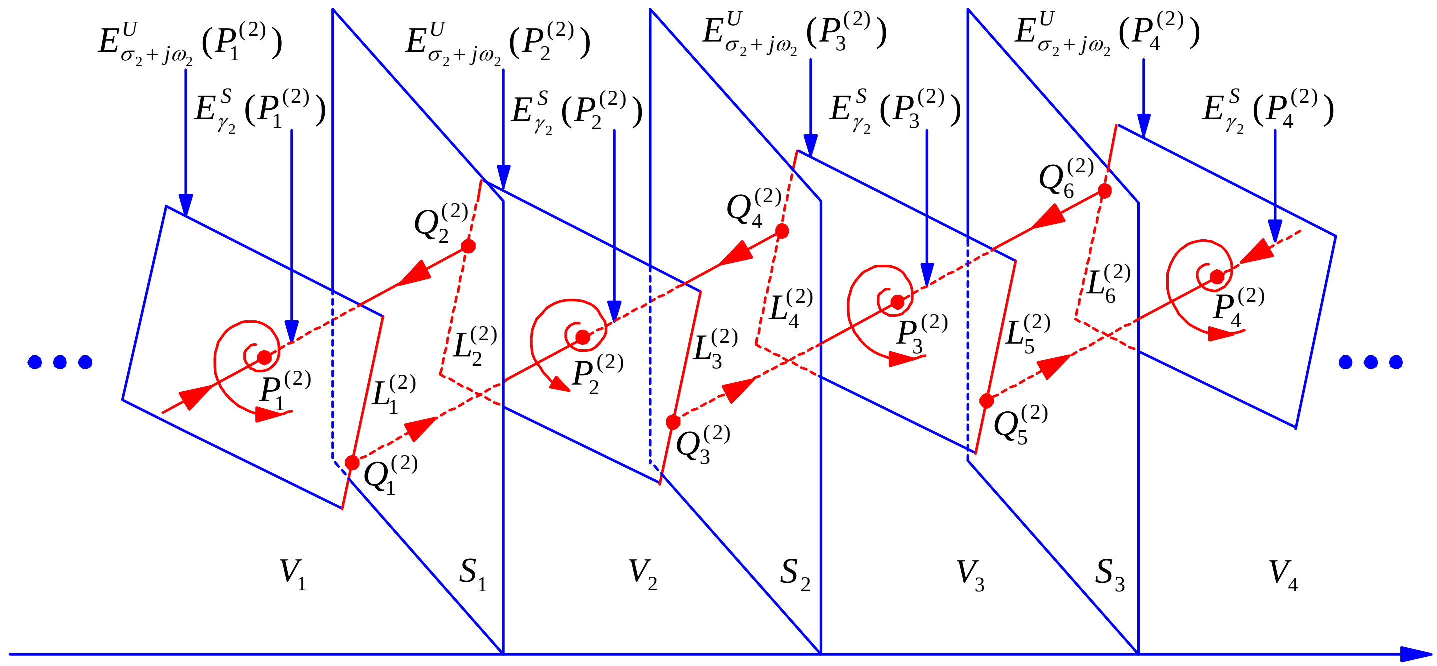
)



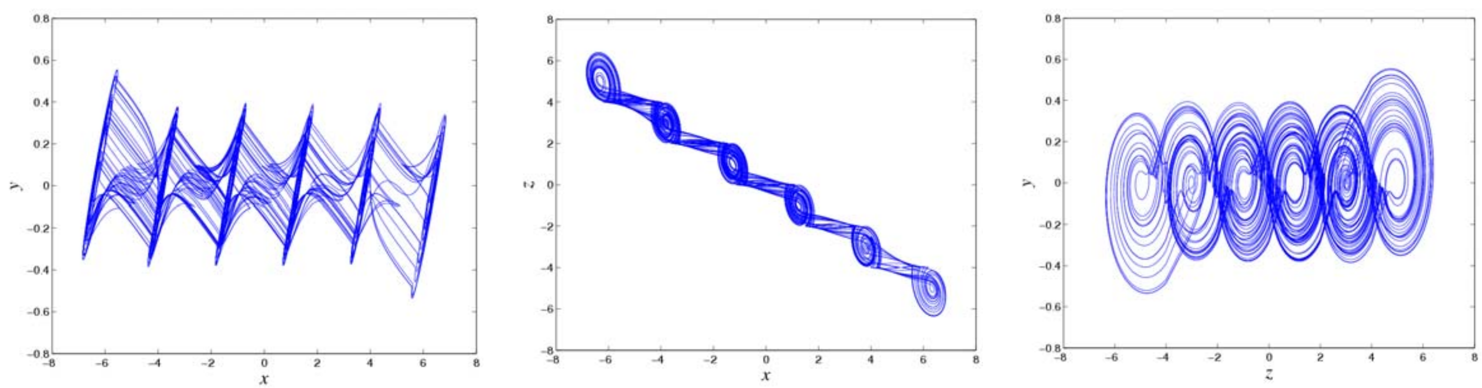
(a) Single heteroclinic orbit



(b) Double-scroll attractor



(a) Multiple heteroclinic orbit



(b) Multi-equilibrium attractors

Multi-Equilibrium Attractors in Nature

Conclusions

In a typical 3D quadratic autonomous chaotic system, such as the Lorenz and Rössler systems, the number of equilibrium points is three or less and the number of scrolls in their attractors is two or less.

Today, we are able to construct a simple 3D quadratic autonomous chaotic system that can have any desired number of equilibrium points.

Such multi-equilibrium chaotic systems all have hidden attractors.

Thank You !