On the Sparsity of a Linear Network Code for Broadcast Systems with Feedback

Chi Wan Sung
Department of Electronic Engineering
City University of Hong Kong
Email: albert.sung@cityu.edu.hk

Kenneth W. Shum
Institute of Network Coding
The Chinese University of Hong Kong
Email: wkshum@inc.cuhk.edu.hk

Ho Yuet Kwan
Department of Electronic Engineering
City University of Hong Kong
Email: hykwan@cityu.edu.hk

***Abstract*—One method for reducing the decoding complexity of network coding in wireless broadcast systems is to generate sparse encoding vectors. The problem of finding the minimal Hamming weight of innovative encoding vectors over large finite field is formulated. By reducing the problem of hitting set to it, we show that the problem is NP-hard. However when the number of users is fixed, the problem can then be solved in polynomial time. A systematic method for solving the problem is provided.**

I. Introduction

Linear network coding [1], [2] provides a promising solution to communications over the broadcast packet erasure channel. Given that a file is divided into N equal-size packets, with linear network coding, the source node transmits linear combinations of those N packets, with coefficients drawn from the finite field $GF(q)$, where q is a power of prime. We call the vector with the N coefficients as components the *encoding vector*. This vector is included in the header of each packet and is sent to the receiving nodes as well. A receiver, after receiving N packets whose encoding vectors are linearly independent over $GF(q)$, can rebuild the original file by solving a set of linear equations. The benefits of linear network coding is investigated in a number of papers [3]–[16].

With network coding, an encoded packet is "useful" to a user if and only if the corresponding encoding vector does not lie in the linear span of the previously received encoding vectors of that user. We call such a packet *innovative to that user*. A packet is said to be *innovative* if it is innovative to all users, and the corresponding encoding vector is also said to be innovative. In [3], it is shown that if the size of the finite field is greater than or equal to the number of users, an innovative packet can always be found. Such a network-coded broadcast scheme is clearly delay optimal.

In addition to the size of the finite field, the design of a network coding scheme also depends on whether user feedback is available. For systems without feedback, random linear network code and LT code are commonly used. For systems with feedback, various transmission schemes have been developed. The authors of [3] suggests the use of Jaggi-Sanders algorithm [17], which is a general network code generation method and is able to find innovative encoding vectors for $q \geq K$. However, the encoding and decoding complexities are relatively high and may not be desirable in real applications. In [5], [11]–[13], the encoding vector is chosen such that the resulting packet is instantly decodable for reducing the decoding complexity. However, as an instantly decodable packet to all users may not exist, the number of packet transmissions is in general larger than that in a system without this extra requirement. Some other heuristics for systems with feedback are proposed and evaluated in [6], [7], [10], [15]. Generally, there is a tradeoff between encoding complexity, decoding delay, and decoding complexity. No transmission scheme is superior in all aspects.

To reduce decoding complexity, the sparsity of the encoding vectors plays a significant role as sparser encoding vectors may facilitate the use of faster decoding algorithms. As an example, the fast algorithm by Wiedemann for solving a system of sparse linear equations can be used [18]. If each encoding vector contains at most w non-zero components, the complexity for solving an $N \times N$ linear system can be reduced from $O(N^3)$ using Gaussian elimination to $O(wN^2)$.

In this paper, we consider the problem of finding a sparsest innovative encoding vector. The computational approach is used to tackle it. In Section II, the system model is introduced and the SPARSITY problem is formulated formally. We prove in Section III that the SPARSITY problem is NP-complete. In Section IV, we illustrate a method to solve the optimization version of SPARSITY in a systematic way using binary integer programming for $q \geq K$. Finally we draw our conclusion in Section V.

II. System Model and Problem Formulation

We consider a wireless broadcast system with one base station and K users. The base station wants to send a file which is divided into N packets to all users via a wireless channel. The wireless channel is modeled as a discrete-time broadcast packet erasure channel. At each time, each packet transmitted by the base station is obtained by linearly combining the original N packets, with coefficients drawn from $GF(q)$. An N-vector whose components are the N coefficients is said to be the *encoding vector* of that packet.

Given an N-vector $\boldsymbol{x} = (x_1, x_2, \dots, x_N)$ over $GF(q)$, the *support* of $\boldsymbol{x}$, denoted by $supp(\boldsymbol{x})$, is the set of indices of the

This work was partially supported by a grant from the University Grants Committee of the Hong Kong Special Administrative Region, China (Project No. AoE/E-02/08).

978-1-61284-140-3/11/$26.00 ©2011 IEEE

non-zero components in $\boldsymbol{x}$, i.e.,

$$supp(\boldsymbol{x}) \triangleq \{i : x_i \neq 0\}.$$

The *Hamming weight* of $\boldsymbol{x}$ is defined as the cardinality of $supp(\boldsymbol{x})$. An encoding vector that has Hamming weight less than or equal to w is said to be w-sparse.

We assume that there is an error-free feedback channel from each user to the base station. Upon receiving a packet successfully, a user sends an acknowledgement (ACK) to the base station without delay or error. The generation of encoded packets depend on all the previous ACKs from the K users. To minimize the delay of each user, it is crucial to generate encoding vectors that are innovative to all users who have not received enough packets for successful decoding.

Suppose that user k, for $k = 1, 2, \ldots, K$, has already received r_k packets whose encoding vectors are linearly independent. Let $\boldsymbol{C}_k$ be the $r_k \times N$ encoding matrix of user k, whose rows are the r_k encoding vectors. Without loss of generality, we assume that $r_k < N$, for otherwise user k can decode the file successfully and can be omitted from our consideration. A vector $\boldsymbol{x}$ is innovative if it does not belong to the row space of $\boldsymbol{C}_k$ for any k. Given $\boldsymbol{C}_k$, for all k's, we would like to characterize the set of all innovative encoding vectors, $\mathcal{I}$. It is well known that $\mathcal{I}$ is non-empty if $q \geq K$. Given that $q \geq K$, we would like to find a vector in $\mathcal{I}$ that has the smallest Hamming weight and the smallest Hamming weight the *sparsity number*, denoted by ω. We state the decision version of the problem formally as follows:

Problem: SPARSITY

Instance: K matrices over $GF(q)$, $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_K$ where $q \geq K$. Each matrix has N columns. n is a positive integer.

Question: Let $\mathcal{I}$ be the complement of the the union of the row spaces,

$$\mathcal{I} \triangleq GF(q)^N \setminus \bigcup_{k=1}^{K} \text{row space}(\boldsymbol{C}_k). \quad (1)$$

Is there a vector $\boldsymbol{x} \in \mathcal{I}$ with Hamming weight less than or equal to n?

Example: Let $q = 2$, $K = 2$, $N = 4$ and $n = 2$. The two matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ over $GF(2)$ are given by

$$\boldsymbol{C}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; \boldsymbol{C}_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The row space of $\boldsymbol{C}_1$ consists of four vectors $[0\ 0\ 0\ 0]$, $[1\ 1\ 0\ 0]$, $[0\ 0\ 1\ 0]$, and $[1\ 1\ 1\ 0]$. The row space of $\boldsymbol{C}_2$ consists of the eight vectors that have even Hamming weight. There are six innovative encoding vectors, and they form the set

$$\mathcal{I} = \{[1\ 0\ 0\ 0], [0\ 1\ 0\ 0], [0\ 0\ 0\ 1], [0\ 1\ 1\ 1], [1\ 0\ 1\ 1], [1\ 1\ 0\ 1]\}.$$

There are three vectors in $\mathcal{I}$ with Hamming weight less than or equal to $n = 2$.

III. Complexity of the Sparsity Problem

In this section, we first characterize the set $\mathcal{I}$. Then we show that SPARSITY is NP-complete.

For $k = 1, 2, \ldots, K$, let V_k be the row space of $\boldsymbol{C}_k$. Denote the *orthogonal complement* of V_k by $V_k^{\perp}$,

$$V_k^{\perp} \triangleq \{\boldsymbol{v} \in GF(q)^N : \boldsymbol{x} \cdot \boldsymbol{v} = 0 \text{ for all } \boldsymbol{x} \in V_k\},$$

where $\boldsymbol{x} \cdot \boldsymbol{v}$ is the inner product of $\boldsymbol{x}$ and $\boldsymbol{v}$. We will use the fact from linear algebra that a vector $\boldsymbol{x}$ is in V_k if and only if $\boldsymbol{x} \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{v} \in V_k^{\perp}$. Let $\boldsymbol{B}_k$ be an $(N - r_k) \times N$ matrix whose rows form a basis of $V_k^{\perp}$. To see whether a vector $\boldsymbol{x}$ is in V_k, it amounts to checking the condition $\boldsymbol{B}_k \boldsymbol{x} = \boldsymbol{0}$; if $\boldsymbol{B}_k \boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{x} \in V_k$, and vice versa.

The following simple result characterizes the set of innovative encoding vectors, $\mathcal{I}$:

Theorem 1. *Given $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_K$, an encoding vector $\boldsymbol{x}$ belongs to $\mathcal{I}$ if and only if $\boldsymbol{B}_k \boldsymbol{x} \neq \boldsymbol{0}$ for all k's.*

Proof: If $\boldsymbol{B}_k \boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{x}$ is not in V_k and therefore, is innovative to user k. It is innovative if $\boldsymbol{B}_k \boldsymbol{x} \neq \boldsymbol{0}$ for all k's.

Conversely, if $\boldsymbol{B}_k \boldsymbol{x} = \boldsymbol{0}$ for some k, then $\boldsymbol{x}$ is in V_k, and hence is not innovative to user k. Therefore, $\boldsymbol{x} \notin \mathcal{I}$. ■

Before we discuss the complexity of SPARSITY, we prove the following two useful lemmas:

Lemma 2. *Consider K linear inequalities of L variables over $GF(q)$ of the form*

$$f_k(\boldsymbol{x}) \triangleq \alpha_{k1} x_1 + \alpha_{k2} x_2 + \cdots + \alpha_{kL} x_L \neq 0, \; k = 1, 2, \ldots, K,$$

where the coefficients α_{kl}'s are elements in $GF(q)$. Suppose for all k, there exists $\alpha_{kl} \neq 0$ for some l. If $q \geq K$, we can always find a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_L) \in GF(q)^L$ such that all the K inequalities hold.

Proof: Let $\mathcal{S}_l$, where $l = 1, \ldots, L$, be the index set such that $k \in \mathcal{S}_l$ if and only if $\alpha_{kl} \neq 0$. If $|\mathcal{S}_l| = K$ for some l, then we can simply let $x_l = 1$ and $x_n = 0$ for $n \neq l$, and all the K inequalities hold.

It remains to consider the case where $|\mathcal{S}_l| < K$ for all l. We present an algorithmic proof. First of all, initialize all variables x_l's to zero for all l. Assign an arbitrary non-zero element in $GF(q)$ to x_1. Clearly, after this assignment, $f_k(\boldsymbol{x}) \neq 0$ for $k \in \mathcal{S}_1$. Next, consider the assignment of x_2. Changing the value of x_2 would only affect the value of f_k for $k \in \mathcal{S}_2$. Since $|\mathcal{S}_2| < K \leq q$, we can always find an element in $GF(q)$ such that after assigning this element to x_2, $f_k(\boldsymbol{x}) \neq 0$ for all $k \in \mathcal{S}_2$. The same process can be applied to the assignment of all the remaining variables. ■

For $k = 1, 2, \ldots, K$, let $\boldsymbol{b}_{k,i}$ be the i-th row of $\boldsymbol{B}_k$. Define $\tilde{\boldsymbol{b}}_k \triangleq \vee_{i=1}^{N-r_k} \boldsymbol{b}_{k,i}$, where $\vee$ denotes the logical-OR operator applied component-wise to the $N - r_k$ vectors, with each non-zero component being regarded as a "1". In other words, the j-th component of $\tilde{\boldsymbol{b}}_k$ is one if and only if the j-th column of $\boldsymbol{B}_k$ is nonzero. Let $\boldsymbol{B}$ be the $K \times N$ matrix whose k-th row is $\tilde{\boldsymbol{b}}_k$. Note that $\boldsymbol{B}$ is a binary matrix and has no zero rows. For a matrix $\boldsymbol{A}$ and a subset $\mathcal{N}$ of the column indices of $\boldsymbol{A}$, let

$\boldsymbol{A}(\mathcal{N})$ be the $K \times |\mathcal{N}|$ submatrix of matrix $\boldsymbol{A}$, whose columns are chosen according to $\mathcal{N}$.

Lemma 3. *Let $\mathcal{N} \subseteq \{1, 2, \ldots, N\}$ be an index set and $q \geq K$. There exists an encoding vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_N) \in \mathcal{I}$ with support inside $\mathcal{N}$ (i.e. $x_n = 0$ for $n \notin \mathcal{N}$) if and only if $\boldsymbol{B}(\mathcal{N})$ has no zero rows.*

Proof: If $\boldsymbol{B}(\mathcal{N})$ has no zero rows, then $\tilde{\boldsymbol{b}}_k(\mathcal{N}) \neq \boldsymbol{0}$ for all k's. Furthermore, for all k's, there must exist $\boldsymbol{b}_{k,j}(\mathcal{N}) \neq \boldsymbol{0}$ for some j. Let $\boldsymbol{x}$ be an N-vector whose components are in $GF(q)$. By Lemma 2, we can find $\boldsymbol{x}(\mathcal{N}) \in GF(q)^{|\mathcal{N}|}$ such that the inner product of $\boldsymbol{b}_{k,j}(\mathcal{N})$ and $\boldsymbol{x}(\mathcal{N})$ is non-zero, for all k's. Let the components of $\boldsymbol{x}$ whose indices do not belong to $\mathcal{N}$ be zero. Then by Theorem 1, $\boldsymbol{x} \in \mathcal{I}$.

Conversely, if $\boldsymbol{x}$ is an innovative vector with $x_n = 0$ for $n \notin \mathcal{N}$, then $\boldsymbol{B}(\mathcal{N})$ cannot have zero rows, for if row k of $\boldsymbol{B}(\mathcal{N})$ is a zero vector, then the k-th inequality in Theorem 1 cannot hold. ■

Now we are ready to prove the NP-completeness of SPARSITY. This will be done by means of the hitting set problem, HITTINGSET. Recall that a problem instance of HITTINGSET consists of a collection $\mathscr{C}$ of subsets of a finite set $\mathcal{U}$. A *hitting set* for $\mathscr{C}$ is a subset of $\mathcal{U}$ such that it contains at least one element from each subset in $\mathscr{C}$. The decision version of this problem is to determine whether we can find a hitting set with cardinality less than or equal to a given value.

Problem: HITTINGSET

Instance: A finite set $\mathcal{U}$, a collection $\mathscr{C}$ of subsets of $\mathcal{U}$ and an integer n.

Question: Is there a subset $\mathcal{S} \subseteq \mathcal{U}$ with cardinality less than or equal to n such that for each $\mathcal{C} \in \mathscr{C}$ we have $\mathcal{C} \cap \mathcal{S} \neq \emptyset$?

It is well known that HITTINGSET is NP-complete [19].

Example: Let $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $\mathscr{C} = \{\{1, 2, 3\}, \{2, 3, 4\}, \{4, 5\}\}$ and $n = 2$.

The minimum-cardinality hitting sets for $\mathscr{C}$ are: $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$ and $\{3, 5\}$.

Theorem 4. SPARSITY *is NP-complete.*

Proof: We are going to reduce HITTINGSET to an instance of SPARSITY. Let the cardinality of $\mathcal{U}$ be N. Label the elements of $\mathcal{U}$ by $1, 2, \ldots, N$. We define $\mathscr{C} \triangleq \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_K\}$, where K is the number of non-empty subsets in $\mathscr{C}$. For $k = 1, 2, \ldots, K$, form an N-vector $\boldsymbol{b}_k \in GF(q)^N$ with its i-th component equal to one if i is in $\mathcal{C}_k$ and zero otherwise, i.e., $\boldsymbol{b}_k$ is the characteristic vector of $\mathcal{C}_k$. Note that $\boldsymbol{b}_k \neq \boldsymbol{0}$ and $\mathscr{C} = \{supp(\boldsymbol{b}_1), supp(\boldsymbol{b}_2), \ldots, supp(\boldsymbol{b}_K)\}$. These $\boldsymbol{b}_k$'s correspond to the degenerate form of $\boldsymbol{B}_k$'s in Theorem 1 with only one row in $\boldsymbol{B}_k$. Let $\boldsymbol{C}_k$ be the encoding matrix of user k, whose row space is the orthogonal complement of the row space of $\boldsymbol{B}_k$ and $\mathcal{I}$ be the innovative vector set defined in (1). In other words, any instance of HITTINGSET can be represented as an instance of SPARSITY and the complexity of this reduction is in $O(KN)$.

It remains to show that there exists a hitting set $\mathcal{H}$ for $\mathscr{C}$ with $|\mathcal{H}| \leq n$ if and only if there exists an $\boldsymbol{x} \in \mathcal{I}$ with Hamming weight $|supp(\boldsymbol{x})| \leq n$. Given the $\boldsymbol{b}_k$'s obtained via the above reduction, suppose there exists $\boldsymbol{x} \in \mathcal{I}$ with $|supp(\boldsymbol{x})| \leq n$. By Theorem 1, we must have $\boldsymbol{b}_k \cdot \boldsymbol{x} \neq 0$ for all k's, which implies $supp(\boldsymbol{b}_k) \cap supp(\boldsymbol{x}) \neq \emptyset$ for all k's. The set $supp(\boldsymbol{x})$ is therefore a hitting set for the given instance. Conversely, given a hitting set $\mathcal{H}$ for $\mathscr{C}$ with $|\mathcal{H}| \leq n$, by definition $supp(\boldsymbol{b}_k) \cap \mathcal{H} \neq \emptyset$ for all k's. Therefore, $\boldsymbol{B}(\mathcal{H})$ has no zero rows. By Lemma 3, there exists an $\boldsymbol{x} \in GF(q)^N$ such that $supp(\boldsymbol{x}) \subseteq \mathcal{H}$. Hence, $|supp(\boldsymbol{x})| \leq n$.

As SPARSITY is verifiable in polynomial time, SPARSITY is in NP. Hence it is NP-complete. ■

Now we define the optimization version of SPARSITY as follows:

Question: Find a vector $\boldsymbol{x} \in \mathcal{I}$ with minimum Hamming weight.

It is easy to see that if a polynomial-time algorithm can be found for solving the optimization version of SPARSITY, then that algorithm can be used for solving the decision version of SPARSITY in polynomial time as well. That means there exists a Turing reduction from the decision version of SPARSITY to its optimization version. Therefore, the optimization version is NP-hard [19, p.114].

With slight abuse of notation, we also use SPARSITY to denote the optimization version. Which version we are referring to can be understood from the context.

IV. MINIMIZING THE HAMMING WEIGHT OF INNOVATIVE ENCODING VECTOR

We have proven that the optimization version of SPARSITY is NP-hard. On the other hand, if K is held fixed, then there exist polynomial-time algorithms to solve it. It is proven in [20] the existence of K-sparse vector in $\mathcal{I}$, if $q \geq K$. By listing all vectors in $GF(q)^N$ with Hamming weight less than or equal to K, we can use Theorem 1 to check whether each of them is in $\mathcal{I}$. For each K-sparse encoding vector, we compute the matrix product $\boldsymbol{B}_k\boldsymbol{x}$ for $k = 1, 2, \ldots, K$. Each matrix product takes $O(NK)$ finite field operations. The total number of finite field operations for each candidate $\boldsymbol{x}$ is $O(NK^2)$. After checking all K-sparse encoding vectors, we can then find one with minimum Hamming weight. The number of non-zero vectors in $GF(q)^N$ with Hamming weight no more than K is equal to $\sum_{k=1}^{K} \binom{N}{k}(q-1)^k$. For fixed K and q, the summation is dominated by the largest term $\binom{N}{K}(q-1)^K$ when N is large, and has order $O(N^K)$. The brute-force method can solve the problem with time complexity of $O(N^K(NK^2))$. As K is held fixed, SPARSITY can be solved in polynomial time in N.

Next we present a systematic method to solve the problem for general K and N. We have to find an index set $\mathcal{N}$ with minimum cardinality, which indicates the non-zero components of an innovative vector. Once $\mathcal{N}$ is found, an innovative vector can be obtained by the method presented in Lemma 2.

Theorem 5. *Given $q \geq K$, the sparsity number, ω, can be obtained by solving the following binary integer program (BIP):*

$$\omega = \min_{\boldsymbol{y}} y_1 + y_2 + \ldots + y_N,$$

subject to

$$\boldsymbol{B}\boldsymbol{y} \geq \mathbf{1}, \text{ and } y_i \in \{0,1\} \text{ for all } i,$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_N)$ *is an* N*-dimensional column vector,* $\mathbf{1}$ *is the* K*-dimensional all-one column vector, and the inequality sign is applied component-wise.*

Proof: $supp(\boldsymbol{y})$ is the required index set $\mathcal{N}$. The cardinality of $\mathcal{N}$ is equal to ω. ∎

Once the problem is formulated in the form of BIP, we can apply general algorithms for solving BIP, for example the cutting plane method, to solve the SPARSITY problem. We refer the readers to [21] for more details on BIP.

The above BIP is in fact equivalent to the minimization version of HITTINGSET. The K rows of $\boldsymbol{B}$ specifies the K subsets of $\mathscr{C}$ in the minimization version of HITTINGSET. Minimizing the sum of y_n's is the same as minimizing the cardinality of a hitting set. In other words, we can also use algorithms designed for solving the minimization version of HITTINGSET to find $\mathcal{N}$.

Example: Let $q = 3$, $K = 3$ and $N = 4$, and the orthogonal complements of V_1, V_2 and V_3 be given by the row spaces of

$$\boldsymbol{B}_1 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \boldsymbol{B}_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix},$$
$$\boldsymbol{B}_2 = \begin{bmatrix} 0 & 2 & 1 & 0 \end{bmatrix}.$$

The vectors $\tilde{\boldsymbol{b}}_k$, for $k = 1, 2, 3$, are

$$\tilde{\boldsymbol{b}}_1 = [1\ 1\ 0\ 1],\ \tilde{\boldsymbol{b}}_2 = [0\ 1\ 1\ 0],\ \tilde{\boldsymbol{b}}_3 = [1\ 0\ 1\ 1].$$

The corresponding instance of the minimization version of HITTINGSET is:

$$\mathcal{U} = \{1, 2, 3, 4\},\ \mathscr{C} = \{\{1, 2, 4\}, \{2, 3\}, \{1, 3, 4\}\}.$$

The solution to both SPARSITY and the minimization version of HITTINGSET can be obtained by solving the following BIP:

$$\min y_1 + y_2 + y_3 + y_4,$$

subject to

$$y_1 + y_2 + y_4 \geq 1,\ y_2 + y_3 \geq 1,\ y_1 + y_3 + y_4 \geq 1,$$
$$y_1, y_2, y_3, y_4 \in \{0, 1\}.$$

One optimal solution is $y_1 = y_2 = 1$ and $y_3 = y_4 = 0$. That means, the sparsity number, ω, is equal to two. Furthermore, according to Lemma 3, a 2-sparse innovative encoding vector can be found, for example, by the method in the proof of Lemma 2.

V. CONCLUSIONS

We investigate the issue of the generation of sparsest innovative encoding vectors. We formulate the problem SPARSITY and show that its decision version is NP-complete while its optimization version is NP-hard. When the number of users K is fixed, we show that SPARSITY can be solved with polynomial time in N. A method based on binary integer programming is proposed, and the connection with the hitting set problem is revealed. In practice, it may not be necessary to find the sparest innovative encoding vector. An encoding vector that is sufficiently sparse is already good enough. Therefore, we may use heuristic algorithms to approximately solve the hitting set problem, and find a sparse encoding vector, though it may not be the sparsest one. The time complexity of the encoding can then be significantly reduced.

Acknowledgements: The authors would like to thank Prof. Wai Ho Mow and Dr. Kin-Kwong Leung for their stimulating discussions.

REFERENCES

[1] S.-Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. on Inf. Theory*, vol. 49, no. 2, pp. 371–381, Feb. 2003.

[2] R. Koetter and M. Médard, "An algebraic approach to network coding," *IEEE/ACM Trans. Networking*, vol. 11, no. 5, pp. 782–795, Oct. 2003.

[3] M. Durvy *et al.*, "Towards reliable broadcasting using ACKs," in *Proc. IEEE Int. Symp. Inform. Theory*, Nice, Jun. 2007, pp. 1156–1160.

[4] L. Keller, E. Drinea, and C. Fragouli, "Online broadcasting with network coding," in *Proc. of the Fouth Workshop on Network Coding, theory, and applications (NetCod '08)*, Hong Kong, Jan. 2008, pp. 68–73.

[5] R. Costa, D. Munaretto, J. Widmer, and J. Barros, "Informed network coding for minimum decoding delay," in *Fifth IEEE Int. Conf. on Mobile Ad-hoc and Sensor Systems (MASS '08)*, Atlanta, 2008.

[6] J. K. Sundararajan *et al.*, "Online network coding for optimal throughput and delay – the three-receiver case," in *Int. Symp. on Inform. Theory and its Application (ISITA)*, Auckland, Dec. 2008, pp. 1–6.

[7] ——, "A feedback-based adaptive broadcast coding scheme for reducing in-order delivery delay," in *Network Coding, Theory and Application (NetCod)*, Lausanne, Jun. 2009, pp. 1–6.

[8] X. Xiao, L. Yang, W. Wang, and S. Zhang, "A broadcasting retransmission approach based on random linear network coding," in *IEEE Int. Conf. Circuits and Systems for Comm.*, May 2008, pp. 782–786.

[9] J. Heide, M. V. Pedersen, F. H. P. Fitzek, and T. Larsen, "Network coding for mobile devices – systematic binary random rateless codes," in *IEEE Int. Conf. Comm. Workshops*, Dresden, Jun. 2009, pp. 1–6.

[10] D. Nguyen *et al.*, "Wireless broadcast using network coding," *IEEE Trans. on Veh. Tech.*, vol. 58, no. 2, pp. 914–925, Feb. 2009.

[11] P. Sadeghi, D. Traskov, and R. Koetter, "Adaptive network coding for broadcast channels," in *Fifth Workshop on Network Coding, Theory and Applications (NetCod '08)*, 2009, pp. 1–6.

[12] S. Sorour and S. Valaee, "On minimizing broadcast completion delay for instantly decodable network coding," in *IEEE Int. Conf. Comm. (ICC '10)*, May 2010, pp. 1–5.

[13] P. Sadeghi, R. Shams, and D. Traskov, "An optimal adaptive network coding scheme for minimizing decoding delay in broadcast erasure channel," *EURASIP J. on Wireless Comm. and Networking*, vol. 2010, 2010, doi:10.1155/2010/61816.

[14] Z. Dong, C. Zhan, and Y. Xu, "Delay aware broadcast scheduling in wireless networks using network coding," in *Proc. of the Second Int. Conf. on Network Security, Wirless Comm. and Trusted Computing*, Hubei, Apr. 2010, pp. 214–217.

[15] L. Lu, M. Xiao, M. Skoglund, L. K. Rasmussen, G. Wu, and S. Li, "Efficient network coding for wireless broadcasting," in *IEEE Wireless Comm. and networking conf. (WCNC '10)*, Sydney, Apr. 2010, pp. 1–6.

[16] C. Studholme and I. F. Blake, "Random matrices and codes for the erasure channel," *Algorithmica*, vol. 56, no. 4, pp. 605–620, 2010.

[17] S. Jaggi *et al.*, "Polynomial time algorithms for multicast network code construction," *IEEE Trans. Inf. Theory*, vol. 51, no. 6, pp. 1973–1982, Jun. 2005.

[18] D. H. Wiedemann, "Solving sparse linear equations over finite fields," *IEEE Trans. Inf. Theory*, vol. 1, no. 32, pp. 54–62, Jan. 1986.

[19] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*. W. H. Freeman and Company, 1979.

[20] H. Y. Kwan, K. W. Shum, and C. W. Sung, "Generation of innovative and sparse encoding vectors for broadcast systems with feedback," to appear in *Proc. IEEE Int. Symp. Inform. Theory*, St. Petersburg, Jul. 2011.

[21] A. Schrijver, *Theory of linear and integer programming*. John Wiley & Son, 1986.