Sector-Disk Codes and Partial MDS Codes with up to Three Global Parities

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***Abstract*— A new construction for sector-disk codes and partial MDS codes up to three sector erasures are proposed. In contrast to existing codes, which are based on the design of parity check matrix, our new code is based on the design of generator matrix. This new approach allows us to construct a code that requires a small field size. In particular, for the case when there is only one sector erasure, our field size requirement is independent of the number of sectors and is close to optimal. For the case when there are three sector erasures, we provide a condition which allows the code be constructed by computer search.**

***Index Terms*—Sector-Disk Codes, Partial MDS codes, RAID, Erasure codes.**

I. Introduction

An $m \times n$ array code is a collection of $m \times n$ arrays, with each entry containing an element in $\mathbb{F}_q$. In the literature of array code, each row is called a *stripe*, and each column corresponds to a *disk*. An entry of the array is also called a *sector*, so that there are m sectors in a disk. Each sector is labeled as a pair (i, j), where $1 \leq i \leq m$ and $1 \leq j \leq n$.

The objective of conventional array codes is to correct column erasures. Even though there is only one erroneous sector in a disk, conventional array codes consider the whole disk as an erasure. Recently, Plank, Blaum and Hafner proposed that one should differentiate sector failures from disk failures, and introduced the notion of Sector-Disk (SD) codes [1]. Roughly speaking, for positive integers r and s, an (r, s)-SD code is defined as an array code which can tolerate any r disk failures plus s additional sector failures. A smaller class of codes, called Partial Maximal-Distance Separable (PMDS) codes, is studied in [2]. For positive integers r and s, the objective of an (r, s)-PMDS codes is to correct all erasure patterns consisting of x_i erased sectors in row i, with $x_i \geq r$ for all i and $\sum_{i=1}^{m} x_i = rm + s$. The requirement of a PMDS is more stringent than the requirement of an SD code, and thus any (r, s) PMDS code is an (r, s) SD code. (Formal definitions of SD codes and PMDS codes are given in the next section.) Some typical erasure patterns which can be corrected by PMDS codes and SD codes are illustrated in Fig. 1. SD codes and PMDS codes are also applicable to storage systems based on solid-state devices, which wear out gradually over time.

In [3]–[5], Blaum *et al.* gave some constructions of PMDS codes which can tolerate any number of disk failures and two sector erasures, i.e. $(r, 2)$-PMDS codes. Some computer search for the parity-check matrix of (r, s)-PMDS codes for $s = 3$ can be found in [6]. Some constructions of (r, s)-PMDS codes for $r = 1$ and $s = 1, 2, 3, 4$ were presented by Gopalan *et al.* in [7]. (In [7], the authors use the terminology *maximal-recoverable local code* instead of PMDS code.) These constructions are all based on the design of parity-check matrix. In [8], Li and Lee relaxed the sector-failure requirement of SD codes and constructed a new family of codes, called STAIR codes, based on existing erasure codes. In [9], Blaum and Hetzler have made similar considerations. By sacrificing some of the erasure-correction capability, they studied Generalized Concatenated (GC) codes, which can be constructed for any range of parameters with a much smaller field size.

In this paper, we focus on the design of SD and PMDS codes. We adopt a different approach from [3]–[7] by designing the generator matrix rather than the parity check matrix. Definitions of SD codes and partial MDS codes are formally given in Section II. Some facts about generalized Vandermonde matrix, which are used in the proof of correctness of our construction, are reviewed in Section III. The constructions of SD codes given in Section IV have the advantage of smaller field size in comparison with existing constructions. The parameters of available constructions of PMDS and SD codes are summarized in Table I. The code parameters of the new constructions presented in this paper are given in the last three rows of the table. In particular, provided that the field size q is larger than or equal to the number of disks, we can construct SD codes which correct $s = 1$ sector erasure and a given number of column erasures. The property that the field size is independent of the number of sectors per disk when $s = 1$ makes the SD code suitable to applications in which the number of sectors is much larger than the number of disks.

II. Definition of SD and PMDS codes

For a power of prime q, we denote a finite field of size q by $\mathbb{F}_q$. For positive integers m, n, r, s, an $m \times n$ array code $\mathcal{C}$ is called a *sector-disk* (SD) code if

(1) $\mathcal{C}$ is a linear $[mn, m(n-r) - s]$ code over $\mathbb{F}_q$;

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TABLE I
AVAILABLE CONSTRUCTIONS FOR SD AND PMDS CODES.

r	s	Explicit?	Type	Field size q	Ref.
1	1	Yes	PMDS	$q > n$	[4] [6]
≥ 1	1	Yes	PMDS	$q \geq n$	[2] [9]
≤ 2	2	Yes	PMDS	$q > mn$	[4] [6]
≥ 1	2	Yes	SD	$q > mn$	[5]
≥ 1	2	Yes	PMDS	$q > m(r+1) \cdot (n-r-1)+m$	[5]
2	3	No	PMDS	N/A	[6]
≤ 3	3	No	PMDS	$q = 2^{32}$ $(n \leq 24, m \leq 24)$	[6]
1	3	Yes	PMDS	$q = 2^t$, $m \leq 2^{(n-1)\kappa}$ $t = (n-1)(1.5\kappa + 1)$	[7]
1	4	Yes	PMDS	$q = 2^t$, $m \leq 2^{(n-1)\kappa}$ $t = (n-1)(2(\kappa+1) + \kappa/3)$, $3 \mid \kappa$	[7]
1	≥ 1	Yes	PMDS	$q > mn$	[2]
≥ 1	1	Yes	PMDS	$q \geq n$	Thm. 1
≥ 1	2	Yes	SD	$q \geq mn$	Cor. 3
≥ 1	3	No	SD	N/A	Thm. 4

(2) each row is a codeword of an $[n, n-r, r+1]$ maximal-distance separable (MDS) code over $\mathbb{F}_q$;

(3) $\mathcal{C}$ can correct up to $s+mr$ erasures, provided that among the $s+mr$ erasures, mr of them occur in r columns.

A sector-disk code satisfying the above requirements is called an $(r, s; m, n)_q$-SD code. If the size of the array and the field size is clear from the context, we will simply write (r, s)-SD code. The second condition requires that if we restrict an SD code to any row, the resulting code is an MDS code of length n. Hence, each row of an SD code is a *local MDS code*. We let $k := n - r$ be the dimension of the local MDS code in a row, and $K := m(n-r) - s = mk - s$ be the dimension of the array code $\mathcal{C}$.

An array code is called an (r, s)-PMDS code if it satisfies the first two conditions in the definition of SD codes and

(3') for any m non-negative integers $s_1, s_2, \ldots, s_m$ with $s_1 + s_2 + \cdots + s_m = s$, the array code $\mathcal{C}$ can correct up to $s_i + r$ erasures in row i, for $i = 1, 2, \ldots, m$.

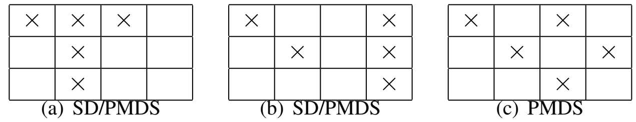
III. PRELIMINARIES ON VANDERMONDE MATRIX

We record a known result about generalized Vandermonde matrix. Let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be a vector whose components are indeterminates. We let $V(\mathbf{X})$ denote the determinant of the $m \times m$ Vandermonde matrix

$$V(\mathbf{X}) := \det \begin{bmatrix} 1 & X_1 & \ldots & X_1^{m-1} \\ 1 & X_2 & \ldots & X_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_m & \ldots & X_m^{m-1} \end{bmatrix},$$

and for $i = 1, 2, \ldots, m$, let $W_i(\mathbf{X})$ be the $m \times m$ matrix

$$W_i(\mathbf{X}) := \det \begin{bmatrix} 1 & \ldots & X_1^{i-1} & X_1^{i+1} & \ldots & X_1^m \\ 1 & \ldots & X_2^{i-1} & X_2^{i+1} & \ldots & X_2^m \\ \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & X_m^{i-1} & X_m^{i+1} & \ldots & X_m^m \end{bmatrix}.$$



(a) SD/PMDS (b) SD/PMDS (c) PMDS

Fig. 1. Erasure patterns which can be corrected by $(1, 2)$-PMDS or SD code. The symbol "$\times$" indicates that the sector is erased.

The matrix in the above line is called a *generalized Vandermonde matrix*. We note that there is no exponent i in the matrix. Also, the determinant $W_m(\mathbf{X})$ is the same as $V(\mathbf{X})$. It is well known that $V(\mathbf{X}) = \prod_{i<j}(x_j - x_i)$. For the generalized Vandermonde matrix, we have the following lemma.

Lemma 1. *For $i = 0, 1, \ldots, m$, we have*

$$W_i(\mathbf{X}) = e_{m-i}(\mathbf{X}) V(\mathbf{X}),$$

where

$$e_i(\mathbf{X}) := \sum_{1 \leq j_1 < \cdots < j_i \leq m} X_{j_1} X_{j_2} \cdots X_{j_i}$$

for $i \geq 1$, is the i^{th} elementary symmetric polynomial with variables in $\mathbf{X}$. (For $i = 0$, we define $e_0(\mathbf{X}) := 1$.)

A proof of Lemma 1 can be found in [10, Lemma 1]. For example, we have

$$W_0(\mathbf{X}) = X_1 X_2 \cdots X_m V(\mathbf{X}), \text{ and}$$
$$W_{m-1}(\mathbf{X}) = (X_1 + X_2 + \cdots + X_m) V(\mathbf{X}).$$

IV. CONSTRUCTION OF SD CODES

The encoding of PMDS and SD codes in this paper takes the following form

$$\begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,k} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,k} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_n^{k-1} \end{bmatrix}. \quad (1)$$

The entries in the first matrix are linear combinations of the K source symbols, and will be chosen such that the K source symbols can be uniquely determined from them. The second matrix is a Vandermonde matrix. We can consider the encoding as a two-step process. We obtain the $m \times k$ matrix $\mathbf{B} = [b_{uv}]$, and then multiply it by the $k \times n$ Vandermonde matrix. The resulting matrix has size $m \times n$, and the (i, j)-entry is stored in the i-th sector of the j-th disk. The entries in the i-th row can be computed by evaluating a polynomial of degree less than or equal to $k - 1$, with $b_{i,1}, \ldots, b_{i,k}$ as the coefficients. This is in accordance with the original definition of Reed-Solomon code [11]. Thus, each row in the array is a codeword of an MDS code of length n.

A. $(r, 1)$-PMDS codes

Theorem 1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n distinct elements in $\mathbb{F}_q$. The $m \times n$ arrays obtained in* (1) *with $k = n - r$ and*

$$b_{1,k} + b_{2,k} + \cdots + b_{m,k} = 0, \quad (2)$$

form an $(r,1;m,n)_q$-PMDS code.

Remark 1: The requirement of field size of the construction Theorem 1 is $q \geq n$. Given the source symbols $s_1,\dots,s_{mn-rm-1}$, we can set the matrix $[b_{u,v}]$ to be the matrix

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_{k-1} & s_k \\ s_{k+1} & s_{k+2} & \cdots & s_{2k-1} & s_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{(m-2)k+1} & s_{(m-2)k+2} & \cdots & s_{(m-1)k-1} & s_{(m-1)k} \\ s_{(m-1)k+1} & s_{(m-1)k+1} & \cdots & s_{mk-1} & \Delta_1 \end{bmatrix}$$

where Δ_1 is defined as

$$\Delta_1 := -(s_k + s_{2k} + \cdots + s_{(m-1)k}), \tag{3}$$

so that the sum of the entries in the last column is equal to zero.

Proof of Theorem 1: It is obvious that the dimension of the array code described in Theorem 1 has dimension $mn - mr - 1$, and each row belongs to an MDS code of length n and dimension $n-1$. It remains to show that it can decode the source symbols if we can correct $rm+1$ erasures with at least r but no more than $r+1$ erasures in each row.

By symmetry, it suffices to show that we can reconstruct the $mn - rm - 1$ source symbols from any $n-r$ symbols from row i, for $i = 1,2,\dots,m-1$ and any $n-r-1$ symbols from the last row. Since each local code in a row can correct r erasures, we can recover the first $m-1$ rows in the $m \times n$ array, as there are precisely r erasures in each of the first $m-1$ rows. We can use any decoding algorithm for MDS code in the recovery of the first $m-1$ rows of matrix $\mathbf{B}$. For example, we can use the Lagrange interpolation formula. Once the first $m-1$ rows of the matrix $\mathbf{B}$ are computed, we can now compute $b_{m,n-r}$ from (2), and subtract $b_{m,n-r} \cdot \alpha_n^{k-1}$ from each of the $n-r-1$ unerased entries in the last row of the $m \times n$ array. The resulting symbols can be regarded as the evaluation of a polynomial of degree at most $n-r-2$. The last row of the $\mathbf{B}$ matrix can be decoded by Lagrange interpolation. After the matrix $\mathbf{B}$ is obtained, the $mn - rm - 1$ source symbols can be read from $\mathbf{B}$ directly. ■

B. $(r,2)$-SD codes

We choose the matrix $[b_{u,v}]$ such that the sum of the entries in the last column is zero, and

$$\beta_1 b_{1,1} + \beta_2 b_{2,1} + \cdots + \beta_m b_{m,1} = 0.$$

The values of the coefficients β_i's will be specified later. For example, we can obtain the matrix $[b_{u,v}]$ by

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_{k-1} & s_k \\ s_{k+1} & s_{k+2} & \cdots & s_{2k-1} & s_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{(m-2)k+1} & s_{(m-2)k+2} & \cdots & s_{(m-1)k-1} & s_{(m-1)k} \\ \Delta_2 & s_{(m-1)k+1} & \cdots & s_{mk-2} & \Delta_1 \end{bmatrix} \tag{4}$$

where Δ_1 is given as in (3) and Δ_2 is defined as

$$\Delta_2 := -\beta_m^{-1}(\beta_1 s_1 + \beta_2 s_{k+1} + \cdots + \beta_{m-1} s_{(m-2)k+1}). \tag{5}$$

We introduce a few more notations. We use j as the index of the columns in the array code. For any subset $\mathcal{J} = \{j_1, j_2, \dots, j_\nu\}$ of $\{1,2,\dots,n\}$, we let $\mathbf{D}_\mathcal{J}$ be the $\nu \times k$ Vandermonde matrix

$$\mathbf{D}_\mathcal{J} := \begin{bmatrix} 1 & \alpha_{j_1} & \alpha_{j_1}^2 & \cdots & \alpha_{j_1}^{k-1} \\ 1 & \alpha_{j_2} & \alpha_{j_2}^2 & \cdots & \alpha_{j_2}^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{j_\nu} & \alpha_{j_\nu}^2 & \cdots & \alpha_{j_\nu}^{k-1} \end{bmatrix}.$$

Theorem 2. *Suppose $n \geq r+2$, and $\alpha_1,\dots\alpha_n$, $\beta_1,\beta_2,\dots,\beta_m$ are elements in $\mathbb{F}_q$ satisfying the conditions*

1) *$\alpha_1,\dots,\alpha_n$ are distinct and non-zero;*
2) *$\beta_1,\dots,\beta_m$ are all non-zero;*
3) *$\beta_{i_1}\alpha_{j_1} \neq \beta_{i_2}\alpha_{j_2}$ for all distinct $i_1,i_2 \in \{1,2,\dots,m\}$ and $j_1,j_2 \in \{1,2,\dots,n\}$. ($j_1$ and j_2 need not be distinct.)*

Then the arrays obtained in (1) with

$$\beta_1 b_{1,1} + \beta_2 b_{2,1} + \cdots + \beta_m b_{m,1} = 0, \text{ and}$$
$$b_{1,k} + b_{2,k} + \cdots + b_{m,k} = 0,$$

form an $(r,2;m,n)_q$-SD code.

Proof: As in the proof of Theorem 1, we only need to show that the array codes can correct any $rm+2$ erasures, with at least r but no more than $r+2$ erasures in each row. We shall consider two cases. In the first case, there is one row with exactly $r+2$ erasures, and in the second case, there are two rows with $r+1$ erasures in each of them. Suppose there are r disks which are totally erased, and let $\mathcal{J}^c$ be the index set of these disks. Let $\mathcal{J}$ be the complement of $\mathcal{J}$ in $\{1,2,\dots,n\}$. So, we have $|\mathcal{J}| = n - r = k$.

Case 1: The proof is very similar to the argument in the proof of Theorem 1. Suppose without loss of generality that there are $r+2$ erasures in the first row of the array code. Let j_1 and j_2, with $j_1 \in \mathcal{J}$ and $j_2 \in \mathcal{J}$, be the locations of the two additional sector errors in the first row.

Since there are precisely r erased symbols in each of the other rows, we can recover $b_{u,v}$ for $u = 2,3,\dots,m$ and $v = 1,2\dots,k$. We can compute $b_{1,1}$ and $b_{1,k}$, and for each $j \in \mathcal{J} \setminus \{j_1,j_2\}$, subtract $b_{1,1} + b_{1,j}\alpha_j^{n-1}$ from the j-th symbol in the first row. We now know the values of

$$\alpha_j \sum_{v=2}^{k-1} b_{1,v} \alpha_j^{v-2}$$

for $j \in \mathcal{J} \setminus \{j_1,j_2\}$. Since $\alpha_j \neq 0$ and $|\mathcal{J} \setminus \{j_1,j_2\}| = k-2$, we can solve for $b_{1,v}$ for $v = 2,3,\dots,k-1$.

Case 2: Suppose that there are $r+1$ sector erasures in each of row 1 and row 2. For $i = 1,2$, let $j_i \in \{1,2,\dots,n\} \setminus \mathcal{J}$ be the location of the erased symbol in row i. Since we can recover all symbols $b_{u,v}$ for $u \geq 3$ and $v = 1,2,\dots,k$, we need to show how to decode the symbols in the first two rows of the matrix $[b_{u,v}]$.

From the unerased symbols in the first two rows of the array, we know the values of the following $2k \times 1$ vector,

$$\left[\begin{array}{ccccc|ccccc} & & \mathbf{D}_{\mathcal{J}\setminus\{j_1\}} & & & & & \mathbf{0} & & \\ \hline & & \mathbf{0} & & & & & \mathbf{D}_{\mathcal{J}\setminus\{j_2\}} & & \\ \hline \beta_1 & 0 & \cdots & 0 & 0 & \beta_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \begin{bmatrix} b_{1,1} \\ \vdots \\ b_{1,k} \\ b_{2,1} \\ \vdots \\ b_{2,k} \end{bmatrix}.$$

The values of $\beta_1 b_{1,1} + \beta_2 b_{2,1}$ and $b_{1,k} + b_{2,k}$ can be obtained from the known values of $b_{u,v}$ for $u = 2, 3, \dots, m$ and $v = 1, k$. Let $\mathbf{A}$ denote the $2k \times 2k$ matrix above. We note that the blocks $\mathbf{D}_{\mathcal{J}\setminus\{j_1\}}$ and $\mathbf{D}_{\mathcal{J}\setminus\{j_2\}}$ have size $(k-1) \times k$, and there are exactly two non-zero entries and $2k-2$ zero entries in each of the last two rows of $\mathbf{A}$. We want to show that $\mathbf{A}$ is non-singular. Let $\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_1\}}$ and $\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_2\}}$ be the vectors

$$\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_1\}} = (\alpha_j)_{j \in \mathcal{J}\setminus\{j_1\}},$$
$$\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_2\}} = (\alpha_j)_{j \in \mathcal{J}\setminus\{j_2\}}.$$

After applying Laplace expansion to the last two rows of $\mathbf{A}$, we can simplify the determinant of $\mathbf{A}$ to

$$\det(\mathbf{A}) = \Big(\frac{\beta_1}{\alpha_{j_1}} - \frac{\beta_2}{\alpha_{j_2}}\Big)\Big(\prod_{j \in \mathcal{J}} \alpha_j\Big) V(\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_1\}}) V(\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_2\}}).$$

By the assumptions in the theorem, the determinant is not equal to zero, and hence we can recover the symbols in the first two rows of the matrix $[b_{u,v}]$. ■

Suppose that the field size q is larger than or equal to the array size mn. Let γ be a primitive element of $\mathbb{F}_q$. We can pick $\beta_i = \gamma^{n(i-1)}$, for $i = 1, 2, \dots, m$, and $\alpha_j = \gamma^{j-1}$, for $j = 1, 2, \dots, n$. Then the values of $\beta_i \alpha_j$ are distinct, and hence $\beta_{i_1}\alpha_{j_1} \neq \beta_{i_2}\alpha_{j_2}$ for distinct i_1 and i_2. For example, we can construct a 3×5 SD code with parameters $r = s = 2$ over $\mathbb{F}_{17}$, by mapping seven source symbols A, B, C, D, E, F and G to a 3×5 array through the following matrix multiplication:

$$\begin{bmatrix} A & B & C \\ D & E & F \\ 2A+10D & G & -C-F \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 8 \end{bmatrix}.$$

With a change of basis, we encode seven source symbols A, B, C, D, E, F and G as in Table II.

We summarize the field size requirements in the following

Corollary 3. *We can construct an $(r, 2; m, n)_q$-SD code if $r \le n-2$ and $q \ge mn$.*

C. $(r, 3)$-SD codes

We choose the matrix $[b_{u,v}]$ such that

$$\beta_1 b_{1,1} + \beta_2 b_{2,1} + \cdots + \beta_m b_{m,1} = 0 \quad (6)$$
$$\gamma_1 b_{1,k-1} + \gamma_2 b_{2,k-1} + \cdots + \gamma_m b_{m,k-1} = 0 \quad (7)$$
$$b_{1,k} + b_{2,k} + \cdots + b_{m,k} = 0, \quad (8)$$

TABLE II

ENCODING OF A $(2,2)$-SECTOR-DISK CODE. THE SYMBOLS A TO G ASSUME VALUES BETWEEN 0 AND 16, AND ARITHMETIC IS PERFORMED MOD 17.

A	B	C	$A-3B$ $+3C$	$3A+9B$ $+6C$
D	E	F	$D-3E$ $+3F$	$3D+9E$ $+6F$
G	$10A+8B$ $-3C+3D$ $-2E+6F$ $+2G$	$2A+B$ $-7C+5D$ $-2E-6F$ $+3G$	$10A-4B$ $+5C+6D$ $-2F+4G$	$10B-C$ $+6D+4E$ $+F+5G$

where β_i's and γ_i's are coefficients to be determined later. The resulting array code has dimension $mk-3$, and each row is a codeword of an $(n, n-3)$ MDS code.

In the rest of this section, we will use the short-hand notation $\mathbf{E}_{\beta,\gamma}$ for the following $3 \times k$ matrix,

$$\mathbf{E}_{\beta,\gamma} := \begin{bmatrix} \beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \gamma & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

for $\beta, \gamma \in \mathbb{F}_q$. Let $\mathcal{J}$ be a set of column indices of size k so that the column with index not in $\mathcal{J}$ are totally erased. We distinguish three types of erasure patterns.

Case 1: Suppose there are r column erasures and 3 more sector erasures in the same row. The decoding of the source symbols is very similar to the proof of Theorem 1 and is omitted.

Case 2: Suppose that there are r column erasures, and the three additional sector erasures are located in two different rows. Without loss of generality, suppose that the three additional sector erasures are located in row 1 and row 2. Suppose that the j_1-th and j_2-th sector in the first row, and the j_3-th sector in the second row are erased. ($j_1, j_2, j_3 \in \mathcal{J}$ and $j_1 \neq j_2$.)

The symbols $b_{u,v}$ for $u = 2, 3, \dots, m$ and $v = 1, 2, \dots, k$ can be obtained from the unerased symbols in row 3 to row m. We want to solve for $b_{u,v}$ for $u = 1, 2$ and $v = 1, 2, \dots, k$. Let $\mathbf{b}_2$ be the $2k \times 1$ column vector

$$\mathbf{b}_2 := (b_{1,1}, \dots, b_{1,k}, b_{2,1}, \dots, b_{2,k})^T$$

obtained by concatenating the first two rows of matrix $[b_{u,v}]$. The value of the following vector of length $2k$ is known,

$$\left[\begin{array}{c|c} \mathbf{D}_{\mathcal{J}\setminus\{j_1,j_2\}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}_{\mathcal{J}\setminus\{j_3\}} \\ \hline \mathbf{E}_{\beta_1,\gamma_1} & \mathbf{E}_{\beta_2,\gamma_2} \end{array}\right] \mathbf{b}_2.$$

Using the determinant of generalized vandermonde matrix, the determinant of the above matrix can be expressed as

$$\Big\{(-1)^k \Big[-\beta_1\gamma_1 \frac{\sum_{j \in \mathcal{J}\setminus\{j_1,j_2\}} \alpha_j}{\alpha_{j_1}\alpha_{j_2}} + \beta_1\gamma_2 \frac{\sum_{j \in \mathcal{J}\setminus\{j_3\}} \alpha_j}{\alpha_{j_1}\alpha_{j_2}} \Big]$$
$$+ \beta_2\gamma_2 \frac{\sum_{j \in \mathcal{J}\setminus\{j_3\}} \alpha_j}{\alpha_{j_3}} \Big\} \Big(\prod_{j \in \mathcal{J}} \alpha_j\Big) V(\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_1,j_2\}}) V(\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_3\}}).$$

TABLE III

ENCODING OF A $(2,3)$-SECTOR-DISK CODE. THE SYMBOLS A TO F ASSUME VALUES BETWEEN 0 AND 16, AND ARITHMETIC IS PERFORMED MOD 17.

A	B	C	$13A+9B$ $+13C$	$12A+13B$ $+10C$
D	E	F	$13D+9E$ $+13F$	$12D+13E$ $+10F$
$3A+5B$ $+2C+5D$ $7E+3F$	$9A+4B$ $+14C+15D$ $9E+8F$	$12A+4B$ $+11C+12D$ $8E+12F$	$4A$ $+6C+16D$ $4E+12F$	$A+16B$ $+10C+D$ $9E+5F$

We can decode the source symbols successfully if the determinant is not equal to zero.

Case 3: Suppose that there are r column erasures, and the three additional sector erasures are located in three different rows. Without loss of generality, suppose that the three additional sector erasures are located in the j_1-th sector in the first row, j_2-th sector in the second row, and j_3-th sector in the third row. Let $\mathbf{b}$ be the $3k\times 1$ column vector

$$\mathbf{b} := (b_{1,1},\dots,b_{1,k},b_{2,1},\dots,b_{2,k},b_{3,1},\dots,b_{3,k})^T$$

obtained by concatenating the first three rows of matrix $[b_{u,v}]$. We know the value of the following $3k\times 1$ column vector,

$$\left[\begin{array}{c|c|c} \mathbf{D}_{\mathcal{J}\setminus\{j_1\}} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}_{\mathcal{J}\setminus\{j_2\}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathcal{J}\setminus\{j_3\}} \\ \hline \mathbf{E}_{\beta_1,\gamma_1} & \mathbf{E}_{\beta_2,\gamma_2} & \mathbf{E}_{\beta_3,\gamma_3} \end{array}\right]\mathbf{b}_3.$$

If we expand the determinant of the above matrix in the last three rows, we get

$$\begin{aligned}&\Big\{\beta_1\gamma_3\frac{\sum_{j\in\mathcal{J}\setminus\{j_3\}}\alpha_j}{\alpha_{j_1}}+\beta_2\gamma_1\frac{\sum_{j\in\mathcal{J}\setminus\{j_1\}}\alpha_j}{\alpha_{j_2}}\\ &+\beta_3\gamma_2\frac{\sum_{j\in\mathcal{J}\setminus\{j_2\}}\alpha_j}{\alpha_{j_3}}-\beta_1\gamma_2\frac{\sum_{j\in\mathcal{J}\setminus\{j_2\}}\alpha_j}{\alpha_{j_1}}\\ &-\beta_2\gamma_3\frac{\sum_{j\in\mathcal{J}\setminus\{j_3\}}\alpha_j}{\alpha_{j_2}}-\beta_3\gamma_1\frac{\sum_{j\in\mathcal{J}\setminus\{j_1\}}\alpha_j}{\alpha_{j_3}}\Big\}\\ &\quad\cdot\Big(\prod_{j\in\mathcal{J}}\alpha_j\Big)V(\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_1\}})V(\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_2\}})V(\boldsymbol{\alpha}_{\mathcal{J}\setminus\{j_3\}}).\end{aligned}$$

We want to choose α's, β's, and γ's such that the determinant is not equal to zero.

Theorem 4. *Suppose that α_j, for $j=1,2,\dots,n$, and β_i and γ_i, for $i=1,2,\dots,m$, satisfy the following conditions:*

1) *For $j=1,\dots,n$, α_j are distinct nonzero elements in $\mathbb{F}_q$.*
2) *For $i=1,\dots,m$, β_i and γ_i are nonzero elements in $\mathbb{F}_q$.*
3) *For each subset $\mathcal{J}$ of $\{1,2,\dots,n\}$ with $|\mathcal{J}|=k$, and $j_1,j_2,j_3\in\mathcal{J}$ with $j_1\neq j_2$,*

$$\begin{aligned}&(-1)^{k+1}\beta_1\gamma_1\frac{\sum_{j\in\mathcal{J}\setminus\{j_1,j_2\}}\alpha_j}{\alpha_{j_1}\alpha_{j_2}}\\ &+(-1)^k\beta_1\gamma_2\frac{\sum_{j\in\mathcal{J}\setminus\{j_3\}}\alpha_j}{\alpha_{j_1}\alpha_{j_2}}+\beta_2\gamma_2\frac{\sum_{j\in\mathcal{J}\setminus\{j_3\}}\alpha_j}{\alpha_{j_3}}\neq 0.\end{aligned}$$

4) *For each subset $\mathcal{J}$ of $\{1,2,\dots,n\}$ with $|\mathcal{J}|=k$, and $j_1,j_2,j_3\in\mathcal{J}$,*

$$\begin{aligned}&\beta_1\gamma_3\frac{\sum_{j\in\mathcal{J}\setminus\{j_3\}}\alpha_j}{\alpha_{j_1}}+\beta_2\gamma_1\frac{\sum_{j\in\mathcal{J}\setminus\{j_1\}}\alpha_j}{\alpha_{j_2}}\\ &+\beta_3\gamma_2\frac{\sum_{j\in\mathcal{J}\setminus\{j_2\}}\alpha_j}{\alpha_{j_3}}-\beta_1\gamma_2\frac{\sum_{j\in\mathcal{J}\setminus\{j_2\}}\alpha_j}{\alpha_{j_1}}\\ &-\beta_2\gamma_3\frac{\sum_{j\in\mathcal{J}\setminus\{j_3\}}\alpha_j}{\alpha_{j_2}}-\beta_3\gamma_1\frac{\sum_{j\in\mathcal{J}\setminus\{j_1\}}\alpha_j}{\alpha_{j_3}}\neq 0,\end{aligned}$$

Then the array codes generated by (1)*, satisfying equations* (6) *to* (8)*, is an $(r,3;m,n)_q$-SD code.*

Altough the construction of $(r,3)$-SD codes given in Theorem 4 is not explicit, the conditions given in Theorem 4 can facilitate the search of $(r,3)$-SD codes by computer. An example of $(2,3)$-SD code of size 3×5 obtained by computer search is given in Table III.

Remark: The construction in this paper can be adapted readily to PMDS codes. The sufficient conditions for PMDS codes are similar to those in Theorems 2 and 4.

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