Characterization of SINR Region for Multiple Interfering Multicast in Power-Controlled Systems

Yi Chen and Chi Wan Sung, *Senior Member, IEEE*

***Abstract*— This paper considers a wireless communication network consisting of multiple interfering multicast sessions. Different from a unicast system where each transmitter has only one receiver, in a multicast system, each transmitter has multiple receivers and broadcasts a common message to all of them. It is a well-known result for wireless unicast systems that the feasibility of a signal-to-interference-plus-noise power ratio (SINR) without power constraint is decided by the spectral radius of a nonnegative matrix. We generalize this result and obtain necessary and sufficient conditions for the feasibility of an SINR in a wireless multicast system with and without power constraint. The feasible SINR region and its geometric properties are studied. Besides, an iterative algorithm is proposed, which can efficiently check the feasibility condition and compute the boundary points of the feasible SINR region.**

***Index Terms*— Wireless multicast system, power control, signal-to-interference-plus-noise power ratio (SINR), SINR region, spectral radius.**

I. Introduction

In WIRELESS communication systems, interference is an inherent phenomenon. Due to the broadcast nature of wireless channels, interference arises whenever multiple transmitter-receiver pairs are active concurrently in the same frequency band, and each receiver is only interested in retrieving information from its own transmitter. For a particular receiver, the received signal is a superposition of its desired signal, interfering signals and background noise. SINR, defined as the power of desired signal divided by the sum of the power of interfering signals and the power of noise, is a widely used performance measure for wireless communication systems. It is analogous to signal-to-noise ratio (SNR) used for single user communication, which has clearly understood implication on the bit error rate (BER) and capacity for additive white Gaussian noise (AWGN) channels. Using SINR as a surrogate for BER and capacity implicitly assumes that the interference is an AWGN. Although there are limitations of this assumption, as reported in [1] and [2], the importance of SINR has never been doubted. Besides, from an information theoretic viewpoint, treating interference as noise is capacity-achieving in Gaussian interference channel with very weak interference [3].

For a system consisting of multiple point-to-point communication sessions, also referred to as a *unicast system*, the SINRs of all receivers form a vector. The feasible SINR region includes all the SINR vectors that can be achieved by some transmission powers. Its geometric properties have been studied in [4]–[6]. Reference [4] proves that in the case of unlimited transmission power, the feasible SINR region is log-convex. In [5], it is shown that under a total power constraint, the infeasible SINR region is not convex. Reference [6] considers a system with only three transmitter-receiver pairs without power constraint, and shows that the feasible SINR region is concave. It also provides certain technical conditions under which a concavity result for systems with a general number of users is established. In [7], for the cases that the transmission powers are subject to arbitrary linear constraints, a mathematical expression for the boundary points of the SINR region is obtained.

In this paper, we consider the feasible SINR region for a system consisting of multiple point-to-multipoint communication sessions, also referred to as a *multicast system*. Multicasting enables data to be delivered from a source node to multiple destination nodes. Practical examples of such configurations include cellular networks and two-way relay networks. In cellular networks, a base station multicasts a common file to multiple mobile devices that request the file at the same time [8]. In two-way relay networks, when the classical three-phase network coding scheme is applied, a relay multicasts the coded packets to two sink nodes in the third phase, which interferes with the tranmission of other relays in the system [9]. Joint power control and scheduling for wireless multicast systems have been studied in [10]–[13] and the references therein. All these works aim to either minimize the system power or maximize the system throughput, subject to the constraint that the SINR of all receivers are larger than a given threshold. The feasibilities of the problems, however, are unknown.

Manuscript received February 9, 2018; revised June 11, 2018 and August 29, 2018; accepted September 19, 2018. Date of publication October 2, 2018; date of current version January 15, 2019. This work is supported in part by the National Natural Science Foundation of China under Grant 61701425, in part by Fundamental Research Project in the Science and Technology Plan of Shenzhen under Grant JCYJ20170306141157272, in part by a grant from the University Grants Committee of the Hong Kong Special Administrative Region, China under Project AoE/E-02/08, and in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China under Project CityU 11216416. The associate editor coordinating the review of this paper and approving it for publication was G. Durisi. *(Corresponding author: Yi Chen.)*

Y. Chen is with the School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen 518000, China, and also with the Shenzhen Research Institute of Big Data, Shenzhen 518000, China (e-mail: yichen@cuhk.edu.cn).

C. W. Sung is with the Department of Electronic Engineering, City University of Hong Kong, Hong Kong (e-mail: albert.sung@cityu.edu.hk).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCOMM.2018.2873304

For a wireless unicast system without power constraint, the feasibility of an SINR vector is determined by the spectral radius of a nonnegative matrix [14], [15]. In this paper, we generalize this result to a wireless multicast system. We first show that an SINR vector is achievable if and only if it is achievable by every embedded unicast system (to be defined later). This seemingly simple result is non-trivial and is proved via a fundamental result in convex analysis called Helly's theorem. Based on this result, we figure out the feasible SINR region by giving its boundary points. The approach is to find the farthest point of the feasible SINR region from the origin in a given direction. An iterative algorithm is proposed to find the farthest point, which is also a distributed power control algorithm to solve the power balancing problem [14] aimed to maximize the minimal SINR of all receivers. The proof technique, however, is different from that used in [14]. The convergence of the algorithm is studied via a new concept called primitive set of nonnegative matrices, which goes beyond the existing literature on power control.

The result described above facilitates the analysis of the geometric properties of the feasible and infeasible SINR regions. We find that the *feasible SINR region of a multicast system* is the *intersection* of the feasible SINR regions of all its embedded unicast systems. Based on the results in [4]–[6] for unicast systems, we show that the feasible SINR region of a multicast system is log-convex, and the infeasible SINR region of a multicast system with two multicast sessions is convex. We also show by an example that, the convexity property of the infeasible SINR region does not hold for a general multicast system with more than two multicast sessions.

Last, we extend the discussion of the SINR feasibility of a multicast system to include some linear constraints on the powers. This extension generalizes the results in [7] which considers unicast systems only.

The rest of the paper is organized as follows: In Section II, the system model and problem formulation are presented. A necessary and sufficient condition on the feasibility of an SINR vector is provided in Section III and its proof is presented in Section IV. Section V gives the characterizations of the SINR region and proposes an iterative algorithm. Section VI studies the geometric properties of the feasible SINR region. Section VII extends the study to include power constraints. Finally, the paper is concluded in Section VIII.

Notation: The following notations are used throughout this paper. Vectors are denoted in bold small letter, e.g., $\mathbf{x}$, with their i-th entry denoted by x_i. They are regarded as column vectors unless stated otherwise. Matrices are denoted by bold capitalized letters, e.g., $\mathbf{X}$, with X_{ij} denoting the (i, j)th entry. Vector and matrix inequalities are component-wise inequalities, e.g., $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ for all i; $\mathbf{X} \geq \mathbf{Y}$ if $X_{ij} \geq Y_{ij}$ for all i and j. The cardinality of a set is denoted by "$|\cdot|$". The Euclidean norm of a vector is denote by "$||\cdot||$". The transpose of a vector or matrix is denoted by $(\cdot)^T$. $\mathbf{I}$ represents an identity matrix with compatible size. $\mathbf{0}$ and $\mathbf{1}$ represent, respectively, all-zero and all-one vectors with compatible size.

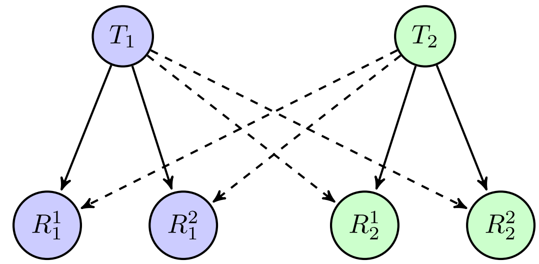


Fig. 1. Example of a multicast network consisting of two multicast sessions. Transmitter T_1 wants to transmit data to both R_1^1 and R_1^2. Transmitter T_2 wants to transmit data to both R_2^1 and R_2^2. Their transmitted signals interfere with each other. The solid lines represent intended links and the dashed lines represent interfering links.

II. System Model

Consider a general wireless communication network consisting of N multicast sessions. The N transmitters are denoted by T_i for $i = 1, \ldots, N$. Each T_i wants to multicast common data packets to K_i receivers, denoted by R_i^k for $k = 1, \ldots, K_i$. If $K_i = 1$ for all i, the scenario reduces to a unicast system. The total number of receivers in the system is $K = \sum_i^N K_i$. Define $\mathcal{K}_i = \{1, 2, \ldots, K_i\}$ for $i = 1, \ldots, N$. Fig. 1 illustrates an example of such a network. Let p_i be the transmission power of transmitter T_i and $\mathbf{p} = [p_1, \ldots, p_N]^T$. The channel gain between T_j and R_i^k is denoted by $g_{i,j}^k$. All the multicast sessions share the same channel and thus interfere with each other. We assume that interference caused by simultaneous transmissions is treated as AWGN with variance identical to the received power. The SINR of receiver R_i^k is given by

$$\gamma_i^k(\mathbf{p}) = \frac{g_{i,i}^k p_i}{\sum_{j \neq i} g_{i,j}^k p_j + \sigma^2}, \tag{1}$$

where σ^2 is the variance of the background noise and is assumed to be identical for all receivers without loss of generality. We define the SINR of the i-th multicast session as

$$\gamma_i(\mathbf{p}) = \min_{k \in \mathcal{K}_i} \left\{ \gamma_i^k(\mathbf{p}) \right\},$$

since the data rate of the i-th multicast session is limited by the minimum SINR of the K_i receivers belonging to that session. The SINR vector of the system is

$$\Gamma(\mathbf{p}) = [\gamma_1(\mathbf{p}), \gamma_2(\mathbf{p}), \ldots, \gamma_N(\mathbf{p})].$$

In this paper, we analyze the feasible SINR region of a multicast system, that is,

$$\Upsilon = \left\{ \Gamma(\mathbf{p}) \in \mathbb{R}_+^N : \mathbf{p} > \mathbf{0}, \mathbf{p} \in \mathbb{R}^N \right\}.$$

Proposition 1: Given a vector $\boldsymbol{\mu} \in \mathbb{R}_+^N$. There exists a power vector $\mathbf{p}^* > \mathbf{0}$ such that $\Gamma(\mathbf{p}^*) = \boldsymbol{\mu}$, if and only if there exists a power vector $\mathbf{p}' > \mathbf{0}$ such that $\Gamma(\mathbf{p}') \geq \boldsymbol{\mu}$.

Proof: The "only if" part is trivial and we show the "if" part. Suppose $\gamma_i(\mathbf{p}') > \mu_i$ for some i. Fix such an i. Since $\gamma_i(\mathbf{p})$ is monotonically decreasing as p_i is decreasing, we can find a $0 < p_i^{(1)} < p_i'$ and let $\mathbf{p}^{(1)} = [p_1', \ldots, p_{i-1}', p_i^{(1)}, p_{i+1}', \ldots, p_N']^T$ such that $\gamma_i(\mathbf{p}^{(1)}) = \mu_i$. On the other hand, since $\gamma_j(\mathbf{p})$ for $j \neq i$

is monotonically increasing as p_i is decreasing, we have $\gamma_j(\mathbf{p}^{(1)}) \geq \mu_j$. By keeping on decreasing the power of transmitters that achieve higer SINR than $\boldsymbol{\mu}$, we obtain a sequence $p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(t)}, \ldots$ for each $i = 1, \ldots, N$. It can be seen that these sequences are monotonically decreasing and bounded below by zero, so they are convergent. Denote the limit point by $\mathbf{p}^*$. For any arbitrarily small $\delta > 0$ and for all i, since $\gamma_i(\mathbf{p})$ is continuous with $\mathbf{p}$, there exists a sufficiently large T, when $t > T$, $|\gamma_i(\mathbf{p}^*) - \gamma_i(\mathbf{p}^{(t)})| < \delta$. Meanwhile, since $\gamma_i(\mathbf{p}^{(t')}) = \mu_i$ for some $t' \geq T$, we have $|\gamma_i(\mathbf{p}^*) - \mu_i| < \delta$. Therefore $\gamma_i(\mathbf{p}^*) = \mu_i$ for all i. □

By Proposition 1, we say that an SINR vector $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_N] > \mathbf{0}$ is *feasible* if and only if there exists a power vector $\mathbf{p} > \mathbf{0}$ such that

$$p_i - \sum_{j \neq i} \mu_i \frac{g_{i,j}^k}{g_{i,i}^k} p_j \geq \mu_i \frac{\sigma^2}{g_{i,i}^k}, \quad \forall k \in \mathcal{K}_i, \ \forall i. \tag{2}$$

In matrix form, it is

$$\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \geq \mathbf{n}(\boldsymbol{\mu}), \tag{3}$$

where

$$\mathbf{A}(\boldsymbol{\mu}) = \begin{bmatrix} 1 & -\mu_1 \frac{g_{1,2}^1}{g_{1,1}^1} & \cdots & -\mu_1 \frac{g_{1,N}^1}{g_{1,1}^1} \\ 1 & -\mu_1 \frac{g_{1,2}^2}{g_{1,1}^2} & \cdots & -\mu_1 \frac{g_{1,N}^2}{g_{1,1}^2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -\mu_1 \frac{g_{1,2}^{K_1}}{g_{1,1}^{K_1}} & \cdots & -\mu_1 \frac{g_{1,N}^{K_1}}{g_{1,1}^{K_1}} \\ -\mu_2 \frac{g_{2,1}^1}{g_{2,2}^1} & 1 & \cdots & -\mu_2 \frac{g_{2,N}^1}{g_{2,2}^1} \\ \vdots & \vdots & \ddots & \vdots \\ -\mu_2 \frac{g_{2,1}^{K_2}}{g_{2,2}^{K_2}} & 1 & \cdots & -\mu_2 \frac{g_{2,N}^{K_2}}{g_{2,2}^{K_2}} \\ \vdots & \vdots & \ddots & \vdots \\ -\mu_N \frac{g_{N,1}^{K_N}}{g_{N,N}^{K_N}} & -\mu_N \frac{g_{N,2}^{K_N}}{g_{N,N}^{K_N}} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^1 \\ \mathbf{a}_1^2 \\ \vdots \\ \mathbf{a}_1^{K_1} \\ \mathbf{a}_2^1 \\ \vdots \\ \mathbf{a}_2^{K_2} \\ \vdots \\ \mathbf{a}_N^{K_N} \end{bmatrix}$$
$$\in \mathbb{R}^{K \times N} \tag{4}$$

and

$$\mathbf{n}(\boldsymbol{\mu}) = \Big[\overbrace{\frac{\mu_1\sigma^2}{g_{1,1}^1}, \frac{\mu_1\sigma^2}{g_{1,1}^2} \ldots, \frac{\mu_1\sigma^2}{g_{1,1}^{K_1}}}^{K_1}, \ldots, \overbrace{\frac{\mu_N\sigma^2}{g_{N,N}^1}, \ldots, \frac{\mu_N\sigma^2}{g_{N,N}^{K_N}}}^{K_N}\Big]^T$$
$$= [n_1^1, n_1^2 \ldots, n_1^{K_1}, \ldots, n_N^1, \ldots, n_N^{K_N}]^T \in \mathbb{R}^{K \times 1}.$$

Each row of $\mathbf{A}(\boldsymbol{\mu})$ corresponds to a receiver. For the convenience of discussion, we use $\mathbf{a}_i^k \in \mathbb{R}^N$ to denote the row of $\mathbf{A}(\boldsymbol{\mu})$ that corresponds to receiver R_i^k. As in the form (3), the feasibility of $\boldsymbol{\mu}$ can be checked through linear programming [16]. However, in a different way, we propose a necessary and sufficient condition on the feasibility, which generalizes the Perron-Frobenius theory for unicast systems (square matrices). This condition is used to explicitly characterize the feasible SINR region Υ and to prove some geometric

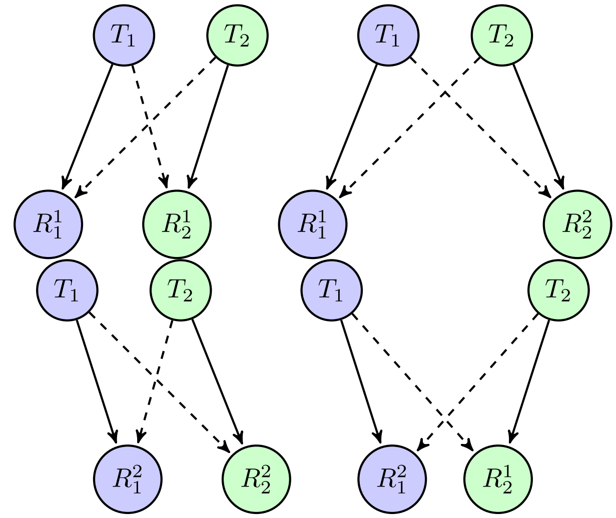


Fig. 2. The four embedded unicast systems for the multicast network shown in Fig. 1

properties of it. Before further discussion, we give some definitions.

Define set

$$\mathcal{G}(\boldsymbol{\mu}) = \left\{ \mathbf{G} = \begin{bmatrix} \mathbf{a}_1^{k_1} \\ \mathbf{a}_2^{k_2} \\ \vdots \\ \mathbf{a}_N^{k_N} \end{bmatrix} \in \mathbb{R}^{N \times N} : k_i \in \mathcal{K}_i \text{ for } i = 1, \ldots, N \right\}.$$

Notice that, for each $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$, only one receiver is involved for each transmitter, which is a unicast scenario. So $\mathcal{G}(\boldsymbol{\mu})$ is the set including all the embedded unicast systems and its size is $\prod_{i=1}^N K_i$. Considering the example in Fig. 1, its four embedded unicast systems are illustrated in Fig. 2 and each corresponds to one $\mathbf{G} \in \mathcal{G}$ given below

$$\mathcal{G}(\boldsymbol{\mu}) = \left\{ \begin{bmatrix} \mathbf{a}_1^1 \\ \mathbf{a}_2^1 \end{bmatrix}, \begin{bmatrix} \mathbf{a}_1^1 \\ \mathbf{a}_2^2 \end{bmatrix}, \begin{bmatrix} \mathbf{a}_1^2 \\ \mathbf{a}_2^2 \end{bmatrix}, \begin{bmatrix} \mathbf{a}_1^2 \\ \mathbf{a}_2^1 \end{bmatrix} \right\}.$$

In subsequent discussion, we also use $\mathbf{k} = (k_1, k_2, \ldots, k_N)$ to specify a $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$. Let $\mathbf{n}_\mathbf{G} = [n_1^{k_1}, n_2^{k_2}, \ldots, n_N^{k_N}]^T$ denote the noise vector with entries of $\mathbf{n}(\boldsymbol{\mu})$ that correspond to the receivers in $\mathbf{G}$. Note that $\mathbf{n}_\mathbf{G} > \mathbf{0}$ for all $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$. Define

$$\mathcal{Z}(\boldsymbol{\mu}) = \{\mathbf{Z} = \mathbf{I} - \mathbf{G} : \mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})\}. \tag{5}$$

Note that $\mathbf{Z} \geq \mathbf{0}$ for all $\mathbf{Z} \in \mathcal{Z}(\boldsymbol{\mu})$ and $\mathcal{Z}(\mathbf{1})$ includes the normalized interference link gain matrices of all the embedded unicast systems. For the simplicity of notation, we sometimes drop the argument $\boldsymbol{\mu}$ of $\mathbf{A}$, $\mathbf{n}$, $\mathcal{G}$ and $\mathcal{Z}$ when the context is clear.

III. Main Result: Feasibility Condition

In this section, we give a necessary and sufficient condition for the feasibility of an SINR vector in a wireless multicast system. We first give some definitions and recall the results for a wireless unicast system.

Definition 1: Let $\mathbf{X}$ *be an* $n \times n$ *matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$*. The* spectral radius *of* $\mathbf{X}$*, denoted by* $\lambda(\mathbf{X})$*, is* $\lambda(\mathbf{X}) = \max_{1 \leq i \leq n} ||\lambda_i||$.

Definition 2: A matrix $\mathbf{X}$ *is called a* Z-matrix *if it is square and all its off-diagonal entries are nonpositive, i.e.,* $X_{ij} \leq 0$ *for all* $i, j, i \neq j$.

Theorem 1 [17], [18]: Let $\mathbf{X}$ *be a* Z*-matrix, the following properties are equivalent:*

1) *There exists a vector* $\mathbf{y} > \mathbf{0}$ *such that* $\mathbf{X}\mathbf{y} > \mathbf{0}$.
2) *There exists a splitting* $\mathbf{X} = \mathbf{A} - \mathbf{B}$ *of* $\mathbf{X}$ *such that* $\mathbf{A}^{-1} \geq \mathbf{0}$, $\mathbf{B} \geq \mathbf{0}$ *and* $\lambda(\mathbf{A}^{-1}\mathbf{B}) < 1$.
3) *For any splitting* $\mathbf{X} = \mathbf{A} - \mathbf{B}$ *of* $\mathbf{X}$ *satisfying* $\mathbf{A}^{-1} \geq \mathbf{0}$, $\mathbf{B} \geq \mathbf{0}$, *it holds* $\lambda(\mathbf{A}^{-1}\mathbf{B}) < 1$.
4) *All principal minors of* $\mathbf{X}$ *are positive.*
5) $\mathbf{X}$ *is nonsingular and* $\mathbf{X}^{-1} \geq \mathbf{0}$.

The feasibility condition of an SINR vector for a wireless unicast system can be derived from Theorem 1 directly, and is summarized as follows.

Theorem 2 [14], [15]: Consider a unicast network setting $\mathbf{G}$. *It is clear that* $\mathbf{G}$ *is a* Z*-matrix. Let* $\mathbf{Z} = \mathbf{I} - \mathbf{G}$. *Then there exists a power vector* $\mathbf{p} > \mathbf{0}$ *such that* $\mathbf{G}\mathbf{p} \geq \mathbf{n}_\mathbf{G} > \mathbf{0}$, *if and only if* $\lambda(\mathbf{Z}) < 1$. *In this case,* $\mathbf{G}$ *is nonsingular and* $\mathbf{G}^{-1} \geq \mathbf{0}$ *exists. Therefore* $\mathbf{p} = \mathbf{G}^{-1}\mathbf{n}_\mathbf{G}$ *exists.*

For a wireless multicast system, since the link gain matrix is no longer a square matrix, Theorem 1 cannot be applied directly. The main result of this paper is the following theorem.

Theorem 3: Consider a multicast network setting $\mathbf{A}(\boldsymbol{\mu})$. *There exists a power vector* $\mathbf{p} > \mathbf{0}$ *such that* $\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \geq \mathbf{n}(\boldsymbol{\mu}) > \mathbf{0}$ *if and only if* $\max_{\mathbf{Z} \in \mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\} < 1$.

Theorem 3 basically says that for a wireless multicast system, an SINR vector $\boldsymbol{\mu}$ is feasible if and only if $\boldsymbol{\mu}$ is feasible to all of its embedded unicast systems. The proof of the forward part is straightforward. Suppose there exists a power vector $\mathbf{p} > \mathbf{0}$ such that $\mathbf{A}\mathbf{p} \geq \mathbf{n}$. Then for all $\mathbf{G} \in \mathcal{G}$, we have $\mathbf{G}\mathbf{p} \geq \mathbf{n}_\mathbf{G}$. By Theorem 2, $\lambda(\mathbf{I} - \mathbf{G}) = \lambda(\mathbf{Z}) < 1$ for all $\mathbf{G}$, which implies $\max_{\mathbf{Z} \in \mathcal{Z}}\{\lambda(\mathbf{Z})\} < 1$.

It is intuitively clear that for the multicast system to be feasible, all its embedded unicast systems must be feasible. The converse, however, is not at all obvious. An embedded unicast system being feasible merely implies that there is a feasible power vector for that particular embedded unicast system. There is no straightforward reason why there is a *common* feasible power vector for *all* embedded unicast system. The elegant fact established by Theorem 3 is that such a common feasible power vector does exist. The proof is rather involved and is presented in the next section.

Corollary 1: When there are only two multicast sessions, i.e., $N = 2$, *the feasibility of* $\boldsymbol{\mu}$ *is determined by the unicast system specified by*

$$\mathbf{G}^* = \begin{bmatrix} \mathbf{a}_1^{k_1^*} \\ \mathbf{a}_2^{k_2^*} \end{bmatrix} \quad \textit{where } k_i^* = \arg\max_{k \in \mathcal{K}_i}\{\mu_i \frac{g_{i,j}^k}{g_{i,i}^k}\}, \quad i = 1, 2, j \neq i.$$

That is, $\boldsymbol{\mu}$ *is feasible if and only if* $\lambda(\mathbf{I} - \mathbf{G}^*) = \lambda(\mathbf{Z}^*) < 1$.

Corollary 1 follows straightforwardly from Theorem 3. Note that when $N = 2$, for any $\mathbf{Z} \in \mathcal{Z}$, we have

$$\lambda(\mathbf{Z}) = \sqrt{\mu_1 \frac{g_{1,2}^{k_1}}{g_{1,1}^{k_1}} \times \mu_2 \frac{g_{2,1}^{k_2}}{g_{2,2}^{k_2}}} \tag{6}$$

for some $k_1 \in \mathcal{K}_1$ and $k_2 \in \mathcal{K}_2$. So $\max_{\mathbf{Z} \in \mathcal{Z}}\{\lambda(\mathbf{Z})\} = \lambda(\mathbf{Z}^*)$.

IV. Proof of the Converse of Theorem 3

In this section, we prove the sufficient condition and assume that $\max_{\mathbf{Z} \in \mathcal{Z}}\{\lambda(\mathbf{Z})\} < 1$. By Theorem 2, it indicates that for each $\mathbf{G} \in \mathcal{G}$, $\mathbf{G}^{-1} \geq \mathbf{0}$ exists, and thus $\mathbf{p} = \mathbf{G}^{-1}\mathbf{n}_\mathbf{G} \geq \mathbf{0}$ exists. For each receiver, define

$$\mathcal{A}_i^k = \{\mathbf{p} \in \mathbb{R}^N : \mathbf{a}_i^k \mathbf{p} \geq n_i^k, \mathbf{p} \geq 0\}.$$

Note that $\mathbf{a}_i^k$ is a row vector as defined in (4). $\mathcal{A}_i^k$ is an intersection of half-spaces and thus is convex. Our proof is based on Helly's theorem given below.

Theorem 4 ***(Helly's Theorem)*** *[19]: Let* $\mathcal{F}$ *be a finite collection of convex sets in* $\mathbb{R}^N$. *The intersection of all the sets of* $\mathcal{F}$ *is non-empty if and only if every* $N + 1$ *of them has non-empty intersection.*

In our case,

$$\mathcal{F} = \{\mathcal{A}_i^k : i = 1, \ldots, N, k \in \mathcal{K}_i\}. \tag{7}$$

There are in total K convex sets in $\mathcal{F}$, each corresponding to one receiver. If every $N + 1$ of them have non-empty intersection, then all of them have non-empty intersection and the intersection points must satisfy $\mathbf{p} > \mathbf{0}$, which implies that the SINR vector is feasible. The number of all combinations of such $N + 1$ sets is $\binom{K}{N+1}$. We first show the proof for $N = 2$. Then we use the mathematical induction to prove the general case.

Lemma 1: Suppose $\mathbf{X} = (X_{ij})$ *is an* $N \times N$ *matrix satisfying* $X_{ij} = 1$ *for* $i = j$ *and* $X_{ij} \leq 0$ *for* $i \neq j$. *Let* $\mathcal{S}$ *be a subset of* $\{1, \ldots, N\}$ *and* $\mathbf{X}'$ *be the matrix by removing the* i*-th row and* i*-th column of* $\mathbf{X}$ *for all* $i \in \mathcal{S}$. *If* $\lambda(\mathbf{I} - \mathbf{X}) < 1$, *then* $\lambda(\mathbf{I} - \mathbf{X}') < 1$.

Proof: Note that all the principal minors of $\mathbf{X}'$ are also the principal minors of $\mathbf{X}$. The lemma follows directly from Theorem 1. □

Lemma 2: Consider $\hat{\mathbf{G}}, \tilde{\mathbf{G}} \in \mathcal{G}$ *such that* $\hat{\mathbf{G}}$ *differs from* $\tilde{\mathbf{G}}$ *only in one row. i.e.,* $\hat{k}_i \neq \tilde{k}_i$ *for one* $i \in \{1, \ldots, N\}$ *and* $\hat{k}_j = \tilde{k}_j$ *for* $j \neq i$. *Let* $\hat{\mathbf{p}} = \hat{\mathbf{G}}^{-1}\mathbf{n}_{\hat{\mathbf{G}}}$ *and* $\tilde{\mathbf{p}} = \tilde{\mathbf{G}}^{-1}\mathbf{n}_{\tilde{\mathbf{G}}}$. *There exists* $\mathbf{p} \in \{\hat{\mathbf{p}}, \tilde{\mathbf{p}}\}$ *such that* $\tilde{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\tilde{\mathbf{G}}}$ *and* $\hat{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\hat{\mathbf{G}}}$.

Proof: Without loss of generality, assume that $\hat{\mathbf{G}}$ and $\tilde{\mathbf{G}}$ differ in the first row, that is, $\hat{k}_1 \neq \tilde{k}_1$ and $\hat{k}_j = \tilde{k}_j$ for $j \neq 1$. Let us partition $\hat{\mathbf{G}}$ into four blocks as follows.

$$\hat{\mathbf{G}} = \left[\begin{array}{c|ccc} 1 & -\mu_1 \frac{g_{1,2}^{\hat{k}_1}}{g_{1,1}^{\hat{k}_1}} & \cdots & -\mu_1 \frac{g_{1,N}^{\hat{k}_1}}{g_{1,1}^{\hat{k}_1}} \\ \hline -\mu_2 \frac{g_{2,1}^{\hat{k}_2}}{g_{2,2}^{\hat{k}_2}} & 1 & \cdots & -\mu_2 \frac{g_{2,N}^{\hat{k}_2}}{g_{2,2}^{\hat{k}_2}} \\ \vdots & \vdots & \ddots & \vdots \\ -\mu_N \frac{g_{N,1}^{\hat{k}_N}}{g_{N,N}^{\hat{k}_N}} & -\mu_N \frac{g_{N,2}^{\hat{k}_N}}{g_{N,N}^{\hat{k}_N}} & \cdots & 1 \end{array}\right] = \left[\begin{array}{c|c} 1 & \mathbf{A} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right].$$

Similarly, $\tilde{\mathbf{G}}$ is partitioned into four blocks as

$$\tilde{\mathbf{G}} = \left[\begin{array}{c|c} 1 & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right].$$

Note that $\hat{\mathbf{G}}$ and $\tilde{\mathbf{G}}$ share the same three blocks: $1, \mathbf{C}$ and $\mathbf{D}$.

Since $\hat{\mathbf{p}} = \hat{\mathbf{G}}^{-1}\mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{G}}^{-1}\mathbf{n}_{\tilde{\mathbf{G}}}$, we have $\hat{\mathbf{G}}\hat{\mathbf{p}} = \mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{p}} = \mathbf{n}_{\tilde{\mathbf{G}}}$. Now we consider $\hat{\mathbf{G}}\tilde{\mathbf{p}} - \mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{G}}\hat{\mathbf{p}} - \mathbf{n}_{\tilde{\mathbf{G}}}$. Since $\mathbf{a}_i^{\tilde{k}_i} = \mathbf{a}_i^{\hat{k}_i}$ and $n_{\tilde{k}_i} = n_{\hat{k}_i}$ for $i = 2, \ldots, N$, $\mathbf{a}_i^{\hat{k}_i}\tilde{\mathbf{p}} = \mathbf{a}_i^{\tilde{k}_i}\tilde{\mathbf{p}} = n_{\tilde{k}_i} = n_{\hat{k}_i}$ and $\mathbf{a}_i^{\tilde{k}_i}\hat{\mathbf{p}} = \mathbf{a}_i^{\hat{k}_i}\hat{\mathbf{p}} = n_{\hat{k}_i} = n_{\tilde{k}_i}$ for $i = 2, \ldots, N$. Therefore we only need to consider $\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} - n_{\hat{k}_1}$ and $\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} - n_{\tilde{k}_1}$. If $\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} = n_{\hat{k}_1}$ or $\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} = n_{\tilde{k}_1}$, then $\hat{\mathbf{G}}\tilde{\mathbf{p}} = \mathbf{n}_{\hat{\mathbf{G}}}$ or $\tilde{\mathbf{G}}\hat{\mathbf{p}} = \mathbf{n}_{\tilde{\mathbf{G}}}$, which implies $\hat{\mathbf{p}} = \tilde{\mathbf{p}}$ and we are done. If $\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} \neq n_{\tilde{k}_1}$ and $\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} \neq n_{\hat{k}_1}$, in the following we prove that, either $\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} > n_{\tilde{k}_1}$ or $\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} > n_{\hat{k}_1}$ but not both, that is, $(\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} - n_{\tilde{k}_1})(\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} - n_{\hat{k}_1}) < 0$.

Since $\hat{\mathbf{G}}^{-1} \geq \mathbf{0}$ exists, by Theorem 1 and Lemma 1, $\mathbf{D}^{-1}$ exists. By block-wise inversion [20], the inverse of $\hat{\mathbf{G}}$ can be written as

$$\hat{\mathbf{G}}^{-1} = \begin{bmatrix} a & -a\mathbf{A}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}a & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}a\mathbf{A}\mathbf{D}^{-1} \end{bmatrix},$$

where $a = (1 - \mathbf{A}\mathbf{D}^{-1}\mathbf{C})^{-1}$. We claim that a must satisfy $a > 0$. Otherwise the entries of the first row of $\mathbf{G}^{-1}$ are all zero, which contradicts with that $\mathbf{G}$ is invertible. Similarly, $\tilde{\mathbf{G}}^{-1}$ is in the same form by replacing $\mathbf{A}$ with $\mathbf{B}$ and replacing a with $b = (1 - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} > 0$. Denote $\mathbf{n}_{\hat{\mathbf{G}}} = \begin{bmatrix} n_{\hat{k}_1} \\ \mathbf{n}' \end{bmatrix}$ and $\mathbf{n}_{\tilde{\mathbf{G}}} = \begin{bmatrix} n_{\tilde{k}_1} \\ \mathbf{n}' \end{bmatrix}$ where $\mathbf{n}' \in \mathbb{R}^{N-1}$. We have

$$\begin{aligned} &\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} - n_{\tilde{k}_1} \\ &\quad = \begin{bmatrix} 1 \ \mathbf{B} \end{bmatrix} \hat{\mathbf{G}}^{-1}\mathbf{n}_{\hat{\mathbf{G}}} - n_{\tilde{k}_1} \\ &\quad = a n_{\hat{k}_1}(1 - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}) - (1 - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})a\mathbf{A}\mathbf{D}^{-1}\mathbf{n}' \\ &\qquad + \mathbf{B}\mathbf{D}^{-1}\mathbf{n}' - n_{\tilde{k}_1} \\ &\quad = -ab^{-1}(\mathbf{A}\mathbf{D}^{-1}\mathbf{n}' - n_{\hat{k}_1}) + (\mathbf{B}\mathbf{D}^{-1}\mathbf{n}' - n_{\tilde{k}_1}). \end{aligned}$$

Similarly, we have

$$\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} - n_{\hat{k}_1} = -ba^{-1}(\mathbf{B}\mathbf{D}^{-1}\mathbf{n}' - n_{\tilde{k}_1}) + (\mathbf{A}\mathbf{D}^{-1}\mathbf{n}' - n_{\hat{k}_1}).$$

Then

$$\begin{aligned} &(\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} - n_{\tilde{k}_1})(\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} - n_{\hat{k}_1}) \\ &= -ab^{-1}(\mathbf{A}\mathbf{D}^{-1}\mathbf{n}' - n_{\hat{k}_1})^2 - ba^{-1}(\mathbf{B}\mathbf{D}^{-1}\mathbf{n}' - n_{\tilde{k}_1})^2 \\ &\quad + 2(\mathbf{A}\mathbf{D}^{-1}\mathbf{n}' - n_{\hat{k}_1})(\mathbf{B}\mathbf{D}^{-1}\mathbf{n}' - n_{\tilde{k}_1}) \\ &= -\left[\sqrt{ab^{-1}}(\mathbf{A}\mathbf{D}^{-1}\mathbf{n}' - n_{\hat{k}_1}) - \sqrt{ba^{-1}}(\mathbf{B}\mathbf{D}^{-1}\mathbf{n}' - n_{\tilde{k}_1})\right]^2 \\ &\leq 0. \end{aligned}$$

Further, since $\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} \neq n_{\tilde{k}_1}$ and $\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} \neq n_{\hat{k}_1}$, $(\mathbf{a}_1^{\tilde{k}_1}\hat{\mathbf{p}} - n_{\tilde{k}_1})(\mathbf{a}_1^{\hat{k}_1}\tilde{\mathbf{p}} - n_{\hat{k}_1}) < 0$. In summary, there exists $\mathbf{p} \in \{\hat{\mathbf{p}}, \tilde{\mathbf{p}}\}$ such that $\tilde{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\tilde{\mathbf{G}}}$ and $\hat{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\hat{\mathbf{G}}}$. □

A. Two Multicast Sessions $N = 2$

If $K = 2$, i.e., $K_1 = K_2 = 1$, it is the unicast scenario and Theorem 3 is true straightforwardly. For other values of K_1 and K_2, we divide the $\binom{K_1+K_2}{3}$ combinations of three sets of $\mathcal{F}$ into two kinds: 1) two sets belong to transmitter T_i and one set belongs to transmitter T_j where $j \neq i$; 2) three sets belong to the same transmitter T_i for $i = 1$ or 2. In the first case, the three sets could be $\mathcal{A}_i^{k_i}, \mathcal{A}_i^{k_i'}, \mathcal{A}_j^{k_j}$ for $i = 1$ or 2 and $j \neq i$. Let

$$\hat{\mathbf{G}} = \begin{bmatrix} \mathbf{a}_i^{k_i} \\ \mathbf{a}_j^{k_j} \end{bmatrix} \quad \text{and } \tilde{\mathbf{G}} = \begin{bmatrix} \mathbf{a}_i^{k_i'} \\ \mathbf{a}_j^{k_j} \end{bmatrix}.$$

By Lemma 2, there exists $\mathbf{p}$ such that $\tilde{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\tilde{\mathbf{G}}}$ and $\hat{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\hat{\mathbf{G}}}$, which implies $\mathbf{p} \in (\mathcal{A}_i^{k_i} \cap \mathcal{A}_i^{k_i'} \cap \mathcal{A}_j^{k_j}) \neq \emptyset$. In the second case, the three sets could be $\mathcal{A}_i^{k_i}, \mathcal{A}_i^{k_i'}, \mathcal{A}_i^{k_i''}$ for $i = 1$ or 2. It can be verified that $p_i = \max\{n_i^{k_i}, n_i^{k_i'}, n_i^{k_i''}\}$ and $p_j = 0$ is one of their intersection points. Overall, we prove that every three sets of $\mathcal{F}$ have a non-empty intersection, and thus the intersection of all sets is non-empty.

B. Multicast Sessions With General N

We use mathematical induction to prove Theorem 3. We already show that it is true when $N = 2$. Assume that the theorem holds for all numbers less than or equal to $N - 1$ and now we prove that it also holds for N. If $K_i = 1$ for all i, it is the unicast scenario and Theorem 3 is true. Otherwise, we categorize the combinations of $N + 1$ sets of $\mathcal{F}$ into N parts: 1) Receivers of N transmitters are involved: $\mathcal{A}_1^{k_1}, \mathcal{A}_2^{k_2}, \ldots, \mathcal{A}_N^{k_N}, \mathcal{A}_i^{k_i'}$. 2) Receivers of $N - 1$ transmitters are involved: $\mathcal{A}_1^{k_1}, \ldots, \mathcal{A}_{j-1}^{k_{j-1}}, \mathcal{A}_{j+1}^{k_{j+1}}, \ldots, \mathcal{A}_N^{k_N}, \mathcal{A}_i^{k_i'}, \mathcal{A}_l^{k_l'}$ where $i, l \neq j$. $\cdots$ D) Receivers of $N - D + 1$ transmitters are involved. $\cdots$ N) Receivers of 1 transmitter are involved. We prove the first part. Let

$$\hat{\mathbf{G}} = \begin{bmatrix} \mathbf{a}_1^{k_1} \\ \vdots \\ \mathbf{a}_i^{k_i} \\ \vdots \\ \mathbf{a}_N^{N_j} \end{bmatrix} \quad \text{and } \tilde{\mathbf{G}} = \begin{bmatrix} \mathbf{a}_1^{k_1} \\ \vdots \\ \mathbf{a}_i^{k_i'} \\ \vdots \\ \mathbf{a}_N^{N_j} \end{bmatrix}.$$

By Lemma 2, there exists $\mathbf{p}$ such that $\tilde{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\tilde{G}}$ and $\hat{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\hat{G}}$, which implies $\mathbf{p} \in (\cap_{j=1}^{N}\mathcal{A}_j^{k_j} \cap \mathcal{A}_i^{k_i'})$.

We prove the D) part for $D = 2, \ldots, N$. Suppose the $D - 1$ transmitters whose receivers are not involved in the $N + 1$ sets, are T_d for $d = d_1, d_2, \ldots, d_{D-1} \in \{1, \cdots, N\}$. We simply let $p_{d_1} = p_{d_2} = \cdots = p_{d_{D-1}} = 0$. The resulting system is equivalent to having $N - D + 1$ multicast sessions characterized by matrix $\mathbf{A}'$, which is constructed by removing the rows in $\mathbf{A}$ that correspond to the receivers of transmitter d and the d-th column of $\mathbf{A}$, for $d = d_1, \ldots, d_{D-1}$. Define $\mathcal{G}' \subset \mathbb{R}^{(N-D+1)\times(N-D+1)}$ for $\mathbf{A}'$. For any $\mathbf{G}' \in \mathcal{G}'$, we can find a $\mathbf{G} \in \mathcal{G}$ such that, $\mathbf{G}'$ is constructed by removing the d-th row and d-th column of $\mathbf{G}$ for all $d = d_1, \ldots, d_{D-1}$. Since $\lambda(\mathbf{I} - \mathbf{G}) < 1$ for all $\mathbf{G} \in \mathcal{G}$, by Lemma 1, $\lambda(\mathbf{I} - \mathbf{G}') < 1$, and therefore $\max_{\mathbf{G}' \in \mathcal{G}'}\{\lambda(\mathbf{I} - \mathbf{G}')\} < 1$. By the inductive hypothesis, we can apply Theorem 3 when $N - D + 1 < N$, and thus there exists $\mathbf{p}' \geq \mathbf{0}$ such that $\mathbf{A}'\mathbf{p}' \geq \mathbf{n}'$, where $\mathbf{n}'$ is obtained by removing the entries of $\mathbf{n}$ that correspond to the receivers of transmitter T_d for $d = d_1, \ldots, d_{D-1}$. By inserting 0 back into $\mathbf{p}'$ at the positions of transmitter T_d for all $d = d_1, \ldots, d_{D-1}$, we get a power $\mathbf{p} \geq \mathbf{0}$ which is in the $N + 1$ subsets.

Overall, we have proved that every $N+1$ subsets of $\mathcal{F}$ has a non-empty intersection. By Helly's theorem, all subsets in $\mathcal{F}$ have an intersection. This completes the proof of Theorem 3.

V. Feasible SINR Region and Algorithm

In this section, we characterize the feasible SINR region of a wireless multicast system by analytically obtaining its boundary points. By Proposition 1, we know that the feasible SINR region is downward comprehensive. That is, if $\boldsymbol{\mu}$ is feasible, then any $\boldsymbol{\mu}'$ satisfying $\mathbf{0} < \boldsymbol{\mu}' \leq \boldsymbol{\mu}$ is also feasible. Therefore, finding the boundary points is enough to figure out the feasible SINR region. Our approach is to find the farthest point from the origin in a given direction. In mathematics, the problem is formulated as

$$\begin{aligned} \sup_{\mathbf{p}} \quad & \beta \\ s.t. \quad & \mathbf{A}(\beta\boldsymbol{\mu})\mathbf{p} \geq \mathbf{n}(\beta\boldsymbol{\mu}) \\ & \mathbf{p} \geq \mathbf{0}, \end{aligned}$$

where $\boldsymbol{\mu}$ is a given direction. By Theorem 3, there is a feasible solution to the above problem if and only if

$$\max_{\mathbf{Z}\in\mathcal{Z}(\beta\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\} = \beta \cdot \max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\} < 1.$$

It is known that for nonnegative matrix $\mathbf{Z} \geq \mathbf{0}$, $\lambda(\mathbf{Z}) \geq \min_{1\leq i\leq N}\sum_{j=1}^{N} Z_{ij}$ and $\lambda(\mathbf{Z}) \geq \min_{1\leq j\leq N}\sum_{i=1}^{N} Z_{ij}$ [21]. So if $\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\} = 0$, then for all $\mathbf{Z} \in \mathcal{Z}$, $\min_{1\leq i\leq N}\sum_{j=1}^{N} Z_{ij} = \min_{1\leq j\leq N}\sum_{i=1}^{N} Z_{ij} = 0$. In other words, for all $\mathbf{Z} \in \mathcal{Z}$, there exists a row and a column whose entries are all zero. It means that there exists an isolated multicast session whose receivers receive no interference from any other multicast session, and an isolated multicast session which generates no interference to any other multicast sessions. Such case is of no practical interests and therefore, we assume that $\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\} > 0$. Then

$$\beta < \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}.$$

Therefore, the optimal value is

$$\beta^*(\boldsymbol{\mu}) = \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}.$$

$\beta^*(\boldsymbol{\mu})\boldsymbol{\mu}$ is a boundary point of the SINR region. The open line segment defined by $\{\alpha\boldsymbol{\mu} : 0 < \alpha < \beta^*(\boldsymbol{\mu})\}$ is in the feasible SINR region Υ, but $\alpha\boldsymbol{\mu}$ is not in the feasible region if $\alpha > \beta^*(\boldsymbol{\mu})$.

We note that the size of $\mathcal{Z}$ is $\prod_{i=1}^{N} K_i$, which grows exponentially with N. It is not an efficient method to calculate the spectral radius of all the embedded unicast systems and find out the maximum one. To circumvent this difficulty, we propose an iterative algorithm to compute $\beta^*(\boldsymbol{\mu})$. For $i = 1, 2, \ldots, N$, let $\mathbf{e}_i$ denote the N-dimensional column vector such that the i-th component of $\mathbf{e}_i$ is 1 while the others are 0. The algorithm is described in Algorithm 1. In line 4 of Algorithm 1, $(\mathbf{e}_i^T - \mathbf{a}_i^{k_i})\mathbf{p}^{(k)}$ is the sum of interference power at receiver $R_i^{k_i}$, and the power of transmitter T_i is updated to the maximum interference power experienced by the receivers in its multicast session. In line 7, the power vector is normalized to be of unit norm. This idea is similar to the distributed power control algorithm for unicast systems [22] to solve the power balancing problem. Recall that in [22], given a normalized interference link gain matrix $\mathbf{Z}$, the algorithm works as $\mathbf{p}^{(k+1)} = \frac{\mathbf{Z}\mathbf{p}^{(k)}}{||\mathbf{Z}\mathbf{p}^{(k)}||}$, where k is the iteration index. It is well known that when $\mathbf{Z}$ is primitive (to be defined later), $||\mathbf{Z}\mathbf{p}^{(k)}||$ converges to the spectral radius, or equivalently the Perron-Frobenius eigenvalue of $\mathbf{Z}$, and $\mathbf{p}^{(k)}$ converges to the corresponding eigenvector. In our proposed algorithm, we are dealing with multicast systems. Given any $\mathbf{p} > 0$, due to the structure of $\mathcal{Z}(\boldsymbol{\mu})$, there always exists $\hat{\mathbf{Z}} \in \mathcal{Z}$ such that $\hat{\mathbf{Z}}\mathbf{p} \geq \mathbf{Z}\mathbf{p}$ for all $\mathbf{Z} \in \mathcal{Z}$. Our algorithm works as $\mathbf{p}^{(k+1)} = \frac{\mathbf{Z}^{(k)}\mathbf{p}^{(k)}}{||\mathbf{Z}^{(k)}\mathbf{p}^{(k)}||}$, where $\mathbf{Z}^{(k)} \in \mathcal{Z}$ is chosen such that $\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \geq \mathbf{Z}\mathbf{p}^{(k)}$ for all $\mathbf{Z} \in \mathcal{Z}$. It differs from the classical distributed power control for unicast systems in that the matrix $\mathbf{Z}^{(k)}$ is not fixed but changes with k. Because of this, the original convergence proof, which is based on the power method, cannot be directly applied. A new concept called *primitive set* is needed to develop the convergence proof. In the rest of this section, we show the convergence of the algorithm.

Algorithm 1 Iterative Algorithm

1: Choose $\mathbf{p}^{(0)} \in \mathbb{R}^N > \mathbf{0}$ and $k \leftarrow 0$
2: **repeat**
3:     **for** $i = 1$ to N **do**
4:         $y_i^{(k)} \leftarrow \max_{k_i\in\mathcal{K}_i}\left\{(\mathbf{e}_i^T - \mathbf{a}_i^{k_i})\mathbf{p}^{(k)}\right\}$
5:     **end for**
6:     $\beta^{(k)} \leftarrow \min_{i=1}^{N}\left\{\frac{p_i^{(k)}}{y_i^{(k)}}\right\}$
7:     $\mathbf{p}^{(k+1)} \leftarrow \frac{\mathbf{y}^{(k)}}{||\mathbf{y}^{(k)}||}$
8:     $k \leftarrow k+1$
9: **until** convergence
10: **return** $\beta^{(k)}$

Lemma 3: The sequence $\{\beta^{(k)}\}$ generated by Algorithm 1 is monotonically increasing and bounded above by $\frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}$, and thus is convergent.

Proof: By Algorithm 1, we have $\mathbf{y}^{(k)} \geq \mathbf{Z}\mathbf{p}^{(k)}$ for all $\mathbf{Z} \in \mathcal{Z}$, and $\mathbf{p}^{(k)} \geq \beta^{(k)}\mathbf{y}^{(k)}$. Then

$$\begin{aligned} \mathbf{Z}\mathbf{p}^{(k+1)} = \mathbf{Z}\frac{\mathbf{y}^{(k)}}{||\mathbf{y}^{(k)}||} &\leq \mathbf{Z}\frac{\mathbf{p}^{(k)}}{\beta^{(k)}||\mathbf{y}^{(k)}||} \\ &\leq \frac{\mathbf{y}^{(k)}}{\beta^{(k)}||\mathbf{y}^{(k)}||} = \frac{\mathbf{p}^{(k+1)}}{\beta^{(k)}}. \end{aligned}$$

Since the above inequality holds for all $\mathbf{Z} \in \mathcal{Z}$, we have

$$\begin{aligned} \beta^{(k)} &\leq \min_{\mathbf{Z}\in\mathcal{Z}}\left\{\min_{i=1}^{N}\left\{\frac{p_i^{(k+1)}}{[\mathbf{Z}\mathbf{p}^{(k+1)^T}]_i}\right\}\right\} \\ &= \min_{i=1}^{N}\left\{\frac{p_i^{(k+1)}}{y_i^{(k+1)}}\right\} = \beta^{(k+1)}. \end{aligned}$$

That is, $\{\beta^{(k)}\}$ is monotonically increasing. On the other hand, since $\mathbf{p}^{(k)} \geq \beta^{(k)}\mathbf{y}^{(k)} \geq \beta^{(k)}\mathbf{Z}\mathbf{p}^{(k)} > \mathbf{0}$ for all $\mathbf{Z} \in \mathcal{Z}$, that is, $(\mathbf{I} - \beta^{(k)}\mathbf{Z})\mathbf{p}^{(k)} \geq \mathbf{0}$, we have $\lambda(\beta^{(k)}\mathbf{Z}) \leq 1$

(ref. to [18, p. 141]). Therefore $\beta^{(k)} \leq \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}$, and thus $\{\beta^{(k)}\}$ is convergent. □

Denote $\lim_{k\to\infty}\beta^{(k)} = \beta^*$. Before we proceed to show that $\beta^* = \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}$, we introduce the concept of *primitive* matrix and *primitive* set.

Definition 3 [21]: A square nonnegative matrix $\mathbf{X}$ *is called* primitive *if there exists a positive integer* n *such that* $\mathbf{X}^n > 0$.

If a square matrix $\mathbf{X}$ is primitive, then its Perron-Frobenius eigenvalue is strictly greater than all other eigenvalues in absolute value. For a unicast system, the primitive condition, that is imposed on the normalized interference link gain matrix $\mathbf{Z}$, guarantees the convergence of the aforementioned distributed power control algorithm for unicast systems (i.e., $\mathbf{p}^{(k+1)} = \frac{\mathbf{Z}\mathbf{p}^{(k)}}{||\mathbf{Z}\mathbf{p}^{(k)}||}$). In our multicast case, we need to use the concept of primitive set, which replaces a single matrix and powers of that matrix with a set of matrices and inhomogeneous products of matrices from the set.

Definition 4 [23]: Let $\mathcal{Z}$ *be a set of square nonnegative matrices. For a positive integer* n*, let* $\Theta(n)$ *be an arbitrary product of* n *matrices from* $\mathcal{Z}$*, with any ordering and with repetitions permitted. Define* $\mathcal{Z}$ *to be a* primitive *set if there is a positive integer* n *such that every* $\Theta(n)$ *is positive.*

It can be seen that a necessary condition for $\mathcal{Z}$ to be primitive is that $\mathbf{Z}$ is primitive for all $\mathbf{Z} \in \mathcal{Z}$. One of the sufficient conditions for $\mathcal{Z}$ to be primitive is that for all $\mathbf{Z} \in \mathcal{Z}$, in each row and each column of $\mathbf{Z}$, there are more than half of the entries that are positive [23]. Interested readers can refer to [23] for more information of the primitive set. For a multicast system being considered, when the system is composed of two multicast sessions, $\mathbf{Z} \in \mathcal{Z}$ are always non-primitive, and therefore $\mathcal{Z}$ cannot be primitive. However, for this case, Corollary 1 already gives an explicit and simple solution to the feasible SINR region. Algorithm 1 works for the systems with more than two multicast sessions and with $\mathcal{Z}$ being a primitive set.

Theorem 5: For a multicast system, if the matrix set $\mathcal{Z}(\boldsymbol{\mu})$ *defined in* (5) *is primitive, then* $\beta^{(k)}$ *converges to* $\beta^* = \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}$ *for any arbitrary initial value* $\mathbf{p}^{(0)} > 0$*. Moreover,* $\mathbf{p}^{(k)}$ *converges to a power vector* $\mathbf{p}^*$ *such that* $\lim_{\alpha\to\infty}\Gamma(\alpha\mathbf{p}^*)$ *achieves the boundary point* $\beta^*\boldsymbol{\mu}$*.*

The proof is provided in Appendix A. Fig. 3 illustrates the typical behavior of Algorithm 1. In this example, we consider a multi-cell network composed of four hexagonal cells with radius 0.1km. Within each cell, there is a transmitter (base station) and three receivers (mobile users), forming a multicast session. The transmitter is located at the center of the cell and the locations of the receivers are generated randomly and uniformly within the cell. The channel gain is modeled as $g^k_{i,j} = A^k_{i,j}/(d^k_{i,j})^4$, where $d^k_{i,j}$ is the distance between T_j and R^k_i, and $A^k_{i,j}$ is the attenuation factor due to shadowing. We assume $A^k_{i,j}$ is lognormal distributed with mean 0dB and standard deviation 4dB. As we can see from the figure, $\beta^{(k)}$ converges within a small number of iterations. By using Algorithm 1, we can efficiently check the feasibility of an SINR vector $\boldsymbol{\mu}$ by checking the value of $\beta^*(\boldsymbol{\mu})$. If $\beta^*(\boldsymbol{\mu}) < 1$, $\boldsymbol{\mu}$ is infeasible, and vice versa. For practical systems to check the feasibility, Algorithm 1 can be implemented in a distributed manner, in the sense that each transmitter only needs to collect the interference information from its intended receivers and update its transmission power to the maximum interference power, without the need to know the interference information of other multicast sessions. Besides checking feasibility, Algorithm 1 can be used to find the optimal solution to the classic power balancing problem for multicast systems, which is to maximize the minimum SINR of all the users, defined in the following form

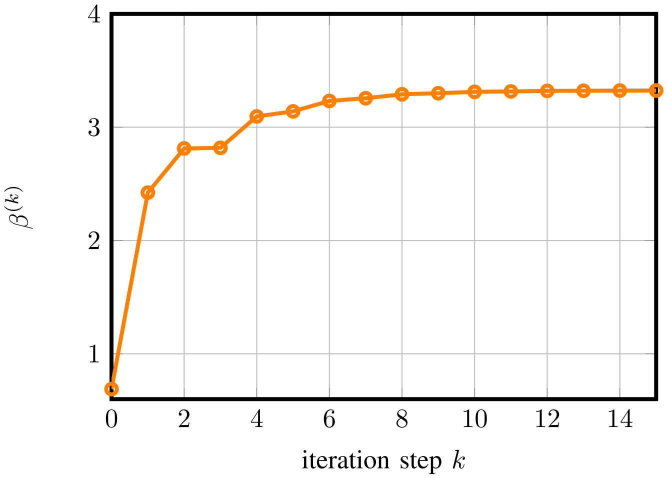


Fig. 3. Convergence of Algorithm 1. In this example, there are four multicast sessions and each has three receivers.

$$\sup_{\mathbf{p}} \min_{i=1}^{N} \gamma_i(\mathbf{p}) \quad s.t.\ \mathbf{p} \geq \mathbf{0},$$

and the optimal value is $\beta^*(\mathbf{1})$. This max min SINR formulation is deployed in many wireless communication applications such as cellular networks.

VI. Geometric Properties of the Feasible SINR Region

In this section, we discuss the geometric properties of the feasible SINR region. Let $D(\boldsymbol{\mu})$ denote the diagonal matrix constructed by

$$D(\boldsymbol{\mu}) = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_N \end{bmatrix}.$$

By Theorem 3, the feasible SINR region is equivalent to

$$\begin{aligned} \Upsilon &= \left\{\boldsymbol{\mu} \in \mathbb{R}^N_+ : \max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\} < 1\right\} \\ &= \left\{\boldsymbol{\mu} \in \mathbb{R}^N_+ : \max_{\mathbf{Z}\in\mathcal{Z}(\mathbf{1})}\left\{\lambda\big(D(\boldsymbol{\mu})\mathbf{Z}\big)\right\} < 1\right\} \\ &= \bigcap_{\mathbf{Z}\in\mathcal{Z}(\mathbf{1})}\left\{\boldsymbol{\mu} \in \mathbb{R}^N_+ : \lambda\big(D(\boldsymbol{\mu})\mathbf{Z}\big) < 1\right\}. \end{aligned}$$

Hence, we have the following nice result:

Theorem 6: The feasible SINR region of a multicast system is the intersection of the feasible SINR regions of all its embedded unicast systems.

Let $\Upsilon^c = \mathbb{R}_+^N \setminus \Upsilon$ denote the complement of Υ in $\mathbb{R}_+^N$, i.e., the infeasible SINR region. Next, we investigate the convexity of Υ^c and the log-convexity of Υ.

A. Convexity of Υ^c

For unicast systems, it has been proved in [6] that the infeasible SINR regions of a general two-user system and a general three-user system are convex. It is also shown in [5] that the convexity of the infeasible SINR region does not hold for a general four-user system. For multicast systems, we have the following observation.

Theorem 7: The infeasible SINR region of a general system consisting of two multicast sessions is convex. The convexity property does not hold for a general system consisting of more than two multicast sessions.

When there are two multicast sessions, by Corollary 1, the feasible SINR region is

$$\Upsilon = \left\{ [\mu_1, \mu_2] \in \mathbb{R}_+^2 : \mu_1 \mu_2 < \frac{g_{1,1}^{k_1^*}}{g_{1,2}^{k_1^*}} \cdot \frac{g_{2,2}^{k_2^*}}{g_{2,1}^{k_2^*}} \right\}.$$

It is ready to verify that Υ^c is convex.

When there are more than two multicast sessions, by Theorem 6, Υ^c is the union of the infeasible SINR regions of all the embedded unicast systems and is in general non-convex. Fig. 4 illustrates the Υ^c for a system consisting of three multicast sessions, where the link gain matrix is given by

$$\begin{array}{c} \\ R_1^1 \\ R_1^2 \\ R_2^1 \\ R_2^2 \\ R_3^1 \\ R_3^2 \end{array} \begin{array}{c} \begin{array}{ccc} T_1 & T_2 & T_3 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0.1 \\ 1 & 0.1 & 0.5 \\ 0.5 & 1 & 0.1 \\ 0.1 & 1 & 0.5 \\ 0.5 & 0.1 & 1 \\ 0.1 & 0.5 & 1 \end{bmatrix} \end{array}.$$

It can be directly observed that its Υ^c is non-convex, which provides a counter-example to show that the infeasible region is in general non-convex.

B. Log-Convexity of Υ

We first introduce the notion of log-convexity. Let $\log(\boldsymbol{\mu}) = [\log \mu_1, \log \mu_2, \ldots, \log \mu_N]$ and $\log(\Upsilon) = \{\log(\boldsymbol{\mu}) : \boldsymbol{\mu} \in \Upsilon\}$. We say a set Υ is log-convex if $\log(\Upsilon)$ is convex. Since $\log(\cdot) : \Upsilon \to \log(\Upsilon)$ is a bijective mapping, by Theorem 6, we have

$$\log(\Upsilon) = \bigcap_{\mathbf{Z} \in \mathcal{Z}(\mathbf{1})} \left\{ \log(\boldsymbol{\mu}) \in \mathbb{R}^N : \lambda\big(D(\boldsymbol{\mu})\mathbf{Z}\big) < 1 \right\}.$$

It has been proved in [4] that the feasible log-SINR region of a unicast system is convex. So $\log(\Upsilon)$, the intersection of the log-SINR regions of all its embedded unicast, is also convex. We conclude this by the following theorem.

Theorem 8: The feasible SINR region of a multicast system is log-convex. In other words, the feasible SINR, expressed in decibels, is a convex set.

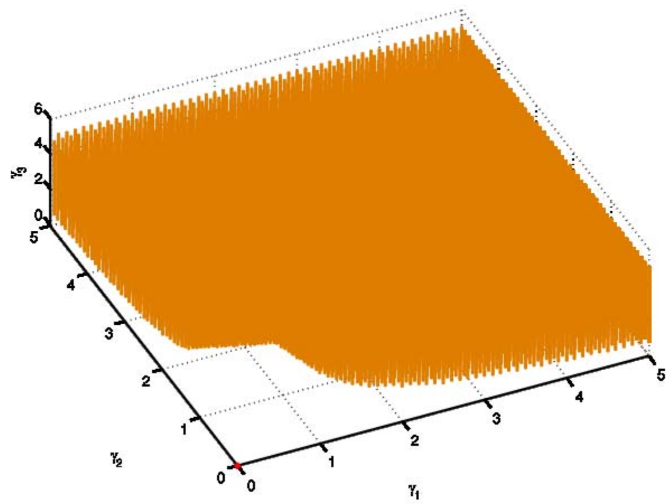


Fig. 4. The shaded area represents the infeasible SINR region of a three-multicast-session system, which is non-convex.

VII. Feasibility of SINR With Power Constraints

So far, we have discussed the feasibility of SINR for a multicast system in the case of unlimited power. In this section, we consider that besides $\mathbf{p} > \mathbf{0}$, the power vector is also subject to some linear constraints

$$\sum_{i \in \Omega_m} p_i \leq \bar{p}_{\Omega_m}, \quad m = 1, \ldots, M,$$

where $\Omega_m \subseteq \{1, \ldots, N\}$ and M is the number of constraints. When $\Omega_m = \{1, \ldots, N\}$, it is a constraint on the total power. When $\Omega_m = \{i\}$, it is a constraint on the individual power of transmitter T_i. Define the power set by

$$\mathcal{P} = \{\mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i \in \Omega_m} p_i \leq \bar{p}_{\Omega_m}, m = 1, \ldots, M\}.$$

Now the feasibility of SINR vector $\boldsymbol{\mu} > \mathbf{0}$ is decided by whether there exists $\mathbf{p} \in \mathcal{P}$ such that $\Gamma(\mathbf{p}) = \boldsymbol{\mu}$. Note that the power vectors in $\mathcal{P}$ are downward comprehensive. That is, if $\mathbf{p}' \in \mathcal{P}$, then $\mathbf{p} \in \mathcal{P}$ if $\mathbf{0} < \mathbf{p} \leq \mathbf{p}'$. Hence using the same argument as in Proposition 1, we know that $\boldsymbol{\mu}$ is feasible if and only if there exits $\mathbf{p} \in \mathcal{P}$ such that $\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \geq \mathbf{n}(\boldsymbol{\mu})$. Our results extend the feasibility condition derived in [7] for a unicast system to a multicast system. Note that in [7], the unicast network setting $\mathbf{G}$ is assumed to be irreducible. A matrix is irreducible if it is not similar via a permutation to a block upper triangular matrix. Here we assume that for any $\mathbf{G} \in \mathcal{G}$, all its the principal submatrix are irreducible. This is equivalent to assume that all the entries in $\mathbf{A}(\boldsymbol{\mu})$ are nonzero.

Definition 5: For a matrix $\mathbf{X} \in \mathbb{R}^{K \times N}$, a vector $\mathbf{y} \in \mathbb{R}^K$ and a set $\Omega \subseteq \{1, \ldots, N\}$, $\psi(\mathbf{X}, \mathbf{y}, \Omega)$ is the operation to add $\mathbf{y}$ to the j-th column of $\mathbf{X}$, for all $j \in \Omega$. That is, $\mathbf{Z} = \psi(\mathbf{X}, \mathbf{y}, \Omega)$, where $Z_{ij} = X_{ij} + y_i$ for all $i \in \{1, \ldots, K\}$ and $j \in \Omega$, and $Z_{ij} = X_{ij}$ for the else.

Theorem 9: Consider a multicast network setting $\mathbf{A}(\boldsymbol{\mu})$ and assume that all the entries in $\mathbf{A}(\boldsymbol{\mu})$ are nonzero. There exists a power vector $\mathbf{p} \in \mathcal{P}$ such that $\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \geq \mathbf{n}(\boldsymbol{\mu})$ if and only *if*

$$\max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})}\max_{m\in\{1,\dots,M\}}\left\{\lambda\Big(\psi\big(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m\big)\Big)\right\}\le 1.$$

Proof: It is already known from [7] that, for a unicast system $\mathbf{G}$ which is irreducible, there exists $\mathbf{p}\in\mathcal{P}$ such that $\mathbf{Gp}\ge\mathbf{n_G}$ if and only if $\max_{m\in\{1,\dots,M\}}\{\lambda(\psi(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m))\}\le 1$. Since all the entries in $\mathbf{A}(\boldsymbol{\mu})$ are nonzero, all $\mathbf{G}\in\mathcal{G}$ and its principal submatrix are irreducible. We first prove the necessary condition. Suppose there exists $\mathbf{p}\in\mathcal{P}$ such that $\mathbf{A}(\boldsymbol{\mu})\mathbf{p}\ge\mathbf{n}(\boldsymbol{\mu})$. Then for any $\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})$, $\mathbf{Gp}\ge\mathbf{n_G}$, which implies $\max_{m\in\{1,\dots,M\}}\{\lambda(\psi(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m))\}\le 1$. Regarding all $\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})$, we have $\max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})}\max_{m\in\{1,\dots,M\}}\{\lambda(\psi(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m))\}\le 1$.

Next we prove the sufficient condition. For any $\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})$, since $\max_{m\in\{1,\dots,M\}}\{\lambda(\psi(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m))\}\le 1$, there exists $\mathbf{p}\in\mathcal{P}$ such that $\mathbf{Gp}\ge\mathbf{n_G}$. Further by proposition 1, there exists $\mathbf{p}\in\mathcal{P}$ such that $\mathbf{Gp}=\mathbf{n_G}$. On the other hand, since $\mathbf{0}\le\mathbf{I}-\mathbf{G}<\psi(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m)$ for all m, by the Perron-Frobenius Theorem for irreducible matrices [17], $\lambda(\mathbf{I}-\mathbf{G})<\lambda(\psi(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m))\le 1$. This implies that $\mathbf{G}^{-1}$ exists and therefore $\mathbf{p}=\mathbf{G}^{-1}\mathbf{n_G}$ belongs to $\mathcal{P}$. The rest of the proof follows the same argument as the proof of Theorem 3. Note that the mathematical induction can be used since all the principal submatrix of $\mathbf{G}\in\mathcal{G}$ are irreducible. □

Note that

$$\max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})}\max_{m\in\{1,\dots,M\}}\left\{\lambda\Big(\psi\big(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m\big)\Big)\right\} = \max_{m\in\{1,\dots,M\}}\max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})}\left\{\lambda\Big(\psi\big(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m\big)\Big)\right\}.$$

Similar to (5), for each of the M linear constraints, define

$$\mathcal{Z}_{\Omega_m}(\boldsymbol{\mu})=\left\{\psi\big(\mathbf{I}-\mathbf{G},\frac{\mathbf{n_G}}{\bar{p}_{\Omega_m}},\Omega_m\big):\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})\right\}.$$

By using Algorithm 1 with $\mathcal{Z}_{\Omega_m}(\boldsymbol{\mu})$, we can find a supremum $\beta^*_{\Omega_m}(\boldsymbol{\mu})$. The farthest point of the SINR region in direction $\boldsymbol{\mu}$ is then $\min_{m=1}^{M}\{\beta^*_{\Omega_m}(\boldsymbol{\mu})\}\boldsymbol{\mu}$. By this approach, the feasible SINR region is characterized. On the other hand, if $\min_{m=1}^{M}\{\beta^*_{\Omega_m}(\boldsymbol{\mu})\}\ge 1$, $\boldsymbol{\mu}$ is feasible. Fig. 5 plots the feasible SINR region of the network example in Fig. 1, with a power constraint on the total power. In this example, the link gain matrix is

$$\begin{array}{c} \\ R_1^1\\ R_1^2\\ R_2^1\\ R_2^2\end{array}\begin{array}{c}\begin{array}{cc}T_1 & \quad T_2\end{array}\\ \begin{bmatrix}0.5326 & 0.6801\\ 0.5539 & 0.3672\\ 0.2393 & 0.8669\\ 0.5789 & 0.4068\end{bmatrix}\end{array},$$

and the power constraint is $p_1+p_2\le 2$. The four dashed lines are the boundary of the feasible SINR regions of the four embedded unicast systems and the solid line is the boundary of the multicast system. It can be seen the SINR region of the multcast system is the intersection of the SINR regions of the embedded unicast systems. Moreover, under power contraint, the infeasible SINR region is not necessarily convex even for a multicast system with two multicast sessions.

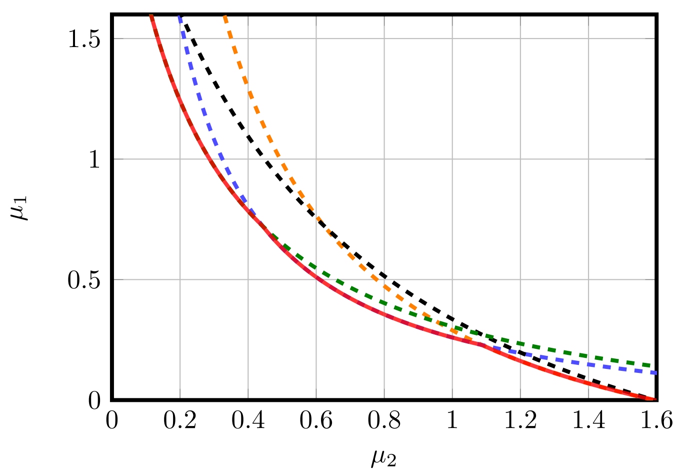


Fig. 5. Feasible SINR region for a multicast system with two multicast sessions, under a total power constraint. The dashed lines correspond to the four embedded unicast systems and the solid line corresponds to the multicast system.

In the end of this section, we introduce an application of our multicast model to a time varying unicast system. Consider a unicast system consisting of N transmitter-receiver pairs, where the channel gains among them vary with time due to the mobility of the receivers. Let $\mathbf{h}_i(t)$ for $i=1,\dots,N$ denote the link gain vector from N transmitters to the i-th receiver at time t. We assume that $\mathbf{h}_i(t)$ are modeled with discrete states, that is, $\mathbf{h}_i(t)$ is randomly selected from a finte set $\{\mathbf{h}_i^1,\mathbf{h}_i^2,\dots,\mathbf{h}_i^{K_i}\}$ for all i. Such model has been adopted in [7] and [24] and is justifiable from a practical perspective. Most wireless communication networks have limited capacity for their feedback links, and therefore only quantized channel state information can be pragmatically sent to the transmitters from the receivers. Our discrete channel model is an approximation to the continuous channel model. The accuracy of the approximation can be improved by choosing a larger channel state set. For this system, we consider the SINR region that is achievable to all channel realizations in their finite sets. To be specific, an SINR vector $\boldsymbol{\mu}$ is said to be zero-outage [7] if there exists a power setting such that no matter which link gain realization in the finite set is, the SINR is achievable all the time. Such a zero-outage SINR problem can be mapped to a feasible SINR problem of a multicast system. The idea is to let one receiver R_i pretend to be K_i receivers, i.e., $R_i^1,\dots,R_i^{K_i}$, and $R_i^{k_i}$ only experiences the k_ith link gain realization $\mathbf{h}_i^{k_i}$. This is analogous to a multicast system where there are N multicast sessions and the ith session has K_i pretended receivers. The feasible SINR region of this artificial multicast system is exactly the zero-outage SINR region of the original time-varying unicast sytem. Theorem 3 and Theorem 9 can be applied.

VIII. Conclusion

While the power control theory for unicast systems has been well developed, there is little, if any, study on power control for multicast systems. It is well known that classical power control is built upon the Perron-Frobenius Theory, which applies to a square non-negative interference matrix. There is no trivia generalization to the multicast case, which deals with non-square interference matrix. In this paper, a necessary and sufficient condition for an SINR vector is obtained via a fundamental result in convex analysis called Helly's theorem. This generalization enriches the theory of power control.

Apart from system feasibility, iterative algorithms to find pareto-maximal SINR points also play a key role in the power control theory. The classical power balancing algorithm is based on the power method, which repeatedly multiply the power vector by a fixed matrix. For the multicast case, a similar algorithm requires repeated multiplication of different matrices, which requires a more in-depth analysis on its convergence. In this regard, the concept of primitive set of nonnegative matrices is needed, and a sufficient condition for its convergence is derived.

Based on the above results, the geometric properties of the feasible SINR region have been analyzed. Furthermore, the feasibility condition is extended to cover the case with additional linear power constraints. All these results give a complete understanding of power control for multicast systems. We hope that our work motivates further enrichment of the power control theory, which is fundamental to interference management in wireless networks, especially when the cross link gains are weak so that interfering signals are appropriately treated as Gaussian noise.

Appendix A
Proof of Theorem 5

Proof: Let $\mathbf{Z}^{(k)}$ denote one of the matrices at the k-th iteration such that $\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \geq \mathbf{Z}\mathbf{p}^{(k)}$ for all $\mathbf{Z} \in \mathcal{Z}$. From the construction of the algorithm, we have

$$\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \leq \frac{1}{\beta^{(k)}}\mathbf{p}^{(k)} \quad \text{for all } k \in \mathbb{N}. \tag{8}$$

Moreover, there exists $1 \leq i \leq N$ such that $[\mathbf{Z}^{(k)}\mathbf{p}^{(k)}]_i = \frac{1}{\beta^{(k)}}[\mathbf{p}^{(k)}]_i$. We note that each vector $\mathbf{p}^{(k)}$ is a unit vector, as $||\mathbf{p}^{(k)}|| = 1$. By the Bolzano-Weierstrass Theorem and the compactness of the unit ball in $\mathbb{R}^N$, there exists a convergent subsequence, that is, $\mathbf{p}^{(k_j)} \to \mathbf{p}^*$. By Lemma 3, $\beta^{(k_j)} \to \beta^*$. Suppose at $\mathbf{p}^*$, $\mathbf{Z}^* \in \mathcal{Z}$ is one of the matrices that satisfy $\mathbf{Z}^*\mathbf{p}^* \geq \mathbf{Z}\mathbf{p}^*$ for all $\mathbf{Z} \in \mathcal{Z}$. Taking the limit of (8) with respect to the subsequence indexed by k_j, we have $\mathbf{Z}^*\mathbf{p}^* \leq \frac{1}{\beta^*}\mathbf{p}^*$. If $\mathbf{Z}^*\mathbf{p}^* = \frac{1}{\beta^*}\mathbf{p}^*$, since $\mathbf{Z}^*$ is primitive and $\mathbf{p}^* > \mathbf{0}$, we have $\beta^* = \frac{1}{\lambda(\mathbf{Z}^*)} \geq \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}$. On the other hand, $\beta^* \leq \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}$ by Lemma 3. Therefore, $\beta^* = \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\boldsymbol{\mu})}\{\lambda(\mathbf{Z})\}}$.

If $\mathbf{Z}^*\mathbf{p}^* \neq \frac{1}{\beta^*}\mathbf{p}^*$, since $\mathcal{Z}$ is primitive, there exists integer n such that an arbitrary product of n matrices from $\mathcal{Z}$ is positive, i.e., $\Theta(n) > \mathbf{0}$, and therefore $\Theta(n)\mathbf{Z}^*\mathbf{p}^* < \Theta(n)\frac{1}{\beta^*}\mathbf{p}^*$. By the continuity of the mapping, there exists $\mathbf{p}^{(k)}$ close enough to $\mathbf{p}^*$ such that $\Theta(n)\mathbf{Z}^*\mathbf{p}^{(k)} < \Theta(n)\frac{1}{\beta^*}\mathbf{p}^{(k)}$ and $\mathbf{Z}^{(k)} = \mathbf{Z}^*$. We now apply the algorithm for n more iterations from $\mathbf{p}^{(k)}$. For $i = 0, 1, \ldots, n-1$ we have the following inequalities

$$\mathbf{Z}^{(k+i+1)}\mathbf{p}^{(k+i)} \leq \mathbf{Z}^{(k+i)}\mathbf{p}^{(k+i)} \tag{9}$$

due to that the selection matrix satisfies $\mathbf{Z}^{(k+i)}\mathbf{p}^{(k+i)} \geq \mathbf{Z}\mathbf{p}^{(k+i)}$ for all $\mathbf{Z} \in \mathcal{Z}$. Meanwhile by Algorithm 1,

$$\begin{aligned}\mathbf{p}^{(k+i)} &= \frac{\mathbf{y}^{(k+i-1)}}{||\mathbf{y}^{(k+i-1)}||} = \frac{\mathbf{Z}^{(k+i-1)}\mathbf{p}^{(k+i-1)}}{||\mathbf{Z}^{(k+i-1)}\mathbf{p}^{(k+i-1)}||} \\ &= \frac{\mathbf{Z}^{(k+i-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}}{||\mathbf{Z}^{(k+i-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}||}.\end{aligned}$$

By substituting $\mathbf{p}^{(k+i)}$ into (9), we get

$$\begin{aligned}&\mathbf{Z}^{(k+i+1)}\mathbf{Z}^{(k+i-1)}\mathbf{Z}^{(k+i-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \\ &\quad \leq \mathbf{Z}^{(k+i)}\mathbf{Z}^{(k+i-1)}\mathbf{Z}^{(k+i-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}.\end{aligned} \tag{10}$$

Let us take a look at these inequalities step by step. By (9) for $i = 0$, $\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. By multiplying $\mathbf{Z}^{(k+2)}$ on both side of the inequality, we have

$$\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+2)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}. \tag{11}$$

By (10) for $i = 1$, $\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. Along with (11), we have

$$\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}.$$

By multiplying $\mathbf{Z}^{(k+3)}$ on both side of the above inequality, we have $\mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. By (10) for $i = 2$, $\mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. So

$$\mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}.$$

By repeating this procedure for $n-1$ times, we can finally get

$$\begin{aligned}&\mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \\ &\quad \leq \mathbf{Z}^{(k+n-1)}\mathbf{Z}^{(k+n-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}.\end{aligned} \tag{12}$$

Since $\Theta(n)\mathbf{Z}^*\mathbf{p}^{(k)} < \Theta(n)\frac{1}{\beta^*}\mathbf{p}^{(k)}$ holds for arbitrary $\Theta(n)$, we let $\Theta(n) = \mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}$. Along with (12), we have

$$\begin{aligned}&\mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)} \\ &\quad = \Theta(n)\mathbf{Z}^*\mathbf{p}^{(k)} \\ &\quad < \Theta(n)\frac{1}{\beta^*}\mathbf{p}^{(k)} \\ &\quad = \frac{1}{\beta^*}\mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \\ &\quad \leq \frac{1}{\beta^*}\mathbf{Z}^{(k+n-1)}\mathbf{Z}^{(k+n-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)}.\end{aligned}$$

By multiplying $\frac{1}{||\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)}||}$ on both side of the inequality, we have

$$\begin{aligned}\mathbf{Z}^{(k+n)}\mathbf{p}^{(k+n)} &= \mathbf{Z}^{(k+n)}\frac{\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)}}{||\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)}||} \\ &< \frac{1}{\beta^*}\frac{\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)}}{||\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)}||} \\ &= \frac{1}{\beta^*}\mathbf{p}^{(k+n)}.\end{aligned}$$

This implies $\beta^{(k+n)} > \beta^*$, which contradicts with that β^* is the limit. Hence there must have $\mathbf{Z}^*\mathbf{p}^* = \frac{1}{\beta^*}\mathbf{p}^*$.

We prove $\mathbf{p}^{(k)} \to \mathbf{p}^*$ by contradiction. Suppose there exists another subsequence such that $\mathbf{p}^{k'_j} \to \mathbf{p}'$ and $\mathbf{p}^* \neq \mathbf{p}'$. Then $\mathbf{Z}^*\mathbf{p}' \leq \frac{1}{\beta^*}\mathbf{p}'$. Meanwhile we already have $\mathbf{Z}^*\mathbf{p}^* = \frac{1}{\beta^*}\mathbf{p}^*$. By the Subinvariance Theorem in [17, p. 23], $\mathbf{p}^* = \mathbf{p}'$, which contradicts with the assumption that $\mathbf{p}^* \neq \mathbf{p}'$. Therefore $\mathbf{p}^{(k)}$ converges to $\mathbf{p}^*$.

Since $\mathbf{Z}^*\mathbf{p}^* = \frac{1}{\beta^*}\mathbf{p}^*$ and $\mathbf{Z}\mathbf{p}^* \leq \frac{1}{\beta^*}\mathbf{p}^*$ for all $\mathbf{Z} \in \mathcal{Z}(\boldsymbol{\mu})$, it is ready to see that $\lim_{\alpha\to\infty} \gamma_i(\alpha\mathbf{p}^*) = \min_{\mathbf{Z}\in\mathcal{Z}(\mathbf{1})}\{\frac{p_i^*}{[\mathbf{Z}\mathbf{p}^*]_i}\} = \beta^*\mu_i$ for all $i = 1, \ldots, N$. So $\lim_{\alpha\to\infty} \Gamma(\alpha\mathbf{p}^*) = \beta^*\boldsymbol{\mu}$. □

References

[1] Y. Chen and W. S. Wong, "Power control for non-Gaussian interference," *IEEE Trans. Wireless Commun.*, vol. 10, no. 8, pp. 2660–2669, Aug. 2011.

[2] Y. Chen, S. Yang, and W. S. Wong, "Exact non-Gaussian interference model for fading channels," *IEEE Trans. Wireless Commun.*, vol. 12, no. 1, pp. 168–179, Jan. 2013.

[3] R. H. Etkin, D. N. C. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.

[4] C. W. Sung, "Log-convexity property of the feasible SIR region in power-controlled cellular systems," *IEEE Commun. Lett.*, vol. 6, no. 6, pp. 248–249, Jun. 2002.

[5] S. Stanczak and H. Boche, "The infeasible SIR region is not a convex set," *IEEE Trans. Commun.*, vol. 54, no. 11, pp. 1905–1907, Nov. 2006.

[6] W. S. Wong, "Concavity of the feasible signal-to-noise ratio region in power control problems," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 2143–2150, Apr. 2011.

[7] H. Mahdavi-Doost, M. Ebrahimi, and A. K. Khandani, "Characterization of SINR region for interfering links with constrained power," *IEEE Trans. Inf. Theory*, vol. 56, no. 6, pp. 2816–2828, Jun. 2010.

[8] H. Won *et al.*, "Multicast scheduling in cellular data networks," *IEEE Trans. Wireless Commun.*, vol. 8, no. 9, pp. 4540–4549, Sep. 2009.

[9] Y. Chen and C. W. Sung, "Resource allocation for two-way relay cellular networks with and without network coding," in *Proc. IEEE Int. Conf. Commun. Syst. (ICCS)*, Nov. 2014, pp. 600–604.

[10] K. Wang, C. F. Chiasserini, R. Rao, and J. Proakis, "A distributed joint scheduling and power control algorithm for multicasting in wireless ad hoc networks," in *Proc. IEEE ICC*, vol. 1, May 2003, pp. 725–731.

[11] E. Karipidis, N. Sidiropoulos, and L. Tassiulas, "Joint QoS multicast power/admission control and base station assignment: A geometric programming approach," in *Proc. IEEE Sensor Array Multichannel Signal Process. Workshop (SAM)*, Jul. 2008, pp. 155–159.

[12] I. Rubin, C.-C. Tan, and R. Cohen, "Joint scheduling and power control for multicasting in cellular wireless networks," *EURASIP J. Wireless Commun. Netw.*, vol. 2012, no. 1, pp. 1–16, 2012.

[13] A. Argyriou, "Link scheduling for multiple multicast sessions in distributed wireless networks," *IEEE Wireless Commun. Lett.*, vol. 2, no. 3, pp. 343–346, Jun. 2013.

[14] J. M. Aein, "Power balancing in systems employing frequency reuse," *Comsat Tech. Rev.*, vol. 3, no. 2, pp. 277–299, 1973.

[15] J. Zander, "Performance of optimum transmitter power control in cellular radio systems," *IEEE Trans. Veh. Technol.*, vol. 41, no. 1, pp. 57–62, Feb. 1992.

[16] P.-C. Lin, "Feasibility problem of channel spatial reuse in power-controlled wireless communication networks," in *Proc. IEEE WCNC*, Apr. 2013, pp. 591–596.

[17] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1991.

[18] M. Fiedler, *Special Matrices and Their Applications in Numerical Mathematics*. Leiden, The Netherlands: Martinus Nijhoff, 1986.

[19] B. Alexander, *A Course Convexity*, vol. 54. Providence, RI, USA: AMS, 2002.

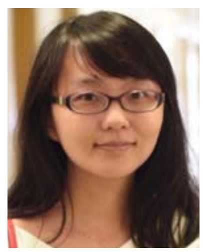
[20] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*. Princeton, NJ, USA: Princeton Univ. Press, 2009.

[21] R. B. Bapat and T. E. S. Raghavan, *Nonnegative Matrices Applications*. Cambridge, U.K.: Cambridge Univ. Press, 1997.

[22] S. A. Grandhi, R. Vijayan, and D. J. Goodman, "Distributed power control in cellular radio systems," *IEEE Trans. Commun.*, vol. 42, no. 234, pp. 226–228, Feb. 1994.

[23] J. E. Cohen and P. H. Sellers, "Sets of nonnegative matrices with positive inhomogeneous products," *J. Linear Algebra Appl.*, vol. 47, pp. 185–192, Oct. 1982.

[24] H. Weingarten, T. Liu, S. Shamai (Shitz), Y. Steinberg, and P. Viswanath, "The capacity region of the degraded multiple-input multiple-output compound broadcast channel," *IEEE Trans. Inf. Theory*, vol. 55, no. 11, pp. 5011–5023, Nov. 2009.



Yi Chen received the B.S. degree in communication engineering from the Beijing University of Posts and Telecommunications in 2007 and the Ph.D. degree in information engineering from The Chinese University of Hong Kong in 2012. She is currently a Research Assistant Professor with the School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen. She is also with the Shenzhen Research Institute of Big Data. Her research interests include wireless communication, resource allocation, and machine learning.

Chi Wan Sung (M'98–SM'16) received the B.Eng., M.Phil., and Ph.D. degrees in information engineering from The Chinese University of Hong Kong in 1993, 1995, and 1998, respectively. He joined the faculty of the City University of Hong Kong in 2000, where he is currently an Associate Professor with the Department of Electronic Engineering. His research interests include wireless communications, resource allocation, power control, network coding, and distributed storage systems. He is on the Editorial Board of the *ETRI Journal* and *Electronics Letters*.