Correspondence optimization in 2D standardized carotid wall thickness map by description length minimization: A tool for increasing reproducibility of 3D ultrasound-based measurements

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Purpose: The previously described 2D standardized vessel-wall-plus-plaque thickness (VWT) maps constructed from 3D ultrasound vessel wall measurements using an arc-length (AL) scaling approach adjusted the geometric variability of carotid arteries and has allowed for the comparisons of VWT distributions in longitudinal and cross-sectional studies. However, this mapping technique did not optimize point correspondence of the carotid arteries investigated. The potential misalignment may lead to errors in point-wise VWT comparisons. In this paper, we developed and validated an algorithm based on steepest description length (DL) descent to optimize the point correspondence implied by the 2D VWT maps.

Methods: The previously described AL approach was applied to obtain initial 2D maps for a group of carotid arteries. The 2D maps were reparameterized based on an iterative steepest DL descent approach, which consists of the following two steps. First, landmarks established by resampling the 2D maps were aligned using the Procrustes algorithm. Then, the gradient of the DL with respect to horizontal and vertical reparameterizations of each landmark on the 2D maps was computed, and the 2D maps were subsequently deformed in the direction of the steepest descent of DL. These two steps were repeated until convergence. The quality of the correspondence was evaluated in a phantom study and an in vivo study involving ten carotid arteries enrolled in a 3D ultrasound interscan variability study. The correspondence quality was evaluated in terms of the compactness and generalization ability of the statistical shape model built based on the established point correspondence in both studies. In the in vivo study, the effect of the proposed algorithm on interscan variability of VWT measurements was evaluated by comparing the percentage of landmarks with statistically significant VWT-change before and after point correspondence optimization.

Results: The statistical shape model constructed with optimized correspondence was more compact and had a better generalization ability than that constructed using the AL approach in both the phantom and in vivo studies. A statistical test on the group-average VWT-Change at each point of the 2D carotid template showed that the group-average VWT-change was significantly different from 0 in 18% of landmarks when the AL approach was used, and this percentage was reduced to 11% after correspondence optimization.

Conclusions: The optimized correspondence resulted in a more compact and generalizable statistical shape model, and the algorithm was shown to reduce interscan variability of point-wise VWT measurements obtained using the previously described arc-length scaling parameterization approach.

Key words: point correspondence, three-dimensional ultrasound, carotid artery, vessel-wall-plus-plaque thickness (VWT), minimum description length (MDL)

1. INTRODUCTION

Stroke is one of the leading causes of death and disability worldwide. Over two thirds of stroke deaths occurred in developing countries. China, as the most populous developing country, has an annual stroke mortality of 1.6 × 10^5, and the mortality rate is more than seven times higher than that in the United States. Carotid atherosclerosis is a major source of thrombosis and subsequent emboli, which travels downstream and may block one of the cerebral arteries, causing ischemic stroke. Management of carotid atherosclerosis for patients at high risk of vascular events through dietary/lifestyle changes and intensive medical therapy has the potential to reduce their risk by 75% to 80%. With the improved strategies to treat atherosclerosis noninvasively, there is a parallel requirement for the development of sensitive, noninvasive, and cost-effective measurement tools or biomarkers for monitoring the progression and regression of atherosclerosis and for assessment of the effect of stroke prevention strategies.

Carotid intima–media thickness (IMT) based on B-mode ultrasound imaging has been established to correlate with an increased risk of stroke and vascular outcomes. However, since the rate of change of IMT is small (0.01–0.03 mm/yr), large samples and long duration are required to establish the effect of treatments. Moreover, IMT is a measurement of vascular wall thickening and not a direct measurement
of atherosclerotic plaque. Some protocols of IMT even require measurements to be taken in a location with no plaque. Recent advances in 3D carotid ultrasound imaging technology have allowed direct quantification of carotid atherosclerosis. 3D atherosclerosis phenotypes, such as total plaque volume and vessel wall volume, have been demonstrated as biomarkers that can detect longitudinal changes in atherosclerotic burden.

Since atherosclerosis is a local disease with plaques predominantly occurring at bends and bifurcations (BFs), monitoring local changes in plaque and vessel wall thickness would allow the development of more sensitive biomarkers, which would identify high-risk patients with rapid plaque progression in a shorter time frame. Alternative treatments could then be considered sooner. In addition, the beneficial effect of a certain treatment would also require a shorter duration to demonstrate, allowing more rapid dissemination of the research findings and decreasing the period of time that beneficial treatments are withheld from the control group.

To quantify the spatial–temporal changes in vessel-wall-plus-plaque thickness (VWT), we have previously proposed pointwise quantification and visualization of VWT-change on the 3D carotid surface, which we referred to as the 3D VWT-change map. To further facilitate the visualization and interpretation of the VWT-Change distribution, we developed a 2D area-preserving mapping approach that allows the distribution to be visualized and interpreted in a single view without having to rotate and interact with the 3D VWT-Change maps. However, our flattening approach shares with other surface flattening approaches in that the shapes of the 2D flattened maps generated depend on the geometry of the original 3D carotid surface. Therefore, although our 2D mapping approach has been applied in many clinical studies, comparison of VWT-Change distributions of subjects who underwent different therapies or of the same subject derived from different imaging modalities were only performed by qualitative visual matching. Thus, there was a requirement to develop a standardized mapping strategy that allows all 3D VWT-Change maps to a carotid template that would allow quantitative comparison between patients or between image modalities. For this reason, we developed a 2D template construction approach based on the arc-length of transverse contours segmented from the 2D transverse image sequence generated by reslicing 3D imaging. This approach would allow quantitative comparison between image modalities. For this reason, we developed a 2D template construction approach based on the arc-length of transverse contours segmented from the 2D transverse image sequence generated by reslicing 3D imaging. This approach would allow quantitative comparison between image modalities.

2. METHODS

Figure 1 shows a flowchart illustrating the proposed point correspondence optimization algorithm with important equations identified. Subsections 2.A–2.D provide a detailed description of each step of the algorithm.

2.A. Initialization

The AL approach for generating the initial 2D maps was described in detail elsewhere and briefly described here. The 3D VWT map was first transformed to a standard 3D coordinate frame with the origin located at the bifurcation.
The longitudinal axis of the common carotid artery (CCA) identified by an expert observer in manual segmentation was aligned with the z-axis. The x-axis was defined by the vector pointing from centroid of external carotid artery (ECA) contour immediately distal to bifurcation [i.e., $C_{ECA_{up}}$ in Fig. 2(a)] to that of internal carotid artery (ICA) contour on the same plane [i.e., $C_{ICA_{up}}$ in Fig. 2(a)]. The y-axis was defined by taking the cross product of the x- and z-axes. A plane cutting the CCA, denoted by $P_{CCA}$, was defined as the plane containing the line connecting the bifurcation and centroid of the CCA contour most proximal to the bifurcation [i.e., $C_{CCA_{up}}$ in Fig. 2(a)] and the y-axis. A plane cutting the ICA, denoted by $P_{ICA}$, was defined by the x-axis together with the line connecting centroid of ICA contour immediately distal to the bifurcation [i.e., $C_{ICA_{up}}$ in Fig. 2(a)] and centroid of the most distal ICA contour [i.e., $C_{ICA_{down}}$ in Fig. 2(a)]. The arterial surface was cut by $P_{CCA}$ and $P_{ICA}$ and unfolded into two connected 2D rectangular domains as shown in Fig. 2(c).

The boundaries of the 3D map after being cut were defined by the original boundaries $\sigma_0$, $\sigma_1$, and $\sigma_2$ and the lines where CCA and ICA were cut, denoted by $C_0$ and $C_1$, respectively [Fig. 2(a)]. These boundaries were mapped to seven edges of the 2D map. The corresponding 3D locations of these edges labeled 1–7 are listed below:

- Edge 1: $y_0 \rightarrow \sigma_0$.
- Edge 2: $y_0 \rightarrow C_0$.
- Edge 3: BF $\rightarrow \sigma_0$.
- Edge 4: BF $\rightarrow C_1$.
- Edge 5: $y_1 \rightarrow \sigma_1$.
- Edge 6: $y_1 \rightarrow \sigma_1 \rightarrow \sigma_0$.
- Edge 7: BF $\rightarrow C_0$.

The transverse contours on the CCA and ICA surfaces were mapped to the 2D map as shown in Fig. 2(c). $P_{CCA}$ cut the CCA surface along $C_0$ on the negative y side and along $C_0'$ on the positive y side [Fig. 2(a)]. Suppose the example transverse contour on the CCA surface with $z = z_0$ shown in Fig. 2(b) intersected with $C_0$ and $C_0'$ at $I_{C_1}$ and $I_{C_1}$, respectively. This contour was mapped to a straight line in the 2D map with $s = z_0$ with (i) the segment of the contour from $I_{C_1}$ to $I_{C_1}$, mapped in the clockwise direction from ($-L_{ECA},z_0$) to ($0,z_0$) and (ii) the segment from $I_{C_1}$ to $I_{C_1}$, mapped from ($0,z_0$) to ($L_{ICA},z_0$) in the same manner. Suppose the example transverse contour on the ICA surface with $z = z_1$ intersects with $C_1$ at $I_{C_3}$ as shown in Fig. 2(b). This contour was mapped in the clockwise direction to a straight line starting from ($0,z_1$) to ($L_{ICA},z_1$). $L_{ECA}$ and $L_{ICA}$ were obtained in a standard way related to the average surface area of the carotid arteries in the study population as previously described in Chiu et al. 

\[\text{Fig. 1. Flowchart of the correspondence optimization algorithm.}\]

\[\text{Fig. 2. Construction of the 2D VWT map using the arc-length scaling approach. (a) shows the two planes $P_{ICA}$ and $P_{CCA}$ cutting the ICA and CCA, respectively, with points defining these planes identified. The lines that the two planes cut, as well as the inlet and outlets of the carotid, are also labeled. (b) shows the same carotid geometry as (a) but with example contours and intersection points between the cutting planes and example contours labeled. (c) 2D VWT map generated by the AL approach.}\]
2.B. Procrustes alignment using landmarks in the 2D VWT map

Each 2D flattened map was sampled in a 0.3 mm interval horizontally and vertically to generate the 2D VWT carotid template. Because of the cylindrical nature of the carotid artery, points lying on C0 were mapped both to Edges 2 and 7 [Fig. 2(b)]. Similarly, points along the cutting line C1 on the ICA surface were mapped to Edges 4 and 6. When excluding the points lying on Edges 6 and 7, the points on the 2D standardized map represent a set of N landmarks with one-to-one mapping with the corresponding 3D map. A 2D standardized map was arbitrarily selected from the data set as a master example and the landmarks on it were fixed. Generalized Procrustes alignment involving translation, rotation, and uniform scaling was applied to align all carotid arteries in an iterative process described in Cootes et al.38 Suppose the set of N landmarks associated with the jth subject are represented by a 3N-dimensional vector \( L_j = [x_{1j} \ldots x_{Nj}, y_{1j} \ldots y_{Nj}, z_{1j} \ldots z_{Nj}]^T \), where the 3D coordinates of the ith landmark were denoted by \((x_{ij}, y_{ij}, z_{ij})\). Each carotid surface was normalized to 1 (i.e., \(|L_i| = 1\)) in the alignment process, resulting in a root mean square (RMS) radius of \(1/\sqrt{N}\) for each carotid surface.

2.C. Statistical shape model and the calculation of description length (DL)

The vectors containing the landmarks of all carotid arteries were concatenated to form the landmark configuration matrix, \( L \):

\[
L = [L_1 \ L_2 \ \ldots \ L_S],
\]

(1)

where \( S \) is the total number of arteries. The principal components of the group of \( S \) arteries, denoted by \( p_m \), can be obtained by eigendecomposition of the covariance matrix of \( L \). The shape model can be described by a linear combination of the mean shape \( \bar{L} \) and the principal components \( p_m \):

\[
L_i = \bar{L} + \sum_m b_m p_m,
\]

(2)

where \( \bar{L} = 1/S \sum_i L_i \) and \( b_m \) are constants specifying the weight of each principal component. The principal components can also be obtained by singular value decomposition (SVD) of the centered and unbiased matrix \( A = (1/\sqrt{S-1}) (L - \bar{L}) \), where \( \bar{L} = [\bar{L} \ \bar{L} \ \ldots \ \bar{L}] \), i.e.,

\[
A = U D V^T.
\]

(3)

\( U \) holds the eigenvectors of the matrix \( A A^T \) (i.e., the principal components \( p_m \)) and \( D \) holds the corresponding eigenvalues (i.e., suppose \( d_m \) is the mth diagonal element of \( D \) and \( \lambda_m \) is the eigenvalue corresponding to the mth eigenvector of \( A A^T \). \( \lambda_m = d_m^2 \)). This SVD approach allows the calculation of gradient information in the optimization process as described below.31

The DL, denoted by \( F \), is defined as

\[
F = \sum_m \xi_m,
\]

(4)

where

\[
\xi_m = \begin{cases} 1 + \log(\lambda_m/\lambda_{cut}), & \lambda_m > \lambda_{cut} \\ \lambda_m/\lambda_{cut}, & \lambda_m \leq \lambda_{cut} \end{cases}
\]

(5)

and \( \lambda_{cut} \) is a cut-off constant which accounts for noise in the training surfaces.31 We adopt the same \( \lambda_{cut} \) value of 0.0032 as Refs. 31, 32, and 35 in our calculation of DL.

2.D. Calculation of the DL gradient and reparameterization on the 2D carotid template

The 2D map introduced in Sec. 2.A has provided an initial parameterization \( p : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and we denote the coordinates in the 2D space by \((r, s)\). Figure 3 shows the notations we use hereafter: labels starting with \( l \) are used to represent landmarks in the 3D space, and their locations in the 2D space are denoted by the mapping \( p \) [e.g., \( p(l) \) in Fig. 3(a)]. To optimize the point correspondences using the MDL criterion, we reparameterized each landmark from an initial position \((r_0, s_0)\) to a new position on the 2D space [i.e., \( \rho^{\text{old}}(l) \rightarrow \rho^{\text{new}}(l) \)]. After this reparameterization, a new landmark \( l^{\text{new}} \) replaced the original landmark \( l \) as the landmark that was parameterized by \((r_0, s_0)\) [i.e., \( (r_0, s_0) = \rho^{\text{new}}(l^{\text{new}}) \)]. We moved \( p \) in the direction of the greatest descent of \( F \), which is opposite to the gradient direction of \( F \) with respect to horizontal and vertical movements in the 2D space, denoted by \( \Delta r, \Delta s \) [i.e., \((\Delta r, \Delta s) = (\partial F/\partial \Delta r, \partial F/\partial \Delta s) \)]. The gradient of \( F \) at each landmark \( l_{ij} \), whose 3D coordinates are \((a_{ij}, a_{i+N,j}, a_{i+2N,j})\) in the matrix.

![Fig. 3. Schematic of the reparameterization approach. (a) shows the 3D landmark \( l \) mapped to 2D space with coordinates \( p(l) = (r_0, s_0) \) and its neighboring points \( p(l^{\text{left}}), p(l^{\text{right}}), p(l^{\text{upper}}), \) and \( p(l^{\text{down}}) \) along \( r \) and \( s \) direction. (b) \( p^{\text{old}}(l) \) moves in the positive \( r \) direction with distance \( \Delta r \) to \( p^{\text{new}}(l) \) after an iteration. The new landmark, denoted by \( l^{\text{new}} \), replaces the \( l \) and is now parameterized by \( (r_0, s_0) \) in the 2D carotid template.](image-url)
A as described in Sec. 2.C, can be written as follows:

$$\frac{\partial F}{\partial \Delta r} = \sum_k \frac{\partial F}{\partial a_{kj}} \frac{\partial a_{kj}}{\partial \Delta r} \quad k \in \{i,j+N,i+2N\},$$

$$\frac{\partial F}{\partial \Delta s} = \sum_k \frac{\partial F}{\partial a_{kj}} \frac{\partial a_{kj}}{\partial \Delta s} \quad k \in \{i,j+N,i+2N\}. \quad (6)$$

To compute $\partial F/\partial a_{ij}$, we need the following result derived by Ericsson and Åström: 42

$$\frac{\partial m}{\partial a_{ij}} = u_{im}v_{jm}, \quad (7)$$

where $u_{im}$ is the $(i,m)$ element of matrix $U$ and $v_{jm}$ is the $(j,m)$ element of matrix $V$ in Eq. (3). Using this result, $\partial F/\partial a_{ij}$ can be expressed as

$$\frac{\partial F}{\partial a_{ij}} = \sum_m \frac{\partial a_m}{\partial a_{ij}} \frac{\partial F}{\partial a_m}, \quad (8)$$

with

$$\frac{\partial a_m}{\partial a_{ij}} = \begin{cases} 2u_{im}v_{jm}/d_{im}, & \lambda_m > \lambda_{cut} \\ 2d_{m+1,m+1}/\lambda_{cut}, & \lambda_m < \lambda_{cut} \end{cases}. \quad (9)$$

It remains to compute the derivative of the coordinates of the landmark $l$ with coordinates $(a_{ij},a_{i+N,j},a_{i+2N,j})$ with respect to the horizontal and vertical displacements, denoted by $\Delta r$ and $\Delta s$, respectively. Consider Fig. 3 where the parameterization of $l$ was moved horizontally in the positive $r$ direction, with the landmark obtained after reparameterization denoted by $l'_{\text{new}}$. The 3D coordinates of the new landmark can be obtained by linear interpolation between the points $p(l^\text{left})$ and $p_{\text{new}}(l)$. The derivation of the derivative $\partial l_{\text{new}}/\partial \Delta r$ is provided in the Appendix, which can be expressed as

$$\frac{\partial l_{\text{new}}}{\partial \Delta r} \bigg|_{\Delta r \to 0} = \frac{\partial (a_{ij},a_{i+N,j},a_{i+2N,j})}{\partial \Delta r} \bigg|_{\Delta r \to 0} = \frac{l_{\text{left}} - l_{\text{right}}}{2d}. \quad (10)$$

Similar analysis for the vertical movement ($\Delta s$) gives

$$\frac{\partial l_{\text{new}}}{\partial \Delta s} \bigg|_{\Delta s \to 0} = \frac{\partial (a_{ij},a_{i+N,j},a_{i+2N,j})}{\partial \Delta s} \bigg|_{\Delta s \to 0} = \frac{l_{\text{down}} - l_{\text{upper}}}{2d}. \quad (11)$$

Substituting Eqs. (10) and (11) into Eq. (6) gives $\partial F/\partial \Delta r$, $\partial F/\partial \Delta s$ at each point $(r_0,s_0)$. Each point $(r_0,s_0)$ in the 2D parameterized space was moved in the direction $-\partial F/\partial \Delta r$, $-\partial F/\partial \Delta s$ for greatest descent of $F$. Self-intersection of the grid could occur if the distance of each movement is larger than or equal to $d/2$. Denoting the maximum magnitude of gradient among the whole 2D map by $m_{\text{max}}$, if $m_{\text{max}}$ is smaller than or equal to $d/2$, each point would move a distance that equals to the gradient magnitude. If $m_{\text{max}}$ is greater than $d/2$, the movement would be scaled by a factor of $d/2m_{\text{max}}$ so that the maximum distance that a point moved was clamped at $d/2$.

In the above description, the movement of each point was determined by four neighboring points (i.e., $l_{\text{left}}, l_{\text{right}}, l_{\text{upper}}$, and $l_{\text{down}}$). Special treatment must be given to boundary points where at least one of the neighboring points was missing. Boundary points were classified into three categories: (1) bifurcation point, (2) points on the horizontal boundary, and (3) points on the vertical boundary. As the bifurcation has been identified by a previously described algorithm from the 3D carotid surface, it was not reparameterized. Points on the horizontal boundaries were allowed to move horizontally only in order to preserve the rectangular shape of the 2D standardized map. Whether to move left or right was determined by Eq. (10).

For each sample point on the vertical Edges 2 and 4, the left neighbor is apparently missing. However, as described previously, landmarks mapped to Edges 2 and 4 were also mapped to Edges 7 and 6, respectively, because of the cylindrical nature of the artery. Mathematically, the inverse mapping of $p$, $p^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$, is a periodic function in $r$:

$$p^{-1}(r,s) = p^{-1}(r + k(L_{ECA} + L_{ICA}), s), \quad k \in \mathbb{Z}.$$ 

In other words, $p$ maps each 3D landmark $l$ to infinitely many locations in the 2D space with adjacent 2D locations separated by $L_{ECA} + L_{ICA}$. However, $p(l)$ is uniquely defined in any interval of $[L_{ECA} + L_{ICA}, 0)$. With these notations defined, a left neighbor for $p_{\text{left}}(l_3)$, denoted by $q(l)$ in Fig. 4(a), can be established, which when mapped back to the 3D surface using $p^{-1}$ results in the same landmark as the left neighbor of $p_{\text{left}}(l_3)$, $p(l_{\text{left}})$. Thus, $p^{-1}(q) = l_{\text{left}}$. The availability of the inverse mapping of this left neighbor allowed us to determine $\partial F/\partial \Delta r, \partial F/\partial \Delta s$ and thus the movement direction of $p_{\text{left}}(l_3)$ according to Eqs. (10) and (11).

After all sampling points in a 2D map were deformed, they were resampled by a regular grid with isotropic intervals of 0.3 mm as shown in Fig. 4(b). The deformation of the 2D map resulted in reparameterization, in which the original parameterization $p$ was changed to $p_{\text{new}}$. Points in Fig. 4(b) were labeled by their mapping $p_{\text{new}}^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ (i.e., the 3D landmarks that would map to the labeled points). Figure 4(b) summarized the interpolation scheme for obtaining new landmarks in three cases. For the case in which the sample point lay inside the border of the 2D map, such as $l_{\text{new}}^\text{left}$, bilinear interpolation was performed based on the four neighbors (e.g., $l_{\text{left}}^\text{left}, l_{\text{left}}^\text{right}, l_{\text{right}}^\text{right},$ and $l_{\text{right}}^\text{left}$ in Fig. 4(b)). For the case in which the sample point lay on the horizontal edges, such as $l_{\text{new}}^\text{left}$, interpolation was based on only two neighbors. Depending on whether the horizontal movement was to the positive or negative side of $r$, the two neighbors could be $l_{\text{left}}^\text{left}$ and $l_2$, or $l_{\text{right}}^\text{left}$ and $l_2$. For the case in which the sample point lay on the vertical edges, such as $l_{\text{new}}^\text{left}$, interpolation was based on four neighbors (e.g., $l_3, l_3^\text{down}, l_3^\text{up},$ and $l_3^\text{left}$ in Fig. 4(b)). For this case, the interpolation result was copied to the corresponding point on the opposite edge. We would emphasize that although movements of points on the 2D map involved change of parameterization $(r,s)$, the underlying 3D surface was not deformed. One iteration of the DL minimization algorithm concluded when 2D maps for all arteries investigated were deformed and resampled as described. Procrutes alignment was performed again based on the new set of landmarks and
improvement of interscan variability of VWT was evaluated in the in vivo 3D ultrasound study as described in Sec. 3.C.

3.A. Generalization and compactness of the statistical models

This set of evaluation metrics quantifies the quality of the statistical shape model constructed from the correspondence set. As models built from more “accurate” point correspondence are expected to be more compact and have a better generalization ability, these two metrics are widely used to assess the quality of the correspondence set.\textsuperscript{29,30} Here, the compactness and the generalization ability of the proposed MDL-based optimization and the AL approaches were compared. Specificity assesses the ability of the model to generate shapes similar to those in the training set and is more important for applications where a newly generated shape needs to be validated, such as model-based deformation and prediction in segmentation.\textsuperscript{29,40} As our study involved analysis of carotid artery geometry and no new carotid shape was generated, specificity was less relevant and not evaluated.

As detailed in Sec. 2.B, each shape is represented by a shape vector generated by concatenating coordinates of correspondence points. Mismatches of correspondence points would lead to an exaggeration of the shape variance. This effect is best demonstrated by an extreme case in which a shape model generated from a set of identical, but differently sampled, shape is associated with apparent shape variation. For this reason, a model generated from anatomically equivalent correspondence tends to be a compact model, which has a small variance and requires a few principal components to describe a shape. Compactness has been widely used to assess the quality of correspondence in statistical model literature.\textsuperscript{29,30} and is quantified by the cumulative eigenvalue

$$C(M) = \sum_{m=1}^{M} \lambda_m,$$

where $\lambda_m$ is the $m$th eigenvalue and the standard error of $C(M)$ could be computed as

$$\sigma_{C(M)} = \sqrt{\frac{2}{S} \lambda_m},$$

where $S$ is the number of shapes involved.

Generalization described the ability of the model to represent unseen shapes and was quantified using the leave-one-out approach. Briefly, models were built using all but one artery in the data set and then fitted to the excluded subject. In each leave-one-out trial, the sum-of-squares approximation error was computed. The average fitting error using different number of modes (i.e., $M$) from $S$ leave-one-out trials, denoted by $G(M)$, was computed to quantify the generalization ability, and the standard error of $G(M)$ was calculated based on the sample standard deviation. Both compactness and generalization were functions of the total number of principal components used in the model (i.e., $M$) and a better model has smaller values. Here, $C(M)$ and $G(M)$ associated with the AL and MDL approaches were compared for a range of $M$.

3. EXPERIMENTAL METHODS

As the purpose of the MDL-based optimization algorithm is to improve the quality of point correspondence for a group of carotid arteries, there is a need to define metrics to assess the “correctness” of the point correspondence generated. Here, we evaluated the performance of the proposed MDL technique in a phantom (Sec. 3.B) and an in vivo study (Sec. 3.C). In both studies, the compactness and generalization ability of the statistical shape model built from the correspondence established were evaluated as described in Sec. 3.A. In addition, the effect of correspondence optimization on the

Fig. 4. Schematic for the resampling process after reparameterizing points inside and on the boundaries of the 2D carotid template. (a) The parameterization of a point inside the carotid template, $l_1$, a point lying on the horizontal edge, $l_2$, and a point lying on the vertical edge, $l_3$, and their neighbors before reparameterization. (b) Resampling was performed after reparameterization with new landmarks established. Here, points are labeled with the new inverse mapping $p_{\text{new}}^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. $l_i^{\text{new}}$ for $i = 1, 2, 3$ denotes landmarks after an iteration had computed.
3.B. Phantom experiment

Three computational phantoms of human carotid arteries were constructed based on the radius and degree of stenosis (defined by NASCET stenosis index) measurements on x-ray angiograms of 62 symptomatic patients with a wide range of stenosis index. The location of the centerline of the vessel from CCA extending to ICA was obtained by averaging the carotid centerlines of the whole group of patients. The radii of four prominent locations along the CCA–ICA segment were expressed as a function of the degree of stenosis. With these radial profiles, the shape of the artery with an arbitrary stenosis value can be computationally reconstructed using spline interpolation as described in Smith et al. In this experiment, the phantom with normal geometry and the 30%– and 50%-stenotic phantoms shown in Fig. 7 were flattened to a 2D plane using the AL approach and subsequently optimized as described in Sec. 2.D. To avoid clustering of correspondence points, the 2D map of the phantom with normal geometry served as the master example with fixed landmarks. The landmarks in the 2D maps of the two stenotic vessels were moved in the direction of greatest descent of DL according to Sec. 2.D. Correspondence relationships obtained using the previously described AL approach and the proposed MDL technique were visually compared and qualitatively assessed. Generalization and compactness associated with the initial and optimized parameterization were compared.

3.C. Experiment using in vivo 3D ultrasound images

Figure 5 shows the flowchart of the experimental procedure with essential equations and notations identified. Each step is detailed in Subsections 3.C.1–3.C.4.
3.C.1. Study subjects, 3D ultrasound image acquisition and segmentation

Ten subjects were involved in a study focusing on the evaluation of interscan reproducibility of 3D ultrasound carotid imaging. These subjects were recruited from the Premature Atherosclerosis Clinic and The Stroke Prevention Clinic at University Hospital (London Health Science Centre, London, Canada) and the Stroke Prevention and Atherosclerosis Research Centre (Robarts Research Institute, London, Canada) and were asymptomatic with carotid stenosis >60% (according to carotid Doppler flow velocities). 3D ultrasound images of this group of subjects were required at baseline and 2 weeks later. No physiological changes were expected for these patients with stable atherosclerosis. The 2-week gap was chosen to maximize variability due to sonographer change, patient repositioning, and different neck orientations, while minimizing the inconvenience of the patients. All subjects provided written informed consent to the study protocol, which was approved by The University of Western Ontario Standing Board of Human Research Ethics.

The 3D ultrasound imaging system used for image acquisition is described in detail elsewhere and briefly summarized here. A conventional transducer (L12-5, Phillips, Bothel, WA) was mounted on a mechanical assembly and translated along the neck for approximately 4.0 cm at a uniform speed of 3 mm/s. The 2D ultrasound frames thus acquired were reconstructed into a 3D ultrasound image with voxel size of approximately 0.1×0.1×0.15 mm³.

The 3D carotid ultrasound images were resliced at 1 mm interslice intervals perpendicular to the CCA axis manually identified. The lumen and outer wall boundaries on each 2D resliced image were segmented five times by an expert observer.

3.C.2. Computation of 3D vessel-wall-plus-plaque thickness (VWT) and its standard error

The mean VWT and its standard error at each point of the contour were computed as described in Chiu et al. and briefly summarized here using the schematic shown in Fig. 6. A mean contour was computed for each of the lumen and outer wall segmentations based on five repeated segmentations. The mean contours were matched on a point-by-point basis, with the distance between each pair of correspondence designated

![Diagram](image)

**Fig. 6.** Method for computing the mean and standard error of VWT. The purple and green contours represent repeated segmentations of lumen and wall boundaries, respectively. Mean lumen and wall boundaries (red contours) are matched on a point-by-point basis. Each pair of corresponding points (e.g., $p_w$ and $p_l$) defines a line that intersects the lumen and wall segmentation contours. The distance between $p_w$ and $p_l$ was defined as mean VWT ($\bar{T}$) and the standard deviation of the lumen and wall could be computed by the intersections (purple and green dots).

![Image](image)

**Fig. 7.** Correspondences between phantoms before and after DL minimization. (a) The semitransparent surface represents the lumen with normal geometry and the solid surface represents the 30%-stenotic lumen. (b) The semitransparent surface represents the lumen with normal geometry and solid surface represents the 50%-stenotic lumen. In both (a) and (b), blue lines represent correspondences before DL minimization and red lines connect pairs of corresponding points after DL minimization.

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<thead>
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<th>Modes ($M$)</th>
<th>$G(M)$</th>
<th>$C(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL</td>
<td>MDL</td>
<td>AL</td>
</tr>
<tr>
<td>1</td>
<td>0.50 (0.30)</td>
<td>0.43 (0.26)</td>
</tr>
<tr>
<td>2</td>
<td>0.50 (0.30)</td>
<td>0.43 (0.26)</td>
</tr>
<tr>
<td>3</td>
<td>0.50 (0.30)</td>
<td>0.43 (0.26)</td>
</tr>
</tbody>
</table>
as the mean VWT, $\bar{T}$. The line connecting each pair of correspondence intersected the repeated outer wall and lumen segmentations. Therefore, the standard deviations of the outer wall and lumen contours at each point can be computed and are denoted by $s_W$ and $s_L$, respectively. The standard error, SE$_T$, and the degree of freedom, $\nu_T$, associated with $\bar{T}$ can be expressed in terms of $s_W$ and $s_L$ as in Eqs. 1 and 2 in Chiu et al.,$^{37}$ respectively. Each vertex on the VWT map was now equipped with three essential quantities (i.e., $\bar{T}$, SE$_T$ and $\nu_T$) required for the $T$-test used to establish whether a significant point-wise VWT-Change was detected between the baseline and follow-up scans.

![Diagram](image.png)

**Fig. 8.** 2D and 3D maps for an example subject with mean VWT ($\bar{T}$) and its standard error (SE$_T$) color-coded and superimposed. The first column shows maps constructed for the baseline image and the second column shows maps constructed for the follow-up image. The first and second rows show the 3D VWT and standard error maps. The third and fourth rows show the 2D VWT and standard error maps constructed using the AL approach. The fifth and sixth rows show the 2D VWT and standard error maps after correspondence optimization.
The first column shows the ΔVWT for the example subject shown in Fig. 8 and the second column shows the results of the statistical test superimposed on the 2D carotid template. In the second column, white indicates statistical significance of ΔVWT, whereas black indicates insignificance. The first row shows the results generated using the AL approach and the second row shows the results after DL minimization.
4. RESULTS

4.A. Phantom experiment

The phantoms with normal geometry, the 30%- and 50%-stenotic phantoms were flattened to 2D planes and resampled so that each map contained 2796 vertices. It took the MDL correspondence optimization algorithm 13 min to converge. The phantom with normal geometry corresponds with the 30%- and 50%-stenotic phantoms as shown in Figs. 7(a) and 7(b), respectively. For the AL technique,28,37 points on contours with the same longitudinal coordinate after standard 3D transform as described in Sec. 2.A were corresponded, as represented by the blue lines in Fig. 7. The red lines show the correspondence obtained after DL minimization. Lines joining correspondence points of two 3D surfaces in a “good” set of point correspondence are expected to be

![Graph showing comparison of ∆VWT generated by the AL and MDL approaches.](image)

**Fig. 10.** Comparison of ∆VWT generated by the AL and MDL approaches. (a) shows the histogram of the percentages of vertices with ∆VWT significantly different from 0 for each subject generated by the AL and MDL approaches. (b) shows the histogram of 27840 ∆VWT measurements (2784 measurements/subject × 10 subjects) obtained using the AL and MDL approaches.

![Comparison of ∆VWT generated by the AL and MDL approaches.](image)

**Fig. 11.** Group-average ∆VWT (first row), its standard error (second row), and the results of statistical test on group-average ∆VWT (third row). The first column shows the results generated using the AL approach and the second column shows the results after correspondence optimization.
Table II. Description length (DL) and percentage of points on average map with \( \Delta VWT \) significantly different from 0 at different stages of optimization. Times are shown in hours and min (h:min). Settings I and II refer to two different experiments with different master examples chosen.

<table>
<thead>
<tr>
<th>Time (h:min)</th>
<th>Description length</th>
<th>% points on average map with ( \Delta VWT ) significantly different from 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>0:00</td>
<td>88</td>
</tr>
<tr>
<td>Intermediate</td>
<td>0:30</td>
<td>71 70</td>
</tr>
<tr>
<td>Convergence</td>
<td>4:00</td>
<td>58 58</td>
</tr>
</tbody>
</table>

approximately perpendicular to both surfaces. In this sense, the correspondence generated by the MDL algorithm is more appropriate than the initial correspondence generated by our previously described AL approach. Table I shows generalization and compactness of statistical shape models associated with the AL and MDL approaches for the first three modes. Smaller values of \( G(M) \) and \( C(M) \) indicated that a more compact model with higher generalization ability could be obtained with DL minimization.

4.B. Experiment on in vivo 3D ultrasound images

Figure 8 provides a visual comparison between the 2D VWT maps generated by the AL and the MDL approaches for an example subject. The left column shows maps generated at baseline and the right column shows maps generated at follow-up. The first and second rows show 3D surfaces with VWT and its SE color-coded and superimposed, respectively. The third and fourth rows show the 2D maps generated by the AL approach with VWT and SE superimposed, and the fifth and sixth rows show the optimized 2D maps with VWT and SE superimposed. The first row of Fig. 9 shows the 2D maps generated by the AL approach with \( \Delta VWT \) and its significance superimposed, with white indicating that \( \Delta VWT \) was significantly different from 0 and black indicating otherwise. The second row shows the optimized 2D maps with \( \Delta VWT \) and its significance superimposed. 21% of the points had \( \Delta VWT \) significantly different from 0 in the maps generated by the AL approach for this subject, compared to 15% in the optimized map. 35% of the points had \( \Delta VWT \) ranging from \(-0.1\) to \(0.1\) mm in the map generated by the AL approach, compared to 39% in the optimized map.

Figure 10(a) shows the percentage of points associated with \( \Delta VWT \) significantly different from 0 for the ten subjects involved in this study. This percentage was 1%–7% lower in the optimized maps, compared to the maps generated by the AL approach. There were 2784 points in each 2D map and each point was equipped with a \( \Delta VWT \) measurement. All point-wise \( \Delta VWT \) measurements obtained for the ten subjects were pooled together, resulting in 27 840 \( \Delta VWT \) measurements for each of the AL and MDL methods. Figure 10(b) shows the histogram of these \( \Delta VWT \) measurements. 50% of \( \Delta VWT \) measurements obtained based on the optimized map ranged from \(-0.1\) to \(0.1\) mm, compared to 45% for the AL approach.

Figure 11 shows the unoptimized and optimized maps of (i) intersubject \( \Delta VWT \) [Eq. (18)], (ii) its SE [Eq. (20)], and (iii) statistical conclusion on whether \( \Delta VWT \) is significantly different from 0 at a significant level of 5%. As mentioned before, a 2D standardized map was arbitrarily selected as the master example and the landmarks on it were not moved throughout the optimization process. Two experiments with different master examples chosen, referred to as Settings I and II in Table II, were performed to investigate the sensitivity of the correspondence to the master example selection. It took approximately 4 h for the MDL algorithm to converge in both settings. Since the 4-h convergence time is long, it is sensible to report how much improvement on the quality of the correspondence had been accomplished in a fraction of the convergence time so that a user of this algorithm could make a decision on how long he/she would spend on the algorithm based on the performance level desired. Table II reported the DL and the statistical results of the group-level statistical analysis described in Sec. 3.C.4 before the algorithm started, after the algorithm had executed for 30 min, which we referred to as the intermediate point, and at the convergence point for both settings. Figure 12 shows a plot of DL against number of iterations for settings I and II. As DL decreased

![Figure 12](image-url)
monotonically with respect to the number of iterations applied, the long time spent was not caused by oscillation around local minima. In addition, DL decreased sharply at the early stage, and the amount of DL reduction was smaller approaching convergence.

Figure 13 shows the generalization ability and compactness of statistical shape models associated with the initial, intermediate, and convergence points for settings I and II. Results shown in Table II, Figs. 12 and 13 show that the performance of the MDL algorithm was not sensitive to the choice of the master example. It is noteworthy that the generalization capabilities up to the seventh modes of the model obtained in the intermediate point were approximately the same as those obtained at convergence for both experimental settings.

5. DISCUSSION AND CONCLUSION

In this paper, we proposed a correspondence optimization algorithm for 2D carotid template construction. We introduced a strategy to reparameterize the boundaries of the 2D map, which include the inlet/outlet boundaries and locations where the CCA and ICA were cut. Most previously described 3D surface correspondence approaches were designed specifically to handle genus-0 closed surfaces, in which there are no boundaries, such as the lateral ventricles and hippocampi of the brains, livers, lungs, and kidneys. In the calculation of the gradient of DL, we also made a contribution in showing that the derivatives of the 3D coordinates of each landmark represented in the 2D map with respect to horizontal and vertical reparameterizations (i.e., $\frac{\partial l_{\text{new}}}{\partial r}$, $\frac{\partial l_{\text{new}}}{\partial s}$) are equivalent to the tangents of the 3D surface along lines with constant horizontal and vertical parameterizations, respectively [Eqs. (10) and (11), Appendix]. This finding lays a mathematical foundation for the computation of the gradient of DL and leads to an efficient implementation of the steepest descent algorithm. Our algorithm also has the advantage that the initial 2D parameterization can be quickly computed using the AL approach (Sec. 2.A).

We showed that the MDL-optimized maps in the phantom and the in vivo studies led to a more compact statistical model and higher generalization ability, which are two criteria introduced by Davies et al. and have since been widely used in evaluating the quality of shape correspondence. In addition to these quantitative measurements, an important criterion driven by our application in carotid imaging is the effect on the reproducibility of VWT measurements. The patients involved in this study were scanned twice within a 2-week period in an interscan study. No physiological change was expected for this group of patients with stable atherosclerosis. The 2-week gap was chosen to maximize the variability due to patient repositioning and sonographer changes while minimizing the inconvenience of the patients. Subject-level statistical analysis shows that the percentage of points with statistically significant VWT-change was reduced after correspondence optimization for all ten subjects. Group-level statistical analysis was applied to evaluate whether the group-average VWT-Change at each point of the template was statistically significant. Our results showed that the percentage of points with group-average VWT-Change significantly different from 0 was reduced. These statistical tests indicated that DL minimization reduced the effects of interscan variability in VWT measurements.

On top of the quality of correspondence of carotid surfaces we focused to optimize in this paper, other sources of variability in VWT measurements include variabilities in segmenting the wall and lumen boundaries as well as in the identification of the bifurcation point by the expert observer, which we incorporated in our evaluation by involving repeated segmentations (Fig. 5). While we acknowledge that landmarks established for each vessel that was used in subsequent generalized Procrustes alignment (Sec. 2.B) would be affected by an aberrant segmentation with in-plane outline and bifurcation selection substantially deviating from the mean, we would
point out that this effect was minimal as landmarks for each subject were established on the mean shape computed by averaging the five repeated segmentations. To handle outliers, a median may be more appropriate than mean as discussed in our previous paper dedicated to the study of the carotid segmentation reproducibility. However, statistical tests on ∆VWT measurements described in Sec. 3.C.3 would no longer be valid if the median were used instead. New statistical tests based on the probability density function of the median in the line of Rider would need to be developed before statistical tests could be performed. The average standard deviation of repeated segmentations in this study was around 0.2–0.3 mm [Figs. 8(g) and 8(h)]. In Chiu et al., we demonstrated that the mean and median contours were not substantially different even with standard deviations of repeated segmentations up to 1.2 mm. For this reason, development of a new set of statistical tests may not be necessary.

Limitations include the use of a small group of subjects in study in which we piloted this approach. The statistical evaluation involved five repeated segmentations of the wall and lumen boundaries for 20 images (10 subjects × 2 time points). As the expert observer required 20–30 min to complete a single segmentation for each image, segmenting 20 images had already taken a considerable amount of time. While we acknowledge that these ten carotid arteries may not be sufficient to represent the variability in the carotid geometry in a broader population, it would be sensible to perform a pilot statistical evaluation of the proposed algorithm in this small population before including more subjects. As we have demonstrated the efficacy of the proposed technique here, it can serve as a module to enhance the capability of our existing VWT analysis pipeline, and can be applied in future clinical studies similar to those previously described. Although the substantial decrease in the percentage of points associated with statistically significant ∆VWT (from 18% to 11%, which amounts to a 38% decrease) showed the effectiveness of the MDL approach in reducing intersegmental variability of ∆VWT measurements, the AL and MDL maps obtained for patients in this pilot study were visually similar as shown in Figs. 8 and 9. This observation raised the question of whether it is clinically important to optimize correspondence using MDL. In this pilot study involving patients scanned within two weeks, we observed that the carotid surfaces obtained at baseline and follow-up were very similar from 0 had been reduced from 18% to 13%, which is comparable to the converged value of 11%. This result suggests that a user has the option of cutting the computational time by one-eighth and making a small compromise on the quality of the correspondence. Two approaches could be applied to reduce the computational time requirement without compromising the correspondence quality, and they can be potentially combined. The maximum distance each landmark can move in each iteration was limited to half of the interval between adjacent landmarks in the sampling grid as discussed in Sec. 2.D to avoid self-intersection of the grid. Prescribing a larger allowable displacement would reduce the number of iterations required to converge, but would require a postprocessing step after each iteration to eliminate self-intersection by ensuring the mapping of landmarks from one iteration to the next iteration is order-preserving both in the vertical and horizontal dimensions. Another full study would be required to develop and validate such a postprocessing technique. Another option would be to apply the parallel processing capability of GPU. In our experiment, there are 2784 landmarks on each 2D map. In each iteration of optimization, gradient directions of 52 896 points (2784 points/map × 19 maps) were required to be computed. Since gradient calculations at different landmarks were independent, parallel computation can be implemented using GPU to reduce computational time.

The motivation of this study stems from the requirement to optimize correspondences implicitly defined by the 2D standardized carotid template constructed and described previously. In this paper, we demonstrated the application...
of the proposed correspondence optimization algorithm on the improvement of the reproducibility of VWT in an interscan study. The algorithm can potentially be applied by increasing the sensitivity of the biomarker previously introduced to assess effects of stroke prevention strategies. Briefly, the biomarker measured mean VWT-Change specific to regions of interest in the 2D carotid template identified by a machine learning algorithm based on the difference in the VWT-Change distribution of the subjects receiving atorvastatin and placebo. We compared the sample sizes required to show statistically significant difference between the atorvastatin and placebo groups using the mean VWT-Changes computed over the regions of interest (denoted by ΔVWT) and the whole 2D maps (denoted by ΔVWT), and showed that the sample size required by ΔVWT was three times smaller than that required by ΔVWT. As carotid atherosclerosis is a local disease with plaques predominantly appearing in bends and bifurcations, ΔVWT, being a biomarker that focuses on VWT-Change within carotid regions of interest, is expected to be more sensitive than the mean VWT computed over the whole 2D map. As regional mismatches produced by the AL approach can be adjusted using the proposed technique, we hypothesize that ΔVWT computed based on the 2D maps with optimized correspondence will yet be more sensitive in quantifying the effect of an intervention than that computed based on the maps generated by the AL approach. We will carry out studies to assess this hypothesis.

The application of this algorithm is not limited to analysis of the ΔVWT distributions. Optimization of correspondence will be important for applications involving the use of the 2D carotid template to adjust for anatomic variability in a population of carotid arteries. These applications include analysis of local segmentation error and variability, correlation between hemodynamic forces and plaque components or geometric properties of the carotid luminal surface, and risk stratification based on fissure of plaque surface.

ACKNOWLEDGMENTS

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CONFLICT OF INTEREST DISCLOSURE

The authors have no COI to report.

APPENDIX: DERIVATION OF THE DERIVATIVE \( \partial l_{\text{new}}/\partial s \) (EQ. (10))

When a grid point \((r_0, s_0)\) in the 2D parameterized space moves in the positive \( r \) direction \((\Delta r > 0)\) as in Fig. 14(a), the 3D landmark it corresponds is changed from \( l \) to \( l_{\text{new}} \), with \( l_{\text{new}} \) determined by linear interpolation and expressed by

\[
l_{\text{new}} = \Delta r + d \frac{l_{\text{left}}}{\Delta r + d} + \frac{d}{\Delta r + d} l,
\]

(A1)

\[
\frac{\partial l_{\text{new}}}{\partial \Delta r} = \left[ \frac{1}{\Delta r + d} - \frac{\Delta r}{\Delta r + d} \right] l_{\text{left}} - \frac{d}{(\Delta r + d)^2} l.
\]

(A2)

Substituting \( \Delta r = 0 \) gives

\[
\left. \frac{\partial l_{\text{new}}}{\partial \Delta r} \right|_{\Delta r \to 0^+} = \frac{l_{\text{left}} - l}{d}.
\]

(A3)

Similarly, when \( \Delta r < 0 \) as in Fig. 14(b), \( l_{\text{new}} \) becomes

\[
l_{\text{new}} = \frac{|\Delta r|}{|\Delta r| + d} l_{\text{right}} + \frac{d}{|\Delta r| + d} l
\]

(A4)

\[
\frac{\partial l_{\text{new}}}{\partial \Delta r} = \left[ \frac{1}{d - \Delta r} - \frac{\Delta r}{(d - \Delta r)^2} \right] l_{\text{right}} + \frac{d}{(d - \Delta r)^2} l.
\]

(A5)

Substituting \( \Delta r = 0 \) gives

\[
\left. \frac{\partial l_{\text{new}}}{\partial \Delta r} \right|_{\Delta r \to 0^-} = \frac{l - l_{\text{right}}}{d}.
\]

(A6)

The difference in the derivatives obtained in Eqs. (A3) and (A6) indicates that \( \partial l_{\text{new}}/\partial \Delta r \) is discontinuous at \((r_0, s_0)\), which is expected for discrete geometry. Since the sampling of the 2D map is fine \((d = 0.3 \text{ mm})\), \( \partial l_{\text{new}}/\partial \Delta r \) can be approximated by the averaging of the two derivatives obtained

\[d\]

\[\Delta r\]

\[l_{\text{new}}(\text{left})\]

\[l_{\text{new}}(\text{right})\]

\[p_{\text{new}}(l_{\text{left}})\]

\[p_{\text{new}}(l_{\text{right}})\]

\[p_{\text{new}}(l)]\]

Fig. 14. Schematic of horizontal movement \((\Delta r)\) on the 2D parameterized space: (a) \( \Delta r > 0 \), (b) \( \Delta r < 0 \).
above

\[ \frac{\partial t^{\text{new}}}{\partial \Delta r} \bigg|_{\Delta r \to 0} = \frac{\partial (a_{ij}, a_{i+N,j}, a_{i+2N,j})}{\partial \Delta r} \bigg|_{\Delta r \to 0} = \frac{\rho_{\text{left}} - \rho_{\text{right}}}{2d}. \]  

(A7)

\[ \partial \]


