Sensitivity of Blocking Probability in the Generalized Engset Model for OBS (Extended Version)
Technical Report

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Abstract

The generalized Engset model can be applied to evaluate the blocking probability at an optical cross connect (OXC) of bufferless optical burst switching (OBS) system. For tractability, previous studies assumed that burst transmission time (*on-time*) and time intervals between bursts provided by the same input channel (*off-time*) are exponentially distributed. Here we aim to study the sensitivity of blocking probability to the shape of these distributions. Extensive numerical results demonstrate that the blocking probability is not very sensitive to on- and off-time distributions in general. We observe certain new effects that higher variance of on- and off-time distributions may lead to better performance.

Keywords: Blocking probability, optical burst switching (OBS), generalized Engset formula, traffic model, sensitivity.

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1. Introduction

Optical burst switching (OBS) [1, 2] is a switching technology proposed for wavelength division multiplexing (WDM) networks. It intends to combine the benefits of optical circuit switching (OCS) and optical packet switching (OPS). In OBS network, traffic is carried by bursts, which consist of IP packets.

There are various versions of OBS, including OBS/JET [3] and OBS/JIT [4], where bursts contending for a group of wavelength channels at each optical cross connect (OXC) may not use a large number of input channels to justify Poisson arrivals. Thus, an OXC cannot be simply regarded as an $M/M/k/k$ queuing system. Moreover, the Engset model is inaccurate for loss based OBS system [5, 6]. Instead, the generalized Engset model [7] could be applied.

Although the model in [5, 8] and [9] gives exact blocking probability solutions at an OXC when, for each input wavelength, on- and off-times are exponentially distributed, it does not provide exact solutions for other distributions. For the Engset model, blocking is insensitive to the shape of these distributions. However, the generalized Engset model does not possess such a property. This raises the importance to investigate *errors* introduced by assuming exponential distribution to evaluate blocking probabilities for other distributions. In this letter, we evaluate such errors when on- and off-time distributions are deterministic, exponential, hypo-exponential, hyper-exponential, Pareto and truncated Gaussian (to avoid negative values). We observe that blocking probability is generally not very sensitive to the shape of distributions but traffic whose on- and off-time distributions have higher variance may have lower blocking probability.

2. Methodology

2.1. Modeling of OBS OXC

As in [5], we focus on a set of output wavelength channels in an output cable of an OXC. Suppose there are F optical fibers in this cable, each of which carries W wavelengths. Without wavelength conversion, an arriving burst on a given wavelength must use the same wavelength at the output, so only F wavelength channels are available. With wavelength conversion, all FW output channels are available for an arriving burst. These available wavelength channels are considered as K *servers*. We use the term *sources* for the relevant input channels in each case. Without wavelength conversion, the sources are the input channels that have the same wavelength as the F output channels. With wavelength conversion, the sources are all the relevant input channels that provide bursts to the FW output channels. The number of sources is denoted by M. Each source transmits bursts as an on/off process, with mean on- and off-time equal to $1/\mu$ and $1/\lambda$, respectively.

If there is no output channel available for an arriving burst, the burst is dumped, in which case, it still occupies ("freezes") the input channel for the entire burst duration. We classify the sources to be free, busy and frozen. A source in its off-time is free and otherwise either busy, when its burst is being transmitted through an output channel, or frozen, when its burst is being dumped.

2.2. Markovian models

Cases where on- and off-times follow exponential, hyper-exponential or hypo-exponential distributions lead themselves to exact Markov chain analyses. Dimensions of these models are listed in Table 1.

Consider a hyper-exponentially distributed random variable with the following probability density

$$f(t) = p_1 f_{\mu_1}(t) + p_2 f_{\mu_2}(t) \quad (p_1 + p_2 = 1)$$

where $f_{\mu_1}(t)$ and $f_{\mu_2}(t)$ are probability densities of exponential distribution with parameters μ_1 and μ_2, respectively. That is, with probability p_i, the hyper-exponential random variable is governed by exponential distribution with parameter μ_i $(i = 1, 2)$. Accordingly, for hyper-exponentially distributed on-time and exponentially distributed off-time, we consider each busy or frozen source has two possible states. Sources transmitting bursts of exponentially distributed lengths with parameter μ_1 and μ_2 are said to be in states 1 and 2, respectively.

Let $\pi_{i,j,k,l}$ be the steady state probability that there are i and j busy sources in states 1 and 2, respectively, and k and l frozen sources in states 1 and 2, respectively $(0 \leq i + j \leq K;\ 0 \leq k + l \leq M - K;\ i, j, k, l \geq 0)$. For a free source, the rates to become busy or frozen in states 1 and 2 are $p\lambda$ and $(1 - p)\lambda$, respectively. The state transition

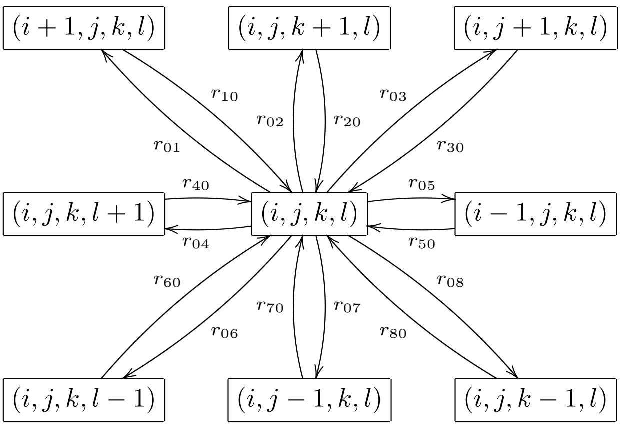
Table 1: Markovian models

On-time Distribution	**Off-time Distribution**	**Dimensions**
Exponential	Exponential	2
Hyper-exponential	Exponential	4
Hypo-exponential	Exponential	4
Exponential	Hyper-exponential	3
Exponential	Hypo-exponential	3
Hyper-exponential	Hyper-exponential	5
Hypo-exponential	Hypo-exponential	5

Table 2: Non-markovian models

On-time Distribution	**Off-time Distribution**
Deterministic	Exponential
Pareto	Exponential
Truncated Gaussian	Exponential
Exponential	Deterministic
Exponential	Pareto
Exponential	Truncated Gaussian
Deterministic	Deterministic
Pareto	Pareto
Truncated Gaussian	Truncated Gaussian

diagram is depicted below.



$$r_{01} = \begin{cases} (M-i-j-k-l)p & i+j<K \\ 0 & i+j=K \end{cases}$$

$$r_{02} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l)p & i+j=K \end{cases}$$

$$r_{03} = \begin{cases} (M-i-j-k-l)(1-p) & i+j<K \\ 0 & i+j=K \end{cases}$$

$$r_{04} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l)(1-p)\lambda & i+j=K \end{cases}$$
$r_{05} = i\mu_1$
$r_{06} = l\mu_2$
$r_{07} = j\mu_1$
$r_{08} = k\mu_2$
$$r_{10} = \begin{cases} (i+1)\mu_1 & i+j<K \\ 0 & i+j=K \end{cases}$$
$r_{20} = (k+1)\mu_1$
$$r_{30} = \begin{cases} (j+1)\mu_2 & i+j<K \\ 0 & i+j=K \end{cases}$$
$r_{40} = (l+1)\mu_2$
$r_{50} = (M-i-j-k-l+1)p\lambda$
$$r_{60} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l+1)(1-p)\lambda & i+j=K \end{cases}$$
$r_{70} = (M-i-j-k-l+1)(1-p)\lambda$
$$r_{80} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l+1)p\lambda & i+j=K \end{cases}$$
Then we have the following steady state equations.
For $i+j<K$,

$$\begin{aligned} &[(M-i-j-k-l)\lambda + (i+k)\mu_1 + (j+l)\mu_2]\pi_{i,j,k,l} \\ &\quad = (M-i+1-j-k-l)p\lambda\pi_{i-1,j,k,l} \\ &\quad +(M-i-j+1-k-l)(1-p)\lambda\pi_{i,j-1,k,l} \\ &\quad +(i+1)\mu_1\pi_{i+1,j,k,l} + (j+1)\mu_2\pi_{i,j+1,k,l} \\ &\quad +(k+1)\mu_1\pi_{i,j,k+1,l} + (l+1)\mu_2\pi_{i,j,k,l+1}. \end{aligned} \tag{1}$$

For $i+j=K$,

$$\begin{aligned} &[(M-K-k-l)\lambda + (i+k)\mu_1 + (j+l)\mu_2]\pi_{i,j,k,l} \\ &\quad = (M-K+1-k-l)p\lambda(\pi_{i-1,j,k,l} + \pi_{i,j,k-1,l}) \\ &\quad + (M-K+1-k-l)(1-p)\lambda(\pi_{i,j-1,k,l} + \pi_{i,j,k,l-1}) \\ &\quad + (k+1)\mu_1\pi_{i,j,k+1,l} + (l+1)\mu_2\pi_{i,j,k,l+1}. \end{aligned} \tag{2}$$

For brevity, in (1) and (2), $\pi_{i,j,k,l}$ values out of the range ($0 \le i+j \le K$; $0 \le k+l \le M-K$; $i,j,k,l \ge 0$) take the value zero. Then we have the normalization equation:

$$\sum_{i=0}^{K}\sum_{j=0}^{K-i}\sum_{k=0}^{M-K}\sum_{l=0}^{M-K-k} \pi_{i,j,k,l} = 1.$$

The offered load is given by

$$T_o = \sum_{i=0}^{K}\sum_{j=0}^{K-i}\sum_{k=0}^{M-K}\sum_{l=0}^{M-K-k} (M-i-j-k-l)\lambda(\frac{p}{\mu_1} + \frac{1-p}{\mu_2})\pi_{i,j,k,l}.$$

The carried load is given by

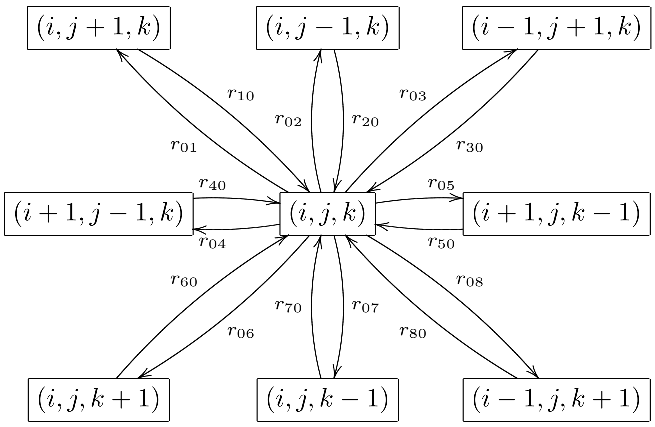
$$T_c = \sum_{i=0}^{K}\sum_{j=0}^{K-i}\sum_{k=0}^{M-K}\sum_{l=0}^{M-K-k}(i\mu_1 + j\mu_2)(\frac{p}{\mu_1} + \frac{1-p}{\mu_2})\pi_{i,j,k,l}.$$

The blocking probability is obtained by

$$B = \frac{T_o - T_c}{T_o}.$$

For hyper-exponentially distributed off-time and exponentially distributed on-time, we consider each free source has two possible states. Sources whose off-time is exponentially distributed with parameter λ_1 and λ_2 are said to be in states 1 and 2, respectively.

Let $\pi_{i,j,k}$ be the steady state probability that there are i free sources in state 1, j busy sources, and k frozen sources ($0 \le i \le M$; $0 \le j \le K$; $0 \le k \le M - max(K, i+j)$). Therefore, the number of free sources in state 2 is $M - i - j - k$. For a busy or frozen source, the rates to become a free source in states 1 and 2 are $p\mu$ and $(1-p)\mu$, respectively. The state transition diagram is depicted below.



$$r_{01} = \begin{cases} (M-i-j-k)\lambda_2 & j < K \\ 0 & j = K \end{cases}$$

$$r_{02} = j(1-p)\mu$$

$$r_{03} = \begin{cases} i\lambda_1 & j < K \\ 0 & j = K \end{cases}$$

$$r_{04} = jp\mu$$

$$r_{05} = kp\mu$$

$$r_{06} = \begin{cases} 0 & j < K \\ (M-i-j-k)\lambda_2 & j = K \end{cases}$$

$$r_{07} = k(1-p)\mu$$

$$r_{08} = \begin{cases} 0 & j < K \\ i\lambda_1 & j = K \end{cases}$$

$$r_{10} = \begin{cases} (j+1)(1-p)\mu & j < K \\ 0 & j = K \end{cases}$$

$r_{20} = (M-i-j-k+1)\lambda_2$

$r_{30} = \begin{cases} (j+1)p\mu & j < K \\ 0 & j = K \end{cases}$

$r_{40} = (i+1)\lambda_1$

$r_{50} = \begin{cases} (i+1)\lambda_1 & j < K \\ 0 & j = K \end{cases}$

$r_{60} = (k+1)(1-p)\mu$

$r_{70} = \begin{cases} 0 & j < K \\ (M-i-j-k+1)\lambda_2 & j = K \end{cases}$

$r_{80} = (k+1)p\mu$

Then we have the following steady state equations:
For $j = 0,\ 1,\ 2,\ ...,\ K-1$,

$$\begin{aligned} &[(M-i-j-k)\lambda_2 + i\lambda_1 + (j+k)\mu]\pi_{i,j,k} \\ &\quad = (M-i-j+1-k)\lambda_2\pi_{i,j-1,k} \\ &\quad +(i+1)\lambda_1\pi_{i+1,j-1,k} + (j+1)(1-p)\mu\pi_{i,j+1,k} \\ &\quad +(k+1)(1-p)\mu\pi_{i,j,k+1} + (j+1)p\mu\pi_{i-1,j+1,k} \\ &\quad +(k+1)p\mu\pi_{i-1,j,k+1}. \end{aligned} \quad (3)$$

For $j = K$,

$$\begin{aligned} &[(M-K-i-k)\lambda_2 + i\lambda_1 + (K+k)\mu]\pi_{i,K,k} \\ &= (i+1)\lambda_1\pi_{i+1,K-1,k} + (i+1)\lambda_1\pi_{i+1,K,k-1} \\ &\quad +(M-K-i-k+1)\lambda_2(\pi_{i,K-1,k} + \pi_{i,K,k-1}) \\ &\quad +(k+1)(1-p)\mu\pi_{i,K,k+1} + (k+1)p\mu\pi_{i-1,K,k+1}. \end{aligned} \quad (4)$$

For brevity, in (3) and (4) $\pi_{i,j,k}$ values out of the range $0 \le i \le M$, $0 \le j \le min(K, M-i)$ and $0 \le k \le M - max(K, i+j)$ take the value zero. Then we also have the normalization equation:

$$\sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{M-max(K,i+j)} \pi_{i,j,k} = 1.$$

The offered load is given by

$$T_o = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{M-max(K,i+j)} \frac{i\lambda_1 + (M-i-j-k)\lambda_2}{\mu}\pi_{i,j,k}.$$

The carried load is given by

$$T_c = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{M-max(K,i+j)} j\pi_{i,j,k}.$$

The blocking probability is obtained by

$$B = \frac{T_o - T_c}{T_o}.$$

Next consider the case with hyper-exponentially distributed on- and off-time. We consider each busy or frozen source have two states. Sources transmitting bursts of exponentially distributed lengths with parameter μ_1 and μ_2 are said to be in states 1 and 2, respectively. We can also consider each free sources into two states. Sources whose off-time is exponentially distributed with parameter λ_1 and λ_2 are said to be in states 3 and 4, respectively.

Let $\pi_{i,j,k,l,m}$ be the steady state probability that there are i free sources in state 3, and j and k busy sources in states 1 and 2, respectively, and l and m frozen sources in states 1 and 2, respectively ($0 \leq i \leq M$; $0 \leq j+k \leq min(K, M-i)$; $0 \leq l+m \leq M - max(K, i+j+k)$; $i,j,k,l,m \geq 0$). Each free source turns to be a busy or frozen source in state 1 with probability p and in state 2 with probability $(1-p)$. Each busy or frozen source turns to be a free source in state 1 with probability q and in state 2 with probability $(1-q)$. The state transition diagram is depicted below.

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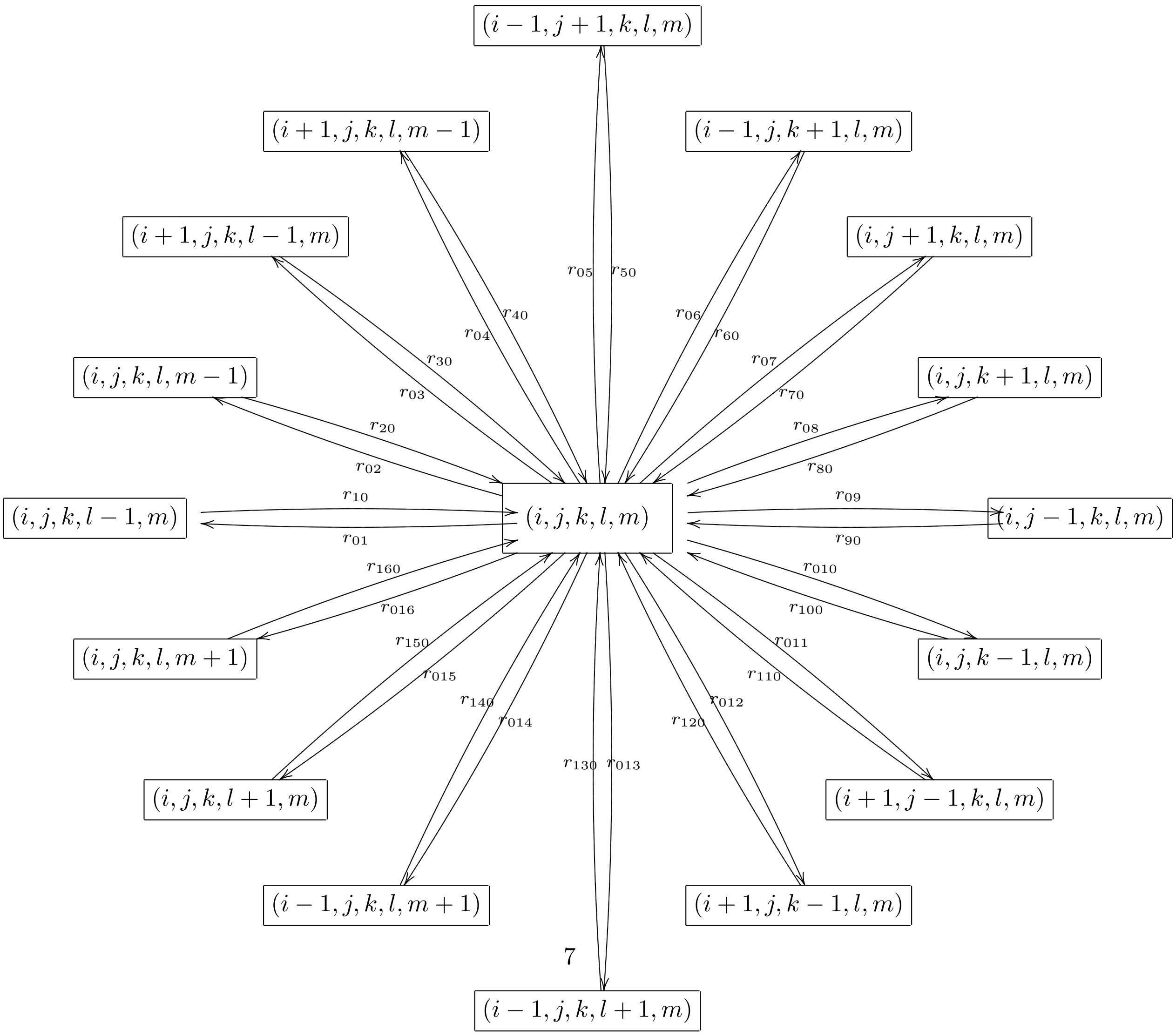


$r_{01} = m(1-q)\mu_1$
$r_{02} = n(1-q)\mu_2$
$r_{03} = mq\mu_1$
$r_{04} = nq\mu_2$
$r_{05} = \begin{cases} ip\lambda_1 & j+k<K \\ 0 & j+k=K \end{cases}$
$r_{06} = \begin{cases} i(1-p)\lambda_1 & i+j<K \\ 0 & i+j=K \end{cases}$
$r_{07} = \begin{cases} (M-i-j-k-m-n)p\lambda_2 & i+j<K \\ 0 & i+j=K \end{cases}$
$r_{08} = \begin{cases} (M-i-j-k-m-n)(1-p)\lambda_2 & i+j<K \\ 0 & i+j=K \end{cases}$
$r_{09} = j(1-q)\mu_1$
$r_{010} = k(1-q)\mu_2$
$r_{011} = jq\mu_1$
$r_{012} = kq\mu_2$
$r_{013} = \begin{cases} 0 & i+j<K \\ ip\lambda_1 & i+j=K \end{cases}$
$r_{014} = \begin{cases} 0 & i+j<K \\ i(1-p)\lambda_1 & i+j=K \end{cases}$
$r_{015} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l-m)p\lambda_2 & i+j=K \end{cases}$
$r_{016} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l-m)(1-p)\lambda_2 & i+j=K \end{cases}$
$r_{10} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l-m+1)p\lambda_2 & i+j=K \end{cases}$
$r_{20} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l-m+1)(1-p)\lambda_2 & i+j=K \end{cases}$
$r_{30} = \begin{cases} 0 & i+j<K \\ (i+1)p\lambda_1 & i+j=K \end{cases}$
$r_{40} = \begin{cases} 0 & i+j<K \\ (i+1)(1-p)\lambda_1 & i+j=K \end{cases}$
$r_{50} = \begin{cases} (j+1)q\mu_1 & i+j<K \\ 0 & i+j=K \end{cases}$
$r_{60} = \begin{cases} (k+1)q\mu_2 & i+j<K \\ 0 & i+j=K \end{cases}$
$r_{70} = \begin{cases} (j+1)(1-q)\mu_1 & i+j<K \\ 0 & i+j=K \end{cases}$

$r_{80} = \begin{cases} (k+1)(1-q)\mu_2 & i+j < K \\ 0 & i+j = K \end{cases}$
$r_{90} = (M-i-j-k-l-m+1)p\lambda_2$
$r_{100} = (M-i-j-k-l-m+1)(1-p)\lambda_2$
$r_{110} = (i+1)p\lambda_1$
$r_{120} = (i+1)(1-p)\lambda_1$
$r_{130} = (m+1)q\mu_1$
$r_{140} = (n+1)q\mu_2$
$r_{150} = (m+1)(1-q)\mu_1$
$r_{160} = (n+1)(1-q)\mu_2$
We have the following steady state equations:
For $j+k = 0,\ 1,\ 2,\ ...,\ K-1$,

$$\begin{aligned}
&[(M-i-j-k-l-m)\lambda_2 + i\lambda_1 + (j+l)\mu_1 + (k+m)\mu_2]\pi_{i,j,k,l,m} \\
&\quad = (M-i-j+1-k-l-m)p\lambda_2\pi_{i,j-1,k,l,m} \\
&\quad +(M-i-j-k+1-l-m)(1-p)\lambda_2\pi_{i,j,k-1,l,m} \\
&\quad +(i+1)p\lambda_1\pi_{i+1,j-1,k,l,m} + (i+1)(1-p)\lambda_1\pi_{i+1,j,k-1,l,m} \\
&\quad +(j+1)q\mu_1\pi_{i-1,j+1,k,l,m} + (j+1)(1-q)\mu_1\pi_{i,j+1,k,l,m} \\
&\quad +(k+1)q\mu_2\pi_{i-1,j,k+1,l,m} + (k+1)(1-q)\mu_2\pi_{i,j,k+1,l,m} \\
&\quad +(l+1)q\mu_1\pi_{i-1,j,k,l+1,m} + (l+1)(1-q)\mu_1\pi_{i,j,k,l+1,m} \\
&\quad +(m+1)q\mu_2\pi_{i-1,j,k,l,m+1} + (m+1)(1-q)\mu_2\pi_{i,j,k,l,m+1}.
\end{aligned} \tag{5}$$

For $j = K$,

$$\begin{aligned}
&[(M-i-K-l-m)\lambda_2 + i\lambda_1 + (j+l)\mu_1 + (k+m)\mu_2]\pi_{i,j,k,l,m} \\
&\quad = (M-i-K+1-l-m)p\lambda_2(\pi_{i,j-1,k,l,m} + \pi_{i,j,k,l-1,m}) \\
&\quad +(M-i-K+1-l-m)(1-p)\lambda_2(\pi_{i,j,k-1,l,m} + \pi_{i,j,k,l,m-1}) \\
&\quad +(i+1)p\lambda_1(\pi_{i+1,j-1,k,l,m} + \pi_{i+1,j,k,l-1,m}) \\
&\quad +(i+1)(1-p)\lambda_1(\pi_{i+1,j,k-1,l,m} + \pi_{i+1,j,k,l,m-1}) \\
&\quad +(l+1)q\mu_1\pi_{i-1,j,k,l+1,m} + (l+1)(1-q)\mu_1\pi_{i,j,k,l+1,m} \\
&\quad +(m+1)q\mu_2\pi_{i-1,j,k,l,m+1} + (m+1)(1-q)\mu_2\pi_{i,j,k,l,m+1}.
\end{aligned} \tag{6}$$

For brevity, in (5) and (6), $\pi_{i,j,k,l,m}$ values out of the range ($0 \le i \le M$; $0 \le j+k \le min(K, M-i)$; $0 \le l+m \le M - max(K, i+j+k)$; $i,j,k,l,m \ge 0$) take the value zero. Then we also have the normalization equation:

$$\sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{min(K,M-i)-j} \sum_{l=0}^{M-max(K,i+j+k)} \sum_{m=0}^{M-max(K,i+j+k)-l} \pi_{i,j,k,l,m} = 1.$$

The offered load is given by

$$T_o = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{min(K,M-i)-j} \sum_{l=0}^{M-max(K,i+j+k)} \sum_{m=0}^{M-max(K,i+j+k)-l}$$
$$[i\lambda_1 + (M-i-j-k-l)\lambda_2](\frac{p}{\mu_1} + \frac{1-p}{\mu_2})\pi_{i,j,k,l,m}.$$

The carried load is given by

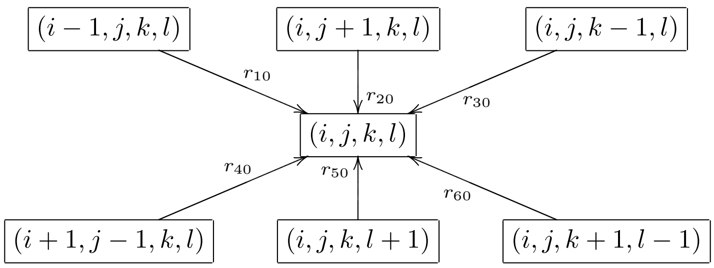
$$T_c = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{min(K,M-i)-j} \sum_{l=0}^{M-max(K,i+j+k)} \sum_{m=0}^{M-max(K,i+j+k)-l} (j\mu_1 + k\mu_2)(\frac{p}{\mu_1} + \frac{1-p}{\mu_2})\pi_{i,j,k,l,m}.$$

The blocking probability is obtained by

$$B = \frac{T_o - T_c}{T_o}.$$

The random variable $X = X_1 + X_2$ is called hypo-exponential random variable if X_1, X_2 are two independent exponentially distributed random variable with parameter μ_1 and μ_2, respectively. Accordingly, for hypo-exponentially distributed on-time and exponentially distributed off-time, the transmission for each burst can be regarded as two successive stages which are both exponentially distributed: stages 1 and 2 with mean $1/\mu_1$ and $1/\mu_2$, respectively. One burst goes through the two stages to finish its transmission. To be consistent with previous notations, each busy or frozen source is said to be in states 1 and 2 when its burst is in stage 1 and 2 of the transmission, respectively.

Let $\pi_{i,j,k,l}$ be the steady state probability that there are i and j busy sources in states 1 and 2, respectively, and k and l frozen sources in states 1 and 2, respectively ($0 \leq i+j \leq K$; $0 \leq k+l \leq M-K$; $i,j,k,l \geq 0$). The state transition diagram is depicted below.



$r_{10} = (M-i-j-k-l+1)\lambda$

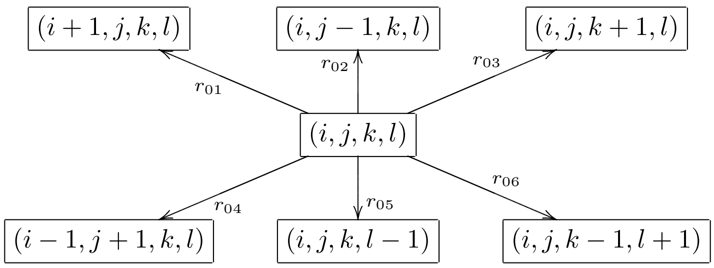
$$r_{20} = \begin{cases} (j+1)\mu_2 & i+j < K \\ 0 & i+j = K \end{cases}$$

$$r_{30} = \begin{cases} 0 & i+j < K \\ (M-i-j-k-l+1)\lambda & i+j = K \end{cases}$$

$r_{40} = (i+1)\mu_1$

$r_{50} = (l+1)\mu_2$

$r_{60} = (k+1)\mu_1$



$r_{01} = \begin{cases} (M-i-j-k-l+1)\lambda & i+j<K \\ 0 & i+j=K \end{cases}$
$r_{02} = j\mu_2$
$r_{03} = \begin{cases} 0 & i+j<K \\ (M-i-j-k-l+1)\lambda & i+j=K \end{cases}$
$r_{04} = i\mu_1$
$r_{05} = l\mu_2$
$r_{06} = k\mu_1$
Then, We have the following steady state equations:
For $i+j = 0,\ 1,\ 2,\ ...,\ K-1$,

$$\begin{aligned} &[(M-i-j-k-l)\lambda + (i+k)\mu_1 + (j+l)\mu_2]\pi_{i,j,k,l} \\ &\quad = (M-i+1-j-k-l)\lambda\pi_{i-1,j,k,l} \\ &\quad +(i+1)\mu_1\pi_{i+1,j-1,k,l} + (j+1)\mu_2\pi_{i,j+1,k,l} \\ &\quad +(k+1)\mu_1\pi_{i,j,k+1,l-1} + (l+1)\mu_2\pi_{i,j,k,l+1}. \end{aligned} \tag{7}$$

For $i+j = K$,

$$\begin{aligned} &[(M-K-k-l)\lambda + (i+k)\mu_1 + (j+l)\mu_2]\pi_{i,j,k,l} \\ &\quad = \quad (M-K+1-k-l)\lambda(\pi_{i-1,j,k,l} + \pi_{i,j,k-1,l}) \\ &\quad + \quad (i+1)\mu_1\pi_{i+1,j-1,k,l} + (k+1)\mu_1\pi_{i,j,k+1,l-1} \\ &\quad + \quad (l+1)\mu_2\pi_{i,j,k,l+1}. \end{aligned} \tag{8}$$

For brevity, in (7) and (8) $\pi_{i,j,k,l}$ values out of the range $0 \le i+j \le K$ and $0 \le k+l \le M-K$ take the value zero.
Then we have the normalization equation:

$$\sum_{i=0}^{K}\sum_{j=0}^{K-i}\sum_{k=0}^{M-K}\sum_{l=0}^{M-K-k}\pi_{i,j,k,l} = 1.$$

The offered load is given by

$$T_o = \sum_{i=0}^{K}\sum_{j=0}^{K-i}\sum_{k=0}^{M-K}\sum_{l=0}^{M-K-k}(M-i-j-k-l)\lambda(\frac{1}{\mu_1} + \frac{1}{\mu_2})\pi_{i,j,k,l}.$$

The carried load is given by

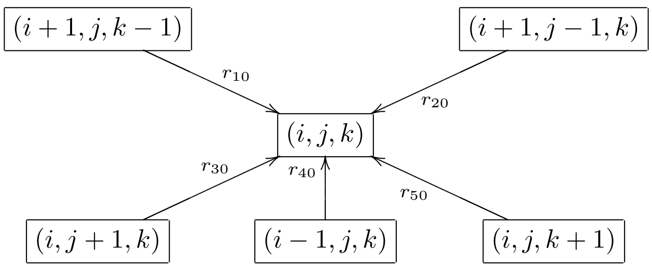
$$T_c = \sum_{i=0}^{K} \sum_{j=0}^{K-i} \sum_{k=0}^{M-K} \sum_{l=0}^{M-K-k} (j\mu_2)(\frac{1}{\mu_1} + \frac{1}{\mu_2})\pi_{i,j,k,l}.$$

The blocking probability is obtained by

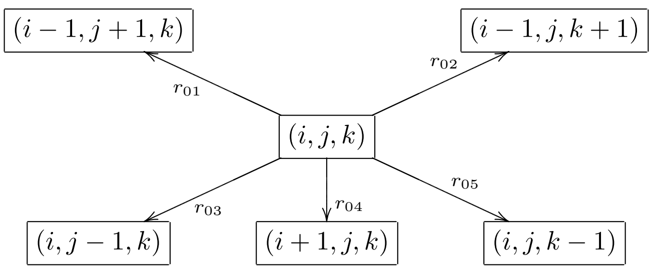
$$B = \frac{T_o - T_c}{T_o}.$$

For hypo-exponentially distributed off-time and exponentially distributed on-time, one free source goes through two stages, of which the durations are both exponentially distributed, to become a busy or frozen source: stages 1 and 2 with parameter λ_1 and λ_2, respectively. To be consistent with above notations, each free source is said to be in states 1 and 2 when it is in stages 1 and 2 of its off-time, respectively.

Let $\pi_{i,j,k}$ be the steady state probability that there are i free sources in stage 2, j busy sources, and k frozen sources ($0 \leq i \leq M$; $0 \leq j \leq min(K, M-i)$; $0 \leq k \leq M - max(K, i+j)$). The state transition diagram is depicted below.



$$r_{10} = \begin{cases} 0 & j < K \\ (i+1)\lambda_2 & j = K \end{cases}$$
$$r_{20} = (i+1)\lambda_2$$
$$r_{30} = \begin{cases} (j+1)\mu & j < K \\ 0 & j = K \end{cases}$$
$$r_{40} = (M - i - j - k + 1)\lambda_1$$
$$r_{50} = (k+1)\mu$$



$$r_{01} = \begin{cases} i\lambda_2\mu & j < K \\ 0 & j = K \end{cases}$$

$$r_{02} = \begin{cases} 0 & j < K \\ i\lambda_2 & j = K \end{cases}$$
$r_{03} = j\mu$
$r_{04} = (M - i - j - k)\lambda_1$
$r_{05} = k\mu$
We have the following steady state equations:
For $j = 0,\ 1,\ 2,\ ...,\ K - 1$,

$$\begin{aligned} &[(M - i - j - k)\lambda_1 + i\lambda_2 + (j + k)\mu]\pi_{i,j,k} \\ &\quad = (M - i + 1 - j - k)\lambda_1\pi_{i-1,j,k} \\ &\quad +(i + 1)\lambda_2\pi_{i+1,j-1,k} + (j + 1)\mu\pi_{i,j+1,k} \\ &\quad +(k + 1)\mu\pi_{i,j,k+1}. \end{aligned} \tag{9}$$

For $j = K$,

$$\begin{aligned} &[(M - K - i - k)\lambda_1 + i\lambda_2 + (K + k)\mu]\pi_{i,K,k} \\ &\quad = \quad (M - K + 1 - k - l)\lambda_1\pi_{i-1,K,k} \\ &\quad + \quad (i + 1)\lambda_2(\pi_{i+1,K-1,k} + \pi_{i+1,K,k-1}) \\ &\quad + \quad (k + 1)\mu\pi_{i,K,k+1}. \end{aligned} \tag{10}$$

For brevity, in (9) and (10) $\pi_{i,j,k}$ values out of the range $0 \leq i \leq M$, $0 \leq j \leq min(K, M - i)$ and $0 \leq k \leq M - max(K, i + j)$ take the value zero.
Then we also have the normalization equation:

$$\sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{M-max(K,i+j)} \pi_{i,j,k} = 1.$$

The offered load is given by

$$T_o = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{M-max(K,i+j)} \frac{i\lambda_2}{\mu}\pi_{i,j,k}.$$

The carried load is given by

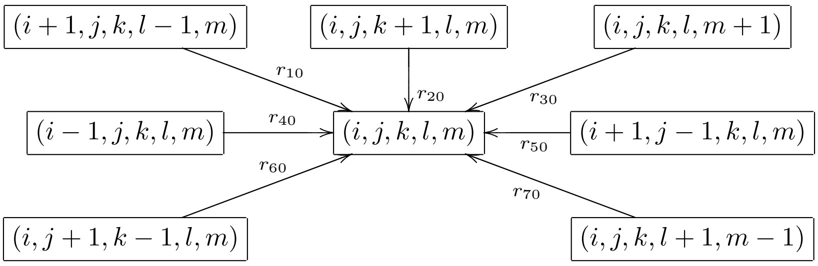
$$T_c = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{M-max(K,i+j)} j\pi_{i,j,k}.$$

The blocking probability is obtained by

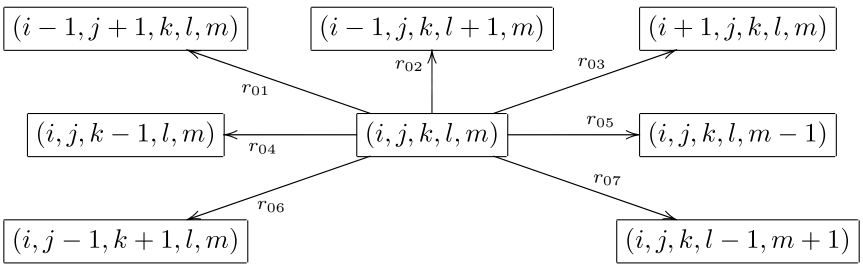
$$B = \frac{T_o - T_c}{T_o}.$$

Finally, we consider the case with hypo-exponentially distributed on- and off-time. Both the arrival process and the service for each burst can be regarded as two states which are exponentially distributed. Busy or frozen sources whose bursts are in stages 1 and 2 of the transmission are said to be in states 1 and 2, respectively. Free sources in stages 1 and 2 of the off-time are said to be in states 3 and 4, respectively.

Let $\pi_{i,j,k,l,m}$ be the steady state probability that there are i free sources in state 4, j and k busy sources in states 1 and 2, respectively, and l and m frozen sources in states 1 and 2, respectively ($0 \le i \le M$; $0 \le j+k \le K$; $0 \le l+m \le M - max(K, i+j+k)$; $i,j,k,l,m \ge 0$). The state transition diagram is depicted below.



$$r_{10} = \begin{cases} 0 & i+j < K \\ (i+1)\lambda_2 & i+j = K \end{cases}$$
$$r_{20} = \begin{cases} (k+1)\mu_2 & i+j < K \\ 0 & i+j = K \end{cases}$$
$r_{30} = (m+1)\mu_2$
$r_{40} = (M-i+1-j-k-l-m)\lambda_1$
$r_{50} = (i+1)\lambda_2$
$r_{60} = (j+1)\mu_1$
$r_{70} = (l+1)\mu_1$



$$r_{01} = \begin{cases} i\lambda_1 & i+j < K \\ 0 & i+j = K \end{cases}$$
$$r_{02} = \begin{cases} 0 & i+j < K \\ j\lambda_1 & i+j = K \end{cases}$$
$r_{03} = (M-i-j-k-l-m)\lambda_1$
$r_{04} = k\mu_2$
$r_{05} = m\mu_2$
$r_{06} = j\mu_1$
$r_{07} = l\mu_1$
We have the following steady state equations:

For $j+k = 0,\ 1,\ 2,\ ...,\ K-1$,

$$\begin{aligned}
&[(M-i-j-k-l-m)\lambda_1 + i\lambda_2 + (j+l)\mu_1 + (k+m)\mu_2]\pi_{i,j,k,l,m} \\
&\quad = (M-i+1-j-k-l-m)\lambda_1\pi_{i-1,j,k,l,m} \\
&\quad +(i+1)\lambda_2\pi_{i+1,j-1,k,l,m} + (j+1)\mu_1\pi_{i,j+1,k-1,l,m} \\
&\quad +(k+1)\mu_2\pi_{i,j,k+1,l,m} + (l+1)\mu_1\pi_{i,j,k,l+1,m-1} \\
&\quad +(m+1)\mu_2\pi_{i,j,k,l,m+1}.
\end{aligned} \tag{11}$$

For $j+k = K$,

$$\begin{aligned}
&[(M-K-k-l-m)\lambda_1 + i\lambda_2 + (j+l)\mu_1 + (k+m)\mu_2]\pi_{i,j,k,l,m} \\
&= \quad (M-K+1-k-l-m)\lambda_1\pi_{i-1,j,k,l,m} \\
&+ \quad (i+1)\lambda_2(\pi_{i+1,j-1,k,l,m} + \pi_{i,j,k,l-1,m}) \\
&+ \quad (j+1)\mu_1\pi_{i,j+1,k-1,l,m} + (l+1)\mu_1\pi_{i,j,k,l+1,m} \\
&+ \quad (m+1)\mu_2\pi_{i,j,k,l,m+1}.
\end{aligned} \tag{12}$$

For brevity, in (11) and (12) $\pi_{i,j,k,l,m}$ values out of the range ($0 \le i \le M$; $0 \le j+k \le K$; $0 \le l+m \le M - max(K, i+j+k)$; $i, j, k, l, m \ge 0$) take the value zero.
Then we also have the normalization equation:

$$\sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{min(K,M-i)-j} \sum_{l=0}^{M-max(K,i+j+k)} \sum_{m=0}^{M-max(K,i+j+k)-l} \pi_{i,j,k,l,m} = 1.$$

The offered load is given by

$$T_o = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{min(K,M-i)-j} \sum_{l=0}^{M-max(K,i+j+k)} \sum_{m=0}^{M-max(K,i+j+k)-l} i\lambda_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})\pi_{i,j,k,l,m}.$$

The carried load is given by

$$T_c = \sum_{i=0}^{M} \sum_{j=0}^{min(K,M-i)} \sum_{k=0}^{min(K,M-i)-j} \sum_{l=0}^{M-max(K,i+j+k)} \sum_{m=0}^{M-max(K,i+j+k)-l} k\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})\pi_{i,j,k,l,m}.$$

The blocking probability is obtained by

$$B = \frac{T_o - T_c}{T_o}.$$

Although the computation time can be reduced by using matrix methods [10], for certain large size problems we rely on Markov chain simulations.

Next we briefly discuss time complexity and space complexity of the method in [10], which uses block LU decomposition, compared with the method which uses Gaussian elimination to solve the steady state equations. For simplicity, we discuss the 2-dimensional model where both on- and off-times are exponentially distributed. Block LU decomposition requires $2/3(K+1)^2(K^2/4 + (K+1)(M-K+1)$ floating point operations for all the LU decompositions [10]. Gaussian elimination requires $[(M*K)^3 +$

$3 * (M * K)^2 + 2 * (M * K)]/3$ floating point operations. For larger M and K, block LU decomposition requires less computation time. However, space complexity of block LU decomposition in this model is $O(M * K * K))$, while space complexity of Gaussian elimination in this model is $O((M - K) * K)$. To solve the steady state equations for larger M and K, time complexity is the main obstacle for Gaussian elimination and space complexity is the main obstacle for block LU decomposition.

2.3. Non-Markovian models

When distributions of on- or off-time are deterministic, Pareto and truncated Gaussian, listed in table 2, the blocking probabilities are obtained by discrete event simulations.

3. Numerical Results

Aiming to investigate the errors introduced by assuming exponential on- and off-time distributions when evaluating blocking probabilities for the other distributions, we present here normalized histograms that estimate the error distributions. The depicted histograms are based on about 40,000 cases of calculations and simulations over a wide range of parameters.

In Fig. 1-3, μ is fixed at 0.1; λ is randomly chosen between 0.01 and 10; M is selected based on a discrete uniform distribution among $3, 4, \ldots 30$ and then K is selected uniformly among $1, 2, \ldots M - 1$. In Fig. 1, we present error distribution histograms for cases where off-times follow exponential distribution and on-times follow the other distributions. In Fig. 2, we present error histograms for cases where on-times follow exponential distribution and off-times follow the other distributions. In Fig. 3, we present error histograms for cases where both on- and off-times follow the other distributions. They all demonstrate that the blocking probability is generally not very sensitive to the shape of on- and off-time distributions. However, it is more sensitive to the shape of off-time distributions compared with on-time distributions. As discussed below, there are cases that give larger blocking probability errors.

From the above figures we observe that when on- or off-time is deterministic, blocking probability is usually higher. To explain this effect consider an example with two sources and one server. Assume that for each of the sources on-time and off-time are deterministic where the on-time is $\Delta + \epsilon$ and the off-time is $\Delta - \epsilon$ for arbitrarily small ϵ. In this case, all the bursts will collide, so arriving bursts will be dumped with probability of 0.5, which is the highest possible blocking probability in a system of two identical sources and one server with the same mean on- and off-time. If we increase the variance of the off-time, the occurrences of longer off-time in one source allow bursts from the other source to access the server without collision, reducing the blocking probability. We have observed similar results in cases with larger values of M and K.

In Table. 3-8, we present blocking probability errors over a wide range of M and K. Cases in all the six tables have the same M/K. In Table. 3-5, λ/μ keeps the same, therefore the normalized traffic $(M\lambda/(K\mu))$ keeps the same. In Table. 6-8, in which blocking probabilities of cases where both on- and off-times are exponentially distributed have negligible differences, we present blocking probability errors of cases which have the same mean on- and off-time with the other distributions.

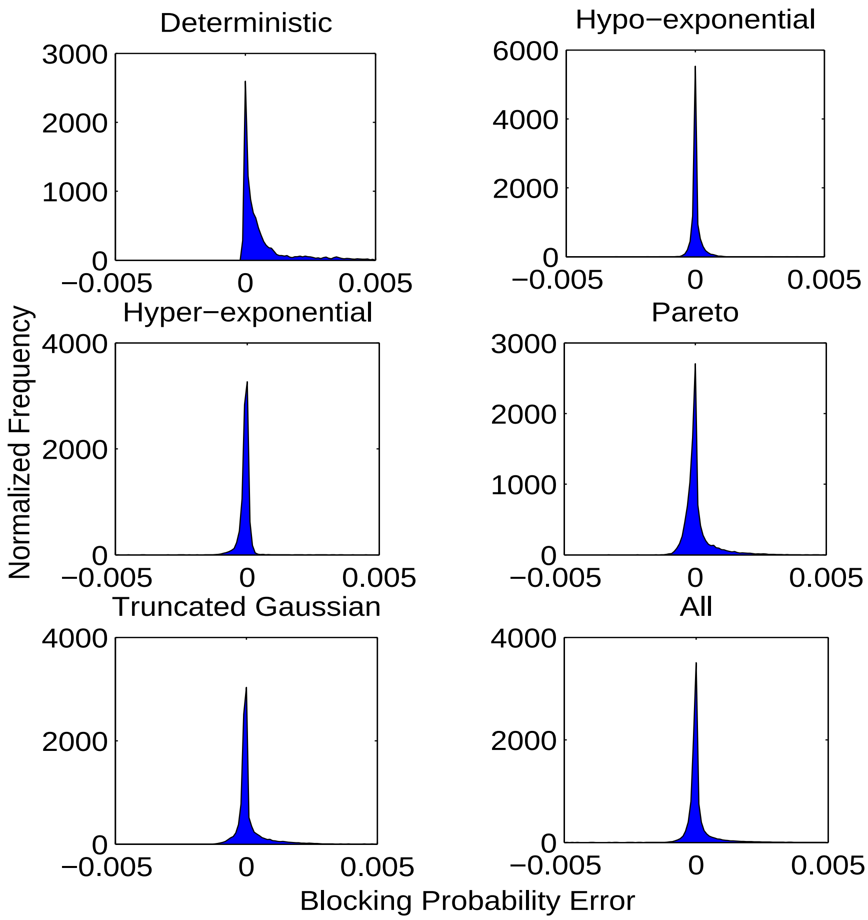


Figure 1: Normalized histograms of blocking probability errors for cases with various on-time distributions and exponential off-time distribution.

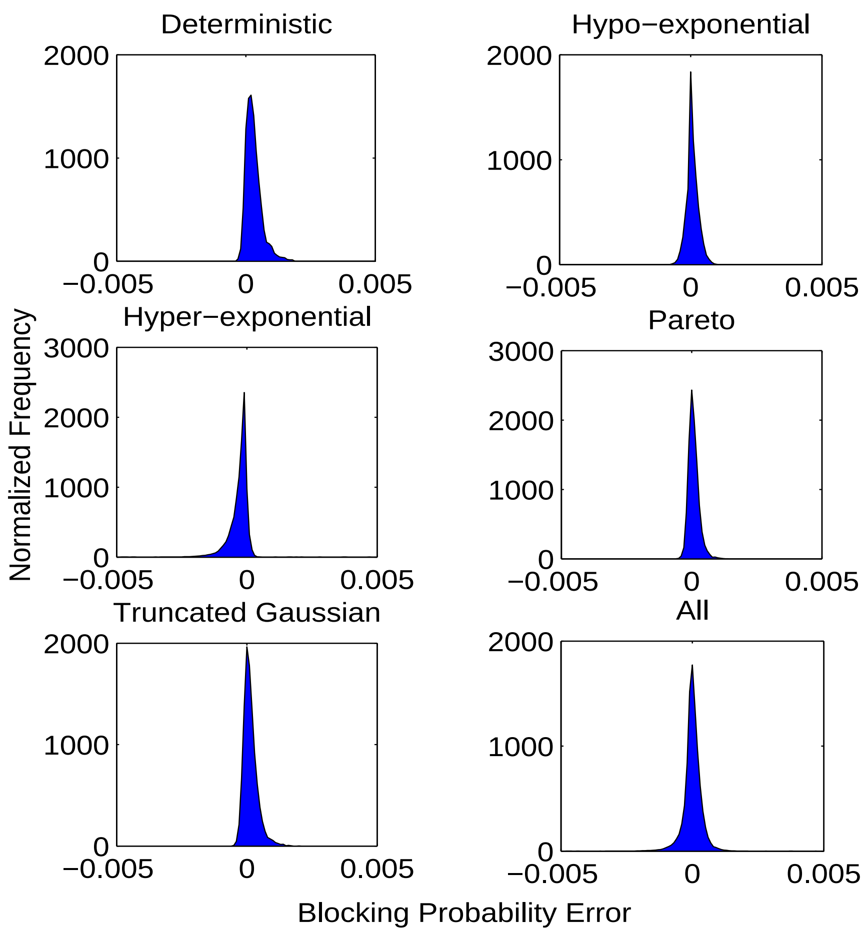


Figure 2: Normalized histograms of blocking probability errors for cases with various off-time distributions and exponential on-time distribution.

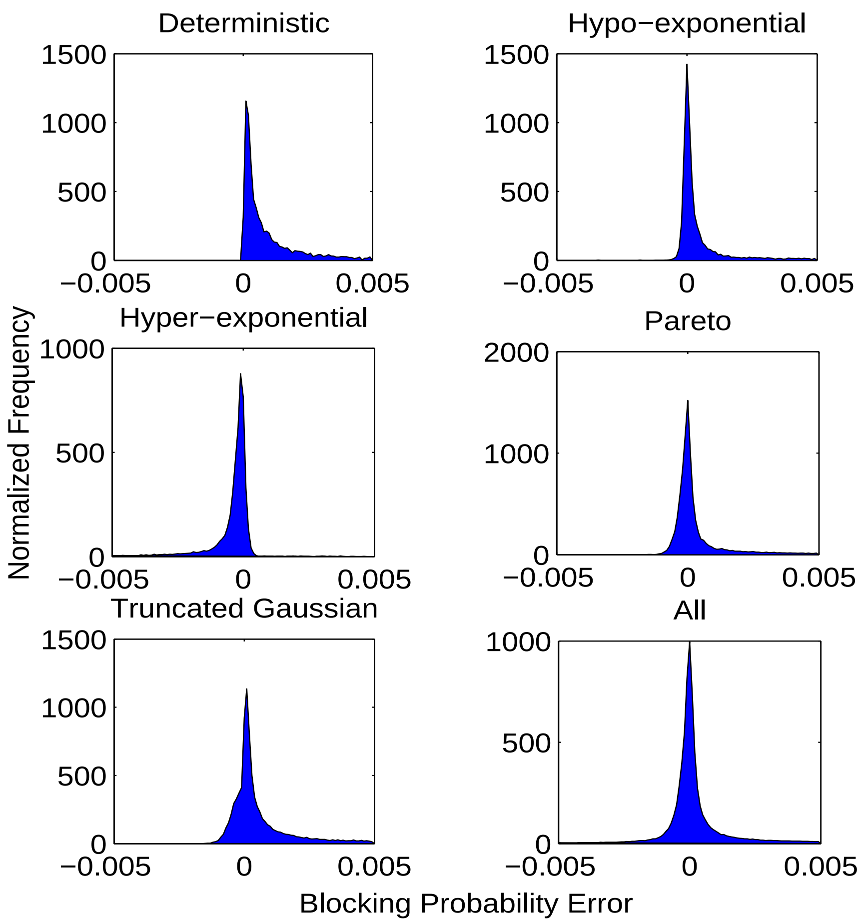


Figure 3: Normalized histograms of blocking probability errors for cases with various on- and off-time distributions.

The reason why we keep the normalized traffic the same is that by keeping the other parameters except M and K the same, we can investigate how M and K affect the errors. However, in practical, the number of sources and servers are designed to keep blocking probability under a certain value. Therefore, we also provide cases with similar blocking probabilities.

In Table. 3, 6, we present blocking probability errors for cases where off-times follow exponential distribution and on-times follow the other distributions. In Table. 4, 7, we present blocking probability errors for cases where on-times follow exponential distribution and off-times follow the other distributions. In Table. 5, 8, we present blocking probability errors for cases where both on- and off-times follow the other distributions. Note that we do not present the case where both on- and off-time are deterministic here because the initial condition (time periods between the beginnings of on-times of different sources) may affect the blocking probability. We also present the 95% confidence intervals based on Student-t distribution for data obtained by simulations. We present exact values without confidence intervals for data obtained by solving the steady state equations.

Table 3: Blocking probabilities for cases with various on-time distributions and exponential off-time distribution.

M		5	50	500
K		4	40	400
λ		0.33333	0.33333	0.33333
μ		0.10000	0.10000	0.10000
$\frac{M\lambda}{K\mu}$		4.12500	4.12500	4.12500
On-time distribution	Variance	Blocking probability (P_{exp}) ($*10^{-5}$)		
Exponential	100	15516	5708	628
On-time distribution	Variance	$P_{other} - P_{exp}$ ($*10^{-5}$)		
Deterministic	0	416±3	784±5	99±5
Hypo-exponential	50	29	98±5	25±5
Hypo-exponential	68	16	54±9	9±7
Hyper-exponential	132	-9.8	-34±5	-5.6±4
Hyper-exponential	228	-33	-166±4	-20±4
Pareto	145	98±12	260±8	37±6
Pareto	476	88±9	207±8	26±7
Truncated Gaussian	36	68±3	211±6	41±5
Truncated Gaussian	100	57±4	177±5	33±4

Table 4: Blocking probabilities for cases with various off-time distributions and exponential on-time distribution.

M		5	50	500
K		4	40	400
λ		0.10000	0.10000	0.10000
μ		0.03000	0.03000	0.03000
$\frac{M\lambda}{K\mu}$		4.12500	4.12500	4.12500
Off-time distribution	Variance	Blocking probability (P_{exp}) ($*10^{-5}$)		
Exponential	100	15516	5708	628
Off-time distribution	Variance	$P_{other} - P_{exp}$ ($*10^{-5}$)		
Deterministic	0	1461±6	588±7	86±7
Hypo-exponential	50	678	307±9	48±5
Hypo-exponential	68	440	226±9	31±7
Hyper-exponential	132	-316	-111±4	-14±7
Hyper-exponential	228	-1320	-572±7	-86±5
Pareto	145	720±7	400±5	58±6
Pareto	476	490±9	345±7	50±8
Truncated Gaussian	36	1060±5	430±5	64±6
Truncated Gaussian	100	948±6	369±9	58±5

Table 5: Blocking probabilities for cases with various on- and off-time distributions.

M		5	50	500
K		4	40	400
λ		0.10000	0.10000	0.10000
μ		0.03000	0.03000	0.03000
$\frac{M\lambda}{K\mu}$		4.12500	4.12500	4.12500
On- and off-time distribution	Variance	Blocking probability (P_{exp}) ($*10^{-5}$)		
Exponential	100	15516	5708	628
On- and off-time distribution	Variance	$P_{other} - P_{exp}$ ($*10^{-5}$)		
Hypo-exponential	50	825	453±6	74±5
Hypo-exponential	68	513	297±5	49±5
Hyper-exponential	132	-316	-139±7	-20±7
Hyper-exponential	228	-1320	-562±10	-86±7
Pareto	145	1169±5	867±7	114±6
Pareto	476	852±15	740±12	95±8
Truncated Gaussian	36	2095±2	1365±4	192±4
Truncated Gaussian	100	1549±3	962±6	142±4

Table 6: Blocking probabilities for cases with various on-time distributions and exponential off-time distribution.

M		5	50	500
K		4	40	400
λ		0.07500	0.25900	0.40000
μ		0.10000	0.10000	0.10000
$\frac{M\lambda}{K\mu}$		0.93750	3.32750	5.00000
On-time distribution	Variance	Blocking probability (P_{exp}) ($*10^{-5}$)		
Exponential	100	2769	2767	2792
On-time distribution	Variance	$P_{other} - P_{exp}$ ($*10^{-5}$)		
Deterministic	0	38±4	345±6	432±7
Hypo-exponential	50	7	57±7	45±10
Hypo-exponential	68	3	27±10	36±9
Hyper-exponential	132	-3.3	-21±4	-18±7
Hyper-exponential	228	-7	-66±3	-63±6
Pareto	145	13±6	120±6	142±9
Pareto	476	2±4	99±7	121±9
Truncated Gaussian	36	17±5	127±3	97±5
Truncated Gaussian	100	16±3	110±4	79±6

Table 7: Blocking probabilities for cases with various off-time distributions and exponential on-time distribution.

M		5	50	500
K		4	40	400
λ		0.10000	0.10000	0.10000
μ		0.13333	0.03861	0.02500
$\frac{M\lambda}{K\mu}$		0.93750	3.32750	5.00000
Off-time distribution	Variance	Blocking probability (P_{exp}) ($*10^{-5}$)		
Exponential	100	2769	2767	2792
Off-time distribution	Variance	$P_{other} - P_{exp}$ ($*10^{-5}$)		
Deterministic	0	465±5	399±4	189±8
Hypo-exponential	50	260	215±10	105±11
Hypo-exponential	68	196	152±10	67±18
Hyper-exponential	132	-78	-75±5	-40±8
Hyper-exponential	228	-440	-382±5	-188±13
Pareto	145	392±4	276±6	130±14
Pareto	476	370±4	240±6	112±12
Truncated Gaussian	36	344±3	297±6	138±8
Truncated Gaussian	100	282±6	247±3	120±6

Table 8: Blocking probabilities for cases with various on- and off-time distributions.

M		5	50	500
K		4	40	400
λ		0.10000	0.10000	0.10000
μ		0.13333	0.03861	0.02500
$\frac{M\lambda}{K\mu}$		0.93750	3.32750	5.00000
On- and off-time distribution	Variance	Blocking probability (P_{exp}) ($*10^{-5}$)		
Exponential	100	2769	2767	2792
On- and off-time distribution	Variance	$P_{other} - P_{exp}$ ($*10^{-5}$)		
Hypo-exponential	50	308	307±6	162±10
Hypo-exponential	68	221	205±7	108±9
Hyper-exponential	132	-78	-91±5	-59±7
Hyper-exponential	228	-440	-369±5	-206±6
Pareto	145	479±6	527±9	350±8
Pareto	476	448±9	442±7	303±9
Truncated Gaussian	36	428±4	758±5	583±5
Truncated Gaussian	100	346±4	550±3	400±6

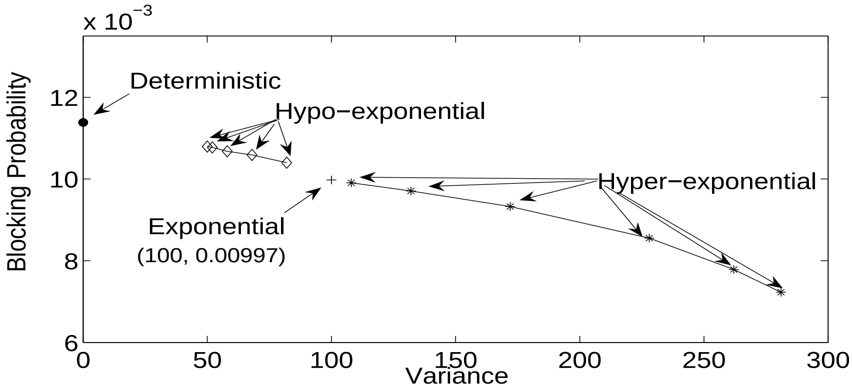


Figure 4: Blocking probability vs. variance of off-time distribution with exponential on-time distribution for $M = 100$, $K = 75$, $\lambda = 0.1$, $\mu = 0.05$.

From the tables we observe that the blocking probability errors of cases with larger M and K are close to cases with smaller M and K. Therefore, the blocking probability is generally not very sensitive to the shape of on- and off-time distributions when M and K are larger. We also observe that blocking probability is higher when on- or off-time is deterministic. This shows that lower variance may also lead to higher blocking probability when M and K are larger.

Fig. 4 depicts blocking probability estimations for cases involving exponential on-time distribution and other off-time distributions. Blocking probability of the case where off-times follow exponential distribution was obtained by solving the steady state equations. The others were obtained by simulations. The 95% confidence intervals based on Student-t distribution are smaller than plotted points and therefore not shown. Their radii are kept below 10^{-4}. We observe that lower variance of the off-time distribution causes certain increase in blocking probability, so clearly, the insensitivity of the Engset model does not apply to the present case. Nevertheless, the variations in the blocking probability are small.

4. Conclusion

We have studied the sensitivity of blocking probability of bursts to the shape of on- and off-time distributions at an OBS OXC. Based on the tests studied, blocking probability is generally not very sensitive to the shape of the distributions of both on- and off-time, which justifies the use of the exponential distributions. Moreover, we have observed and explained the interesting phenomenon that lower variance may lead to higher blocking probability.

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