A Proof of the OPC Conjecture (extended version)*

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Abstract
The overflow priority classification approximation (OPCA) furnishes an estimate of the call blocking probability for a distributed server network, that is more accurate than the conventional approach using Erlang’s fixed-point approximation (EFPA). In relation to this, we present here a complete proof of the OPC conjecture recently posed in [25], which states that OPC offers a better approximation for the blocking probability than EFPA in a certain symmetric overflow loss system. Specifically, we prove $P_{\text{EFPA}} \leq P_{\text{OPCA}} \leq P_{\text{exact}}$, for all possible Poisson offered loads, $a \geq 0$, and for all possible number of servers comprising the network, $N = 0, 1, 2, \ldots$. The first inequality has been proved elsewhere using induction on $N$. A proof of the second inequality is provided in this paper. This is a remarkably tight inequality and, at least in the proof we have obtained, requires very complicated arguments. Our strategy is to show that all of the $2N + 1$ coefficients of the Maclaurin series for $p(aN) \triangleq P_{\text{exact}} - P_{\text{OPCA}}$ are nonnegative, thereby ensuring $p(aN) \geq 0$. The proof of the OPC conjecture holds promise for the use of an OPCA-like approach in performance evaluation of practical overflow loss networks for which $P_{\text{exact}}$ is unknown.

1 Introduction
The aim of this paper is to complete a justification of the so-called overflow priority classification approximation (OPCA) [16, 21, 25] for blocking probability evaluation in overflow networks. OPCA works by using a surrogate second system and estimating the blocking probability in the second system with Erlang’s fixed-point approximation (EFPA). Numerical comparisons have demonstrated the improvement achieved by OPCA over EFPA for a distributed video on demand system in [21] and for circuit-switched trunk reservation networks in [16].

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A combination of numerical and theoretical results are given in [25] to show that OPCA offers a better blocking probability estimate than EFPA for a certain overflow loss system. Our aim in this paper is to replace the numerical arguments given there by a rigorous mathematical justification that OPCA is more accurate than EFPA for that particular system.

Overflow loss networks represent a large and important subclass of loss networks. They are pervasive in stochastic models of various computer and telecommunications systems and networks. The classic example is that of circuit-switched networks using alternative routing [1, 3, 4, 7, 12, 11, 22, 23, 24, 20]. Other examples are optical networks [26], telephony call centers [2], and multiprocessor systems [8]. Generally speaking, an overflow loss network is a loss network with the additional feature that calls (jobs) that are blocked at one server group may be permitted to overflow to another server group.

For a purpose of performance evaluation, overflow loss networks are usually modeled as a multidimensional Markov process that does not lead to a product-form solution. Although the blocking probability can, in principle, be obtained by numerically solving a set of steady-state equations, this approach is usually not practical because the high dimensionality of the state-space.

Approximations are therefore important in estimating blocking probabilities in overflow loss networks. One simple and commonly used approach is to estimate the blocking probability where any stream of calls is assumed to follow a Poisson process characterized by its arrival rate. Under this approach, the total traffic offered to any server group of $N$ servers is assumed to follow a Poisson process with rate equal to the sum of the rates of all individual input streams. This gives rise to an $M/M/N/N$ model, so that the blocking probability perceived by the combined stream as well as by each of the individual streams is then estimated by the Erlang B formula [6]. We will use the notation $E(\lambda, N)$ for the blocking probability of an $M/M/N/N$ system with arrival rate $\lambda$ and $N$ servers. Letting $\lambda_i$ be the original rate of the $i$th stream, the overflow of each individual stream $i$ may go on to offer an arrival rate of $\lambda_i E(\sum_i \lambda_i, N)$ to a subsequent server group.

This approximation, Erlang’s fixed-point approximation (EFPA), was first proposed in [4] in 1964 for the analysis of circuit-switched networks and has remained a cornerstone of telecommunications networks and systems analysis to this day. Its basic idea is to decompose the overflow loss system into server-group subsystems and treat each subsystem as if it were an independent Erlang B sub-system. A fixed-point solution is then used to ensure that each subsystem has the right load and blocking probability. See [1, 3, 7, 12, 11, 14, 15, 17, 18, 22, 23, 24, 20] and references therein for applications of EFPA.

EFPA may be inaccurate for overflow loss networks due to the following two distinct sources of error:

1. The Poisson error – EFPA assumes that the traffic offered by any stream follows a Poisson process while it is known that the traffic offered by an overflow stream has higher peakedness than that of a Poisson process.

2. The independence error – EFPA calculates the distribution of the number of busy servers on a server group as if it were mutually independent of any other server group, while normally, there is statistical dependence.

Numerous approaches have been suggested to strengthen EFPA by combating the presence of one or the other of these two errors, see for example [9, 10, 13, 19]. In [25] a new approximation, namely OPCA, for estimating blocking probability in overflow loss networks was presented, which is fundamentally different from EFPA and its strengthened formulations. As mentioned
above, under OPCA, given a system for which an estimate of blocking probability is sought, a second system acts as a surrogate for the original system and OPCA estimates the blocking probability in the second system using an Erlang Approximation (EA) to it. EA is similar to EFPA except that it does not require a fixed point solution. It still relies on the same two fundamental assumptions EFPA relies on. The surrogate system is defined by regarding an overflow loss network as if it were operating under a preemptive priority regime where each call is classified according to the number of times it has overflowed. Then, a call that has overflowed \(n\) times is given strict preemptive priority over a call that has overflowed \(m\) times, \(n < m\). If a call arrives at a fully occupied server group, it will preempt a call that has the lowest priority among all the calls served in the server group. Also, a call does not visit servers it has already visited. This means that if a lower priority call is preempted from service by a higher priority call the lower priority call will not revisit servers already visited by that same lower priority call. This way, the various server groups are no longer treated independently. A comprehensive set of intuitive arguments and numerical results were used in [16, 21, 25] to explain and demonstrate the benefit of OPCA over EFPA.

Rigorous arguments for the benefit of OPCA over EFPA can be provided for a certain distributed server system considered in [25] for which the exact blocking probability is known. For that example, we prove in [25] that the OPCA blocking probability estimate is always higher than or equal to that of EFPA. We also numerically demonstrated there that the OPCA estimate is equal to or lower than the exact solution and we referred to this statement as the OPCA Conjecture. In this paper we provide a proof of this conjecture. This means that we now have a proof that OPCA is always either equal to or better than EFPA as an estimate for the blocking probability in this case. Through the course of the proof, it will be demonstrated, time and time again, that the difficulty in proving the OPCA conjecture is due to the fact that OPCA gives a result that is amazingly very close to the exact result.

Finally we say a word about the very involved proof presented here. While some simplification may be possible, we believe, as we have already stated, that this is a fundamentally difficult inequality because of its tightness. The proof given here involves dividing the problem into many cases, and some computer checking. We emphasize that the computer checking involves only finitely many calculations, and is only there to eliminate many tedious hand calculations. At no stage have we relied on the computer to do simulations or calculations that would require assumptions about the validity of the software or about the representativeness of the simulations. In places where have we use a computer, it is merely as an arbitrary (or at least sufficiently high) precision calculator. In other words, modulo the relatively small number of hand calculations, the descriptions of which are relegated to the appendices, and for which we have used the package Maxima, (http://maxima.sourceforge.net/index.shtml) permitting arbitrary precision in arithmetical operations, this proof is mathematically rigorous. We have clearly noted in the text wherever Maxima is used to do these calculations. This version of the paper, the extended version is provided to help the reader who wishes to see all aspects of the proof by giving explicit details of all hand calculations. The journal version has abbreviated the description of these arithmetical calculations.

2 Problem formulation

We first describe the distributed server model used in [25] for comparison between EFPA and OPCA and its exact blocking probability obtained by Erlang B formula. Then, we show how EFPA and OPCA are applied to evaluate blocking probabilities in this particular problem, which in turn leads to the formulation of the OPCA Conjecture.
2.1 The distributed server model

Consider the following simplified model of an overflow loss network that arose during a study of a video-on-demand distributed-server network [5]. The network comprises $N$ identical servers. Calls (i.e., requests to download video streams) initiated by users are offered to servers according to mutually independent time-homogeneous Poisson processes each with rate $a$. A call that arrives at a busy server overflows to one of the other $N-1$ servers with equal probability and without delay. A call continues to overflow until it either encounters an idle server, in which case it engages that server until its service period is complete, or has sought to engage all $N$ servers exactly once but found all $N$ servers busy, in which case it is blocked and never returns. The search for an idle server is assumed to be conducted instantaneously. Service periods are independent and exponentially distributed with normalized unit mean.

An $n$-call (for $1 \leq n \leq N-1$) is defined as a call that overflows $n$ times before engaging the $(n+1)$th server in its search for an idle server. An $N$-call is a call that is blocked and cleared and a 0-call is defined as a call initiated by a user (exogenous call).

This distributed-server model can be viewed as an $M/M/N/N$ queue that is offered an arrival rate of $aN$. This allows for exact calculation of blocking probability using the Erlang B formula as

$$P_{\text{exact}} = E(aN, N).$$

Therefore, $E(aN, N)$ provides a benchmark for comparison of blocking probability evaluation.

2.2 Erlang’s Fixed-Point Approximation

We make the following simplifying assumptions:

1. independence — states of servers are mutually independent;
2. Poisson — arrivals of $n$-calls to any server follow a Poisson process,

and write $b$ for the probability that a server is busy. We let $a$ be the arrival rate of new calls at a server and $a_n$ the arrival rate of $n$-calls at a server. Then, from the independence assumption,

$$a_n = ab^n, \quad n = 0, 2, \ldots, N-1,$$

and from the Poisson assumption, the probability of a server being busy gives rise to an $M/M/1/1$ model for which the probability is

$$b = \frac{\sum_{n=0}^{N-1} a_n}{1 + \sum_{n=0}^{N-1} a_n}.\quad (2)$$

Substituting (1) into (2), we obtain

$$b = a(1 - b^N),$$

which forms a fixed-point equation from which $b$ can be solved, giving an EFPA blocking probability estimate

$$P_{\text{EFPA}} = b^N.\quad (4)$$
2.3 OPCA

As in the EFPA case, we assume that events related to the $N$ different servers are mutually independent and that the servers are statistically equivalent. Also as in the EFPA case, we let $a$ be the arrival rate of new calls at a server and $a_n$ the arrival rate of $n$-calls at a server. Thus, $a_0 = a$. The stream formed by $n$-calls, $N - 1 \geq n \geq 0$, arriving at a server, is assumed to follow a Poisson stream with rate $a_n$. The preemptive priority regime defined by OPCA gives priority to $n_h$-calls over $n_l$-calls for any $0 \leq n_h < n_l \leq N - 1$. Accordingly, the $a_n$ values can be obtained recursively by

$$a_{n+1} = \mathbb{E}(\sum_{i=0}^{n} a_i, 1) \sum_{i=0}^{n} a_i - \sum_{i=1}^{n} a_i,$$

for all $n = 0, \ldots, N - 1$, where $a_N$ is defined as the rate of the stream formed by calls that are blocked and cleared.

The OPCA estimate of blocking probability is given by

$$P_{OPCA} = \frac{a_N}{a}.$$  \hspace{1cm} (6)

Eq. (6) has a clear physical interpretation. The $a_N/a$ ratio is the proportion of calls that made it to become $N$-calls (blocked calls) which is the OPCA estimate for the blocking probability.

Graphs are shown in Fig. 1 demonstrating the tightness of $P_{OPCA}$ and $P_{EFPA}$ as lower bounds for $P_{exact}$.

We are now able to state the main theorem of this paper.

**Theorem 1.**

$$P_{EFPA} \leq P_{OPCA} \leq P_{exact}.$$  \hspace{1cm} (7)

The first inequality, namely $P_{EFPA} \leq P_{OPCA}$, was proved in [25], where the second inequality is stated as the *OPC Conjecture*. It is the purpose of this paper to provide a mathematical proof of the OPC conjecture.
By the Erlang B formula, we have

\[ P_{\text{exact}} = \frac{(aN)^N}{N!} \sum_{i=0}^{N} \frac{(aN)^i}{i!}. \tag{8} \]

Therefore, by (6) – (8), the proof of Theorem 1 will be completed when we have proved the following inequality for all \( a > 0 \) and \( N = 1, 2, 3, \ldots \)

\[ \frac{a_N}{a} \leq \frac{(aN)^N}{N!} \sum_{i=0}^{N} \frac{(aN)^i}{i!}. \]  

(9)

The remainder of this paper is given over to establishing this innocuous looking inequality.

3 A Sufficient Condition for the OPC Conjecture

We begin the proof by recalling the definition of Appell polynomials

\[ S_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}, \tag{10} \]

the truncated Maclaurin series of the exponential function, and define a polynomial that will play a key role in the proof of the conjecture. At this point we make the convention that the symbol \( N \) is taken to range over all positive integers unless otherwise stated, so that any statement made about \( N \) without further constraints is assumed to hold for \( N = 1, 2, 3, \ldots \).

\[
P(\lambda) = \left( \frac{\lambda}{N} - 2 \right) \left( \frac{N}{N+1} \right)^N S_N(\lambda) S_N \left( \lambda \left( 1 + \frac{1}{N} \right) \right) + S_N(\lambda)^2 - \frac{\lambda^{N+1}}{NN!} S_N(\lambda) \]

\[
+ \left( 1 - \frac{\lambda}{N} \right) \left( \frac{N}{N+1} \right)^N \frac{\lambda^N}{N!} S_N \left( \lambda \left( 1 + \frac{1}{N} \right) \right) + \frac{\lambda^{2N+1}}{N(N+1)} (11)
\]

The following result reveals its importance:

**Proposition 1.** If \( P(\lambda) \geq 0 \) for all \( \lambda \in [0, N] \) then the OPC conjecture is true.

We shall prove this result in the remainder of this section and then spend the rest of the paper on the apparently much more difficult task of showing that its hypothesis is valid to complete the proof of the OPC conjecture.

We begin the proof of Proposition 1 with a few more formulae involving \( a_n \). Recall that

\[ a_n = \begin{cases} 
  a, & n = 0, \\
  ab(0) \ldots b(n-1), & n = 1, \ldots, N - 1,
\end{cases} \tag{12} \]

where

\[ b(n) = \frac{a_{n+1}}{a_n} \]

\[ = \frac{\text{E}(\sum_{i=0}^{n} a_i, 1) \sum_{i=0}^{n} a_i - \sum_{i=1}^{n} a_i}{a_n}. \tag{13} \]

It follows from (12) and (13) that

\[ a_n = \frac{\left( \sum_{i=0}^{n-1} a_i \right)^2}{1 + \sum_{i=0}^{n-1} a_i} - \sum_{i=1}^{n-1} a_i = a - 1 + \frac{1}{1 + \sum_{i=0}^{n-1} a_i}. \tag{14} \]
For the purposes of this section, we write down the following inequality
\[
\frac{1}{aP_{\text{exact}}(N) + 1 - a} + aP_{\text{exact}}(N) \geq \frac{1}{aP_{\text{exact}}(N + 1) + 1 - a}.
\] (15)

**Lemma 1.** If (15) holds, then Proposition 1 is true.

**Proof.** Our proof of Proposition 1 is going to be by induction on \(N\) with equation (15) providing the induction step. We assume that \(P(\lambda) > 0\) for all \(\lambda \in [0, N]\). We define
\[
\nu_n = \begin{cases} 
1, & n = -1, \\
1 + \sum_{i=0}^{n} a_i, & n > -1,
\end{cases}
\] (16)
and note
\[
a_n = \nu_n - \nu_{n-1} \quad n = 0, 1, 2, \ldots.
\] (17)

As a result,
\[
a_n = a - 1 + \frac{1}{1 + \sum_{i=0}^{n-1} a_i} = a - 1 + \frac{1}{\nu_{n-1}}.
\] (18)

It then follows from equations (18), (9), and (8) that the OPC conjecture is equivalent to
\[
\frac{1}{\nu_{N-1}} \leq aP_{\text{exact}}(N) + 1 - a;
\] (19)
in other words,
\[
\nu_{N-1} \geq \frac{1}{aP_{\text{exact}}(N) + 1 - a} \geq 1.
\] (20)

We now begin the inductive proof, and we use this equation as the statement to be proved by induction. For the case \(N = 1\), \(\nu_0 = 1 + a\) and
\[
P_{\text{exact}}(1) = \frac{a^2}{1 + a} + 1 - a,
\] (21)
so the inequality is in fact an equality for this case.

Assume that (20) is true for \(N - 1\). Using the fact that the function \(x + (1/x)\) is increasing for \(x > 1\), we have
\[
\frac{1}{\nu_{N-1}} + \frac{1}{\nu_{N-1}} \geq \frac{1}{aP_{\text{exact}}(N) + 1 - a} + aP_{\text{exact}}(N) + 1 - a.
\] (22)

Adding \(a - 1\) to both sides we obtain
\[
\frac{1}{\nu_{N-1}} + a - 1 \geq \frac{1}{aP_{\text{exact}}(N) + 1 - a} + aP_{\text{exact}}(N),
\] (23)
and with (18), this yields
\[
\nu_N \geq \frac{1}{aP_{\text{exact}}(N) + 1 - a} + aP_{\text{exact}}(N).
\] (24)

From (20) we see that to complete the proof of the induction step it is enough to show
\[
\frac{1}{aP_{\text{exact}}(N) + 1 - a} + aP_{\text{exact}}(N) \geq \frac{1}{aP_{\text{exact}}(N + 1) + 1 - a}.
\] (25)
This completes the proof of Lemma 1. □
Proof of Proposition 1. Our strategy here is to prove that (25) holds. We recall that

\[ P_{\text{exact}}(N + 1) = P_{\text{exact}} = \frac{(a(N+1))^{N+1}}{(N+1)!} \sum_{i=0}^{N+1} \frac{[(a(N+1))^i]}{i!}, \]  

and write

\[ p_N = \frac{(a(N+1))^N}{\sum_{i=0}^{N} (a(N+1))^i}. \]  

Then

\[ P_{\text{exact}}(N + 1) = \frac{a(N+1)^N}{(a(N+1))^{N+1} + \sum_{i=0}^{N} (a(N+1))^i} = \frac{ap_N}{1 + ap_N}. \]  

We make this substitution for \( P_{\text{exact}}(N + 1) \) in (25) to see that it is enough to prove

\[ \frac{1}{aP_{\text{exact}}(N)} + 1 - a + aP_{\text{exact}}(N) \geq \frac{1}{a \frac{ap_N}{1 + ap_N} + 1 - a}, \]  

and the right side of this equation is

\[ \frac{1 + ap_N}{1 - a + ap_N}. \]  

Thus we have to prove

\[ \frac{aP_{\text{exact}}(N)(aP_{\text{exact}}(N) + 1 - a) + 1}{aP_{\text{exact}}(N) + 1 - a} \geq \frac{1 + ap_N}{1 - a + ap_N}. \]  

Since both denominators are positive, we reduce the problem to proving

\[ (aP_{\text{exact}}(N)(aP_{\text{exact}}(N) + 1 - a) + 1)(1 - a + ap_N) \geq (1 + ap_N)(aP_{\text{exact}}(N) + 1 - a). \]  

This reduces to

\[ P_{\text{exact}}(N)^2(1 - a + ap_N) + P_{\text{exact}}(N)(a - ap_N - 2) + p_N \geq 0. \]  

Dividing by \( P_{\text{exact}}(N)p_N \), we require

\[ \frac{a - 2}{P_{\text{exact}}(N)p_N} + \frac{1}{P_{\text{exact}}(N)^2} - \frac{a}{P_{\text{exact}}(N)} + \frac{1 - a}{p_N} + a \geq 0. \]  

Next we convert to the Appell polynomials referred to earlier by noting that

\[ \frac{1}{P_{\text{exact}}(N)} = S_N(aN) \frac{N!}{(aN)^N}, \]  

and

\[ \frac{1}{p_N} = S_N(a(N + 1)) \frac{N!}{(a(N + 1))^N}. \]  

After substitution of these two into (34), it is enough to prove that

\[ \frac{(a - 2)(N!)^2}{(aN)^N(a(N + 1))} S_N(aN)S_N(a(N + 1)) + \frac{(N!)^2}{(aN)^{2N}} S_N(aN)^2 
- \frac{aS_N(aN)N!}{(aN)^N} + \frac{(1 - a)N!}{(a(N + 1))^N} S_N(a(N + 1)) + a \geq 0. \]
Now we multiply through by \(\frac{(aN+N!)^2}{N^2}\) and replace \(aN\) by \(\lambda\) to obtain that
\[
P(\lambda) = (\frac{\lambda}{N} - 2)(\frac{N}{N+1})^NS_N(\lambda)S_N(\lambda(1 + \frac{1}{N})) + S_N(\lambda)^2 - \frac{(\lambda)^{N+1}}{N!}S_N(\lambda)
\]
\[
+ (1 - \frac{\lambda}{N})(\frac{N}{N+1})^N\frac{\lambda^N}{N!}S_N(\lambda(1 + \frac{1}{N})) + \frac{\lambda^{2N+1}}{N(N!)^2} \geq 0 \tag{38}
\]
is enough to prove the OPC conjecture. This proves Proposition 1. \(\square\)

4 Deconstructing \(P(\lambda)\)

We note that \(P(\lambda)\) is a polynomial of degree \(2N + 1\). We write it then as its (finite) Maclaurin series:
\[
P(\lambda) = P(0) + P'(0)\lambda + \frac{P''(0)}{2!}\lambda^2 + \cdots + \frac{P^{(2N+1)}(0)}{(2N+1)!}\lambda^{2N+1}. \tag{39}
\]
According to Proposition 1 the proof of the OPC conjecture will be complete when we have shown that \(P(\lambda) \geq 0\) for all \(\lambda \in [0, N]\). In fact, we show that \(P(\lambda)\) is positive on the whole positive real line and we do that by showing that each of its derivatives at 0 is positive. It will simplify the formulae involving the derivatives of \(P(\lambda)\) if we define \(\alpha = \frac{N}{N+1}\). Then we have
\[
P(\lambda) = \frac{\lambda}{N}\alpha^NS_N(\lambda)S_N(\frac{\lambda}{\alpha}) - 2\alpha^NS_N(\lambda)S_N(\frac{\lambda}{\alpha}) + S_N(\lambda)^2
\]
\[
- \frac{\lambda^{N+1}}{N!}S_N(\lambda) + \frac{\alpha^N\lambda^N}{N!}S_N(\frac{\lambda}{\alpha}) - \frac{\alpha^N\lambda^{N+1}}{N!}S_N(\frac{\lambda}{\alpha}) + \frac{\lambda^{2N+1}}{N(N!)^2}. \tag{40}
\]

Now we approach the task of calculating the derivatives of each of the terms in this sum. The following equations will help here.
\[
S'_n(x) = S_{n-1}(x) \quad n = 0, 1, \ldots, \tag{41}
\]
where \(S_{-1}(x)\) is identically zero, and so
\[
S_n(0) = \begin{cases} 0 & n = -1 \\ 1 & \text{otherwise.} \end{cases} \tag{42}
\]

We record as a series of lemmas the derivatives of each of the terms in \(P(\lambda)\). In proving these lemmas, we will rely on (41) in addition to Leibnitz rule in the guise
\[
\frac{d^n}{dx^n}x^c f(ax)g(bx) = \sum_{i=0}^{n} \binom{n}{i} c(c-1)\cdots(c-i+1)x^{c-i} \frac{d^{n-i}}{dx^{n-i}} f(ax)g(bx) \tag{43}
\]
where
\[
\frac{d^n}{dx^n} f(ax)g(bx) = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} g^{(i)}(ax)f^{(n-i)}(bx),
\]
and \(f^{(n)}(x)\) is shorthand for \(d^n f(x)/dx^n\) and \(c\) is integer.
Lemma 2.

\[
\frac{d^n}{d\lambda^n} N \alpha^N S_N(\lambda) S_N \left( \frac{\lambda}{\alpha} \right) = \frac{\alpha^N}{N} \left( \sum_{i=0}^{n-1} \binom{n-1}{i} \lambda^i S_{N-i} \left( \frac{\lambda}{\alpha} \right) S_{N-n+i}(\lambda) \left( \frac{1}{\alpha} \right)^i \right) 
\]

\[
+ \lambda \sum_{i=0}^{n} \binom{n}{i} S_{N-i} \left( \frac{\lambda}{\alpha} \right) S_{N-n+i}(\lambda) \left( \frac{1}{\alpha} \right)^i,
\]

\[
\left. \frac{d^n}{d\lambda^n} \frac{\lambda}{N^\alpha} S_N(\lambda) S_N \left( \frac{\lambda}{\alpha} \right) \right|_{\lambda=0} = \begin{cases} 
\frac{\alpha^N}{N} \sum_{i=0}^{n-1} \binom{n-1}{i} \left( \frac{1}{\alpha} \right)^i, & 0 \leq n \leq N+1, \\
\frac{\alpha^N}{N} \sum_{i=n-N}^{N} \binom{N}{i} \left( \frac{1}{\alpha} \right)^i, & N+2 \leq n \leq 2N+1, \\
0 & n > 2N+1.
\end{cases}
\]

Proof. The first equality follows immediately from a rewriting of (43). In particular, we set 
\( f = g = S_N, a = 1/\alpha, b = 1 \) and \( c = 1 \).

We evaluate at \( \lambda = 0 \). The second term is multiplied by \( \lambda \) and is thus immediately eliminated.

The form of the first term depends on \( n \). Three cases arise. For \( 0 \leq n \leq N+1 \), the product \( S_{N-i}(0)S_{N-n+i}(0) = 1 \), as a consequence of (41).

For \( n \geq N+2 \), some of the terms in the summation

\[
\frac{n\alpha^N}{N} \sum_{i=0}^{n-1} \binom{n-1}{i} S_{N-i}(0)S_{N-n+i}(0) \left( \frac{1}{\alpha} \right)^i
\]

fail to make a nonzero contribution. In particular, the lower terminal of the summation can be increased from \( i = 0 \) to \( i = N - N - 1 \), since \( S_{N-n+i}(0) = 0 \) for \( i = 0, \ldots, n - N - 2 \). The upper terminal can be decreased from \( i = n - 1 \) to \( i = N \), since \( S_{N-i}(0) = 0 \) for \( i = N + 1, N + 2, \ldots, n - 1 \). Therefore, for \( n \geq N+2 \), we can write

\[
\frac{n\alpha^N}{N} \sum_{i=0}^{n-1} \binom{n-1}{i} S_{N-i}(0)S_{N-n+i}(0) \left( \frac{1}{\alpha} \right)^i = 0 \]

Finally, for \( n \geq 2N+1 \), all the terms fail to make a nonzero contribution. This can be seen in (45), where the summation runs over no terms when \( n \geq 2N+1 \), since the lower terminal is larger than the upper terminal. Therefore, for \( n \geq 2N+1 \), we have

\[
\frac{n\alpha^N}{N} \sum_{i=0}^{n-1} \binom{n-1}{i} S_{N-i}(0)S_{N-n+i}(0) \left( \frac{1}{\alpha} \right)^i = 0.
\]

Note that \( S_{N-1}(0) = 0 \) for \( i = 0, \ldots, N - 1 \), while \( S_{N-n+i}(0) = 0 \) for \( i = n - N - 1, \ldots \). The two sets \( \{0, \ldots, N - 1\} \) and \( \{n - N - 1, \ldots\} \) are disjoint when \( n \geq 2N+1 \).

\[
\square
\]

Lemma 3.

\[
\frac{d^n}{d\lambda^n} \left( -2\alpha^N S_N(\lambda) S_N \left( \frac{\lambda}{\alpha} \right) \right) = -2\alpha^N \sum_{i=0}^{n} \binom{n}{i} S_{N-i} \left( \frac{\lambda}{\alpha} \right) S_{N-n+i}(\lambda) \left( \frac{1}{\alpha} \right)^i
\]

\[
\left. \frac{d^n}{d\lambda^n} \left( -2\alpha^N S_N(\lambda) S_N \left( \frac{\lambda}{\alpha} \right) \right) \right|_{\lambda=0} = \begin{cases} 
-2\alpha^N \sum_{i=0}^{n} \binom{n}{i} \left( \frac{1}{\alpha} \right)^i, & 0 \leq n \leq N, \\
-2\alpha^N \sum_{i=n-N}^{N} \binom{N}{i} \left( \frac{1}{\alpha} \right)^i, & N + 1 \leq n \leq 2N, \\
0, & n > 2N.
\end{cases}
\]
Proof. The first equality follows immediately from a rewriting of (43). In particular, we set \( f = g = S_N, a = 1/\alpha, b = 1 \) and \( c = 0 \).

To evaluate at \( \lambda = 0 \), we observe that the summation featuring in the proof of Lemma 2 is of the same form as the summation at hand

\[
-2\alpha^N \sum_{i=0}^{n} \binom{n}{i} S_{N-i} \left( \frac{\lambda}{\alpha} \right) S_{N-n+i}(\lambda) \left( \frac{1}{\alpha} \right)^i.
\]

The two differences are that the term \( n\alpha^N/N \) multiplying the summation featuring in the proof of Lemma 2 has now been replaced with \( -2\alpha^N \) and \( n \leftarrow n - 1 \). Therefore, we can reuse the proof of Lemma 2 simply by replacing \( n\alpha^N/N \) with \( -2\alpha^N \) in (45) and then \( n \leftarrow n \).

Lemmas 4-8 involve taking a derivative of the general form given by (43). This can be accomplished in the same way as detailed in the proofs of Lemmas 2 and 3. We provide a summary in Table 1, but omit the proofs of Lemmas 4-8 due to their repetitiveness.

<table>
<thead>
<tr>
<th>Table 1: Form of (43) for Lemmas 2-8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
</tr>
<tr>
<td>Lemma 2</td>
</tr>
<tr>
<td>Lemma 3</td>
</tr>
<tr>
<td>Lemma 4</td>
</tr>
<tr>
<td>Lemma 5</td>
</tr>
<tr>
<td>Lemma 6</td>
</tr>
<tr>
<td>Lemma 7</td>
</tr>
<tr>
<td>Lemma 8</td>
</tr>
</tbody>
</table>

Lemma 4.

\[
\frac{d^{(n)}}{d\lambda^n} S_N(\lambda) S_N(\lambda) = \sum_{i=0}^{n} \binom{n}{i} S_{N-i}(\lambda) S_{N-n+i}(\lambda)
\]

<table>
<thead>
<tr>
<th>Lemma 5.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d^{(n)}}{d\lambda^n} - \frac{1}{N N!} \lambda^{N+1} S_N(\lambda) = -\frac{1}{N N!} \sum_{i=0}^{n} \binom{n}{i} \frac{(N+1)!}{(N+1-i)!} \lambda^{N+1-i} S_{N-n+i}(\lambda) )</td>
</tr>
</tbody>
</table>

\[
\frac{d^{(n)}}{d\lambda^n} - \frac{1}{N N!} \lambda^{N+1} S_N(\lambda) \bigg|_{\lambda=0} = \begin{cases} 
0, & 0 \leq n \leq N, \\
-\frac{(N+1)!}{N N!} \binom{n}{N+1}, & N + 1 \leq n \leq 2N + 1, \\
0, & n > 2N + 1.
\end{cases}
\]
Lemma 6.

\[
\frac{d^{(n)}}{d\lambda^{(n)}} \frac{\alpha^N}{N!} \lambda^N S_N(\frac{\lambda}{\alpha}) = \frac{\alpha^N}{N!} \sum_{i=0}^{n} \binom{n}{i} \frac{N!}{(N-i)!} \lambda^{N-i} S_{N-i}(\frac{\lambda}{\alpha}) \left( \frac{1}{\alpha} \right)^{n-i}
\]

\[
\left. \frac{d^{(n)}}{d\lambda^{(n)}} \frac{\alpha^N}{N!} \lambda^N S_N(\frac{\lambda}{\alpha}) \right|_{\lambda=0} = \begin{cases} 
0, & 0 \leq n \leq N-1, \\
\alpha^N \left( \frac{1}{\alpha} \right)^{n-N}, & N \leq n \leq 2N \\
0, & n > 2N.
\end{cases}
\]

Lemma 7.

\[
\left. \frac{d^{(n)}}{d\lambda^{(n)}} \left( -\frac{\alpha^N}{N!} \lambda^{N+1} S_N(\frac{\lambda}{\alpha}) \right) \right|_{\lambda=0} = \begin{cases} 
0, & 0 \leq n \leq N, \\
-\alpha^N \frac{N+1}{N} \left( \frac{1}{\alpha} \right)^{n-N-1}, & N+1 \leq n \leq 2N+1 \\
0, & n > 2N+1.
\end{cases}
\]

Lemma 8.

\[
\left. \frac{d^{(n)}}{d\lambda^{(n)}} \frac{\lambda^{2N+1}}{N(N!)} \right|_{\lambda=0} = \begin{cases} 
\frac{(2N+1)!}{N(N!)^2}, & n = 2N+1 \\
0, & \text{otherwise}.
\end{cases}
\]

The next step is to collect all of these terms together to obtain \( P^{(n)}(0) \). The formula is different for different ranges of \( n \), as the previous lemmas indicate, and the ranges depend on \( N \).

Lemma 9.

\[
P^{(n)}(0) = \begin{cases} 
\alpha^n n \left( 1 + \frac{1}{\alpha} \right)^{n-1} - 2\alpha^n \left( 1 + \frac{1}{\alpha} \right)^n + 2^n & 0 \leq n < N \\
\alpha^n n \left( 1 + \frac{1}{\alpha} \right)^{n-1} - 2\alpha^n \left( 1 + \frac{1}{\alpha} \right)^n + 2^n + \alpha^N & n = N \\
\frac{\alpha^n}{N} (N+1) \left( 1 + \frac{1}{\alpha} \right)^N - 2\alpha^n \sum_{i=1}^{N} \binom{N+1}{i} \left( \frac{1}{\alpha} \right)^i + \sum_{i=1}^{N} \binom{N+1}{i} & n = N+1 \\
-\frac{N+1}{N} + \alpha^{N-1}(N+1) - \alpha^{N+1} + \alpha^n n \sum_{i=n-N}^{N-1} \binom{n-i}{N} \left( \frac{1}{\alpha} \right)^i - 2\alpha^n \sum_{i=n-N}^{N} \binom{N}{i} \left( \frac{1}{\alpha} \right)^i + \sum_{i=n-N}^{N} \binom{n}{i} & N+2 \leq n \leq 2N \\
0 & \text{otherwise}.
\end{cases}
\]

5 The Easy Cases

We now have to verify that each of the derivatives of \( P(\lambda) \) is non-negative to complete the proof of the OPC conjecture. The cases are

\[
\{ 0 \leq n < N \}, \{ N \}, \{ N+1 \}, \{ N+2 \leq n \leq 2N \}.
\]

The most difficult case turns out to be the last of these: \( \{ N+2 \leq n \leq 2N \} \), and we leave it till the next and subsequent sections. We deal with all of the three other cases here. It simplifies formulae a little to consider \( n! P^{(n)} \) in each case.

Lemma 10.

\[
\frac{n+1}{N} n \left( 1 + \frac{1}{\alpha} \right)^{n-1} - 2\alpha^n \left( 1 + \frac{1}{\alpha} \right)^n + 2^n \geq 0 \quad n = 1, 2, ..., N-1.
\]
Proof. We divide by $2^n$, replace $\alpha$ by $\frac{N}{N+1}$ and rearrange to obtain in place of the left side of (54) that
\[
1 + \frac{n}{2N} \left(1 + \frac{1}{N}\right)^N \left(1 + \frac{1}{2N}\right)^{n-1} - 2 \left(1 + \frac{1}{N}\right)^N \left(1 + \frac{1}{2N}\right)^n
= 1 - \left(1 + \frac{1}{N}\right)^N \left(1 + \frac{1}{2N}\right)^{n-1} \left[2 - \frac{n-2}{2N}\right]. \quad (55)
\]
We need to show that this is non-negative for $0 \leq n < N$. To achieve this we show that the logarithm (to base $e$) of the second term is less than 0; that is,
\[
-N \log\left(1 + \frac{1}{N}\right) + (n-1) \log\left(1 + \frac{1}{2N}\right) + \log 2 - \log\left(1 - \frac{n-2}{4N}\right) < 0. \quad (56)
\]
Using the inequalities
\[
x - \frac{x^2}{2} < \log(1 + x) < x, \quad \log\left(\frac{1}{1 - x}\right) > x \text{ for } 0 \leq x \leq 1. \quad (57)
\]
we see that the expression in (56) does not exceed
\[
-N \left(1 - \frac{1}{2N^2}\right) + \frac{(n-1)}{2N} + \log 2 - \frac{(n-2)}{4N}
\leq (\log 2 - 1) + \frac{n+2}{4N} \leq \log 2 - 1 + \frac{N+1}{4N}. \quad (58)
\]
which is clearly negative for $N \geq 10$. The remaining small collection of cases of the original inequality can be checked by hand. We have used Maxima to check the cases $N < 10$. \qed

Lemma 11.
\[\alpha^N \left(1 + \frac{1}{\alpha}\right)^{N-1} - 2\alpha^N \left(1 + \frac{1}{\alpha}\right)^N + 2^N + \alpha^N \geq 0. \quad (59)\]

Proof. For $N \geq 2$, $\alpha^N < \frac{1}{2}$, so that
\[
\alpha^N \left(1 + \frac{1}{\alpha}\right)^{N-1} - 2\alpha^N \left(1 + \frac{1}{\alpha}\right)^N + 2^N + \alpha^N
= 2^N + \alpha^N - \alpha^N \left(1 + \frac{1}{\alpha}\right)^N \left[\frac{\alpha + 2}{\alpha + 1}\right]
= 2^N + \alpha^N - (\alpha + 1)^N \left[\frac{\alpha + 2}{\alpha + 1}\right]
= 2^N + \alpha^N - 2^N \left(\frac{3N + 2}{2N + 1}\right) \left(1 - \frac{1}{2(N+1)}\right)^N
\geq 2^N \left(1 - \frac{3N + 2}{2N + 1} \left(1 - \frac{1}{2(N+1)}\right)^N\right). \quad (60)
\]

As in the previous lemma, we do this by showing that the logarithm of $\left(\frac{3N+2}{2N+1}\right) \left(1 - \frac{1}{2(N+1)}\right)^N$ is negative using the inequalities given in (57). This reduces to
\[
\log \frac{3}{2} + \log \left(1 + \frac{1}{6N+3}\right) + N \log \left(1 - \frac{1}{2(N+1)}\right)
\leq \log \frac{3}{2} - \frac{1}{2} + \frac{1}{2(N+1)} + \frac{1}{6N+3}. \quad (61)
\]
which is negative for \( N \geq 7 \). The remaining few cases can be checked by hand (on the original inequality (59)), and we have used Maxima to do this. It is an indicator of the tightness of this inequality that the first four cases all give the value zero.

**Lemma 12.**

\[
\alpha^N \frac{N}{N+1} \left( 1 + \frac{1}{\alpha} \right)^N - 2\alpha^N \sum_{i=1}^{N} \left( \frac{N+1}{i} \right) \left( \frac{1}{\alpha} \right)^i + \sum_{i=1}^{N} \left( \frac{N+1}{i} \right) - \frac{N+1}{N} + \alpha^N \left( \frac{N+1}{N} \right) \left( \frac{1}{\alpha} \right) - \alpha^N \frac{N+1}{N} \geq 0. \tag{62}
\]

**Proof.** We begin by rewriting the expression as

\[
\alpha^{N-1} \left( 1 + \frac{1}{\alpha} \right)^N - 2\alpha^N \left[ \left( 1 + \frac{1}{\alpha} \right)^{N+1} - 1 - \frac{1}{\alpha^{N+1}} \right] + 2N^2 - \frac{1}{\alpha} + \alpha^{N-1} (N+1) - \alpha^{N-1} \\
= 2N - \alpha^{N-1} \left( 1 + \frac{1}{\alpha} \right)^N \left[ 1 + 2\alpha \right] + 2\alpha^N + \frac{1}{\alpha} - 2 + \alpha^{N-1}N \\
= 2N - \alpha^{N-1} \left( 1 + \frac{1}{\alpha} \right)^N \left[ 1 + 2\alpha \right] + (N+3)\alpha^N + \frac{1}{\alpha} - 2. \tag{63}
\]

Now we use the inequality \( \alpha^N > \frac{1}{e} \) to obtain

\[
2N - \alpha^{N-1} \left( 1 + \frac{1}{\alpha} \right)^N \left[ 1 + 2\alpha \right] + (N+3)\alpha^N + \frac{1}{\alpha} - 2 \\
\geq 2N - (\alpha + 1)^N \left[ \frac{1}{\alpha} + 2 \right] + \frac{N+3}{e} - 2 \\
\geq 2N - \left[ 2 - \left( \frac{19}{6} \right) \left( 1 - \frac{1}{2(N+1)} \right) \right] \tag{64}
\]

for \( N \geq 6 \) and the final term is positive for \( N \geq 9 \), because \( \left( 1 - \frac{1}{2(N+1)} \right)^N \) is decreasing in \( N \) and is less than \( \frac{12}{19} \) for \( N = 9 \). Again the remaining few terms can be checked by hand, and again we have done this using Maxima. \( \square \)

## 6 The Final Term

The proof of the OPC conjecture will be complete when we have proved the following lemma. This proof will occupy the remainder of the paper.

**Lemma 13.**

\[
\alpha^N \frac{n}{N} \sum_{i=0}^{n-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) \left( \frac{1}{\alpha} \right)^i - 2\alpha^N \sum_{i=n-N}^{N} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \frac{1}{\alpha} \right)^i + \sum_{i=n-N}^{N} \left( \begin{array}{c} n \\ i \end{array} \right) - \frac{N+1}{N} \left( \begin{array}{c} n \\ N+1 \end{array} \right) \\
+ \alpha^N \left( \frac{n}{N} \right) \left( \frac{1}{\alpha} \right)^{n-N} - \alpha^N \frac{N+1}{N} \left( \frac{n}{N+1} \right) \left( \frac{1}{\alpha} \right)^{n-N-1} \geq 0 \tag{65}
\]

for \( n = N+2, N+3, ..., 2N \).
First we replace \( n \) by \( n - N - 1 \) to obtain for the left side of the above inequality:

\[
\frac{n + 1 + N}{N} \alpha^{N} \sum_{i=n}^{N} \binom{n + N}{i} \alpha^{-i} - 2\alpha^{N} \sum_{i=n+1}^{N} \binom{n + N + 1}{i} \alpha^{-i} + \sum_{i=n+1}^{N} \binom{n + N + 1}{i} - \alpha^{-1} \binom{n + N + 1}{N + 1} + \alpha^{N-n-1} \left( \binom{n + N + 1}{N} - \binom{n + N + 1}{N + 1} \right)
\]

where now we have to prove this non-negative for \( 1 \leq n \leq N - 1 \), and as before \( \alpha = \frac{N}{N+1} \).

We fix \( n + N + 1 = R \) and define

\[
\Omega(n, N) = \frac{n + 1 + N}{N} \alpha^{N} \sum_{i=n}^{N} \binom{n + N}{i} \alpha^{-i} - 2\alpha^{N} \sum_{i=n+1}^{N} \binom{n + N + 1}{i} \alpha^{-i} + \sum_{i=n+1}^{N} \binom{n + N + 1}{i} - \alpha^{-1} \binom{n + N + 1}{N + 1} + \alpha^{N-n-1} \left( \binom{n + N + 1}{N} - \binom{n + N + 1}{N + 1} \right).
\]

The idea of the proof here is to show that for the “diagonal” this expression is zero, and that when we move away from the diagonal the value increases. The first assertion is (more precisely) expressed in the following lemma.

**Lemma 14.**

\[
\Omega(K - 1, K) = \Omega(K - 1, K + 1) = 0, \quad K = 1, 2, \ldots
\]

**Proof.** This is a straightforward calculation:

\[
\Omega(K - 1, K) = \binom{2K}{K} \left( \frac{K}{K+1} - 1 \right) + \binom{2K}{K} \left( 1 - \frac{K}{K+1} \right) = 0.
\]

\[
\Omega(K - 2, K) = \binom{2K-1}{K-1} \left( \frac{K}{K+1} - 1 \right) + \binom{2K-1}{K-1} \left( \frac{K-1}{K+1} - 2 \right) \frac{K}{K+1} + 1\right) \binom{K}{K+1} + 1\right) + \frac{K}{K+1} \binom{2K-1}{K} \left( 1 - \frac{K-1}{K+1} \right) = 0.
\]

\[\square\]

Now let \( \Upsilon(n, N) = \Omega(n - 1, N + 1) - \Omega(n, N) \). The remainder of the proof is concerned with the following lemma. Once it is completed, combined with Lemma 14, it gives the required result.

**Lemma 15.** For \( 0 \leq n < N \) \( \Upsilon(n, N) \geq 0 \).
The proof of this result in itself will require several lemmas. It appears to be a non-trivial computation to show that this result is true. To simplify the expressions, we let \( \beta = \frac{N + 1}{N + 2} \). Then a straightforward and omitted calculation shows that

\[
\Upsilon(n, N) = \left( \frac{N + n + 1}{N + 1} \right) \left( \beta - \beta^{N-n+1} \right) - \left( \frac{N + n + 1}{N} \right) \left( 1 - \frac{n + 1}{N + 1} \right) \alpha^{N-n-1}
\]

(71)

\[
+ \sum_{i=0}^{N-n-1} \left( \frac{N + n + 1}{N - i} \right) \left[ \left( \frac{N - i}{N + 2} - 2 \right) \beta^{i+1} - \left( \frac{N - i}{N + 1} - 2 \right) \alpha^{i} \right].
\]

Now we note that the sum in this expression for \( \Upsilon(n, N) \) is (the details are provided in Appendix I):

\[
\sum_{i=0}^{N-n-1} \left( \frac{N + n + 1}{N - i} \right) \left[ \left( \frac{N - i}{N + 2} - 2 \right) \beta^{i+1} - \left( \frac{N - i}{N + 1} - 2 \right) \alpha^{i} \right] = \frac{1}{2} \left\{ \sum_{i=0}^{N-n-1} \left( \frac{N + n + 1}{N - i} \right) \left[ \left( \frac{N - i}{N + 2} - 2 \right) \beta^{i+1} - \left( \frac{N - i}{N + 1} - 2 \right) \alpha^{i} \right] \right\}.
\]

(72)

This yields

\[
\Upsilon(n, N) = \left( \frac{N + n + 1}{N + 1} \right) \left( \beta - \beta^{N-n+1} \right) - \left( \frac{N + n + 1}{N} \right) \left( 1 - \frac{n + 1}{N + 1} \right) \alpha^{N-n-1}
\]

\[
+ \sum_{i=0}^{N-n-1} \left( \frac{N + n + 1}{N - i} \right) S(i) \alpha^{N-n-1-i}
\]

(73)

where

\[
S(i) = \left[ \left( \frac{N - i}{N + 2} - 2 \right) \beta^{i+1} - \left( \frac{N - i}{N + 1} - 2 \right) \alpha^{i} \right]
\]

\[
+ \left[ \left( \frac{i + n + 1}{N + 2} - 2 \right) \beta^{N-n-i} - \left( \frac{i + n + 1}{N + 1} - 2 \right) \alpha^{N-n-1-i} \right].
\]

(74)

**Lemma 16.** \( S(i) \) is increasing in \( i \) for \( 0 \leq i \leq \frac{1}{2}(N - n - 1) \).

**Proof.** Let

\[
Q(i) = \left[ \left( \frac{N - i}{N + 2} - 2 \right) \beta^{i+1} - \left( \frac{N - i}{N + 1} - 2 \right) \alpha^{i} \right].
\]

(75)

Then

\[
S(i) = Q(i) + Q(N - n - 1 - i).
\]

(76)

Differentiating with respect to \( i \) we have

\[
S'(i) = Q'(i) - Q'(N - n - 1 - i).
\]

(77)
To show, then, that $S(i)$ is increasing for $0 \leq i \leq \frac{1}{2}(N - n - 1)$, we shall show that $Q'(i)$ is decreasing for $0 \leq i \leq N - n - 1$. We observe that $\beta^{i+1} \leq \alpha^i$ for all real $i$ in the range in question, and hence that

$$Q''(i) = -\frac{2}{N+2} \beta^{i+1} \ln \beta + \left( \frac{N - i}{N+2} - 2 \right) \beta^{i+1} (\ln \beta)^2$$

$$+ \frac{2}{N+1} \alpha^i \ln \alpha - \left( \frac{N - i}{N+1} - 2 \right) \alpha^i (\ln \alpha)^2$$

$$= \left[ 2 - (N + 4 + i) \ln \left( \frac{N + 2}{N + 1} \right) \right] \frac{\beta^{i+1}}{N+2} \ln \left( \frac{N + 2}{N + 1} \right)$$

$$- \left[ 2 - (N + 2 + i) \ln \left( \frac{N + 1}{N} \right) \right] \frac{\alpha^i}{N+1} \ln \left( \frac{N + 1}{N} \right)$$

$$\leq - \left[ (N + 4 + i) \ln \left( \frac{N + 2}{N + 1} \right) - (N + 2 + i) \ln \left( \frac{N + 1}{N} \right) \right] \frac{\alpha^i}{N+1} \ln \left( \frac{N + 1}{N} \right) \leq 0,$$

which is enough to complete the proof.

The next lemma now follows from the preceding one.

**Lemma 17.**

$$\Upsilon(n, N) \geq \left( \frac{N + n + 1}{N + 1} \right) \left( \beta - \beta^{N-n+1} \right) - \left( \frac{N + n + 1}{N} \right) \left( 1 - \frac{n + 1}{N + 1} \right) \alpha^{N-n-1}$$

$$+ \left( \frac{N + n + 1}{N} \right) S_0 + \frac{S_{n-2}}{2} \left( \frac{N + n + 1}{N - i} \right).$$

(79)

Now define

$$U_n = \sum_{i=1}^{\left\lfloor \frac{N-n-1}{2} \right\rfloor} \left( \frac{N + n + 1}{N - i} \right).$$

(80)

Then we have the following lemma.

**Lemma 18.**

$$\frac{U_n}{\left( \frac{N + n + 1}{N} \right)} \geq \left( N + 3n + 5 \right) \left( \frac{3N + n + 3}{2N - 2n - 2} \right) \left( \frac{N_n}{2N + 3n + 5} \right) - 1.$$ $$- 1. (81)$$

In other words, the following is true.

$$\Upsilon(n, N) \geq \left( \frac{N + n + 1}{N + 1} \right) \left( \beta - \beta^{N-n+1} \right) - \left( \frac{N + n + 1}{N} \right) \left( 1 - \frac{n + 1}{N + 1} \right) \alpha^{N-n-1}$$

$$+ \left( \frac{N + n + 1}{N} \right) S_0$$

$$+ \left( \frac{N + n + 1}{N} \right) \left[ \left( \frac{N + 3n + 5}{2N - 2n - 2} \right) \left( \frac{3N + n + 3}{N + 3n + 5} \right) - 1 \right] S_1.$$ $$- 1.$$ (82)
Proof. This is proved as follows:

\[
\frac{U_n}{\binom{N+n+1}{N}} = \sum_{i=1}^{\frac{N-n-1}{2}} \prod_{k=1}^{i} \left( \frac{N+1-k}{n+1+k} \right)
= \sum_{i=1}^{\frac{N-n-1}{2}} \left( \prod_{k=1}^{i} \frac{(N+1-k)(N+1+k-i-1)}{(n+1+k)(n+1-k+i+1)} \right)^{\frac{1}{2}}
= \sum_{i=1}^{\frac{N-n-1}{2}} \left( \prod_{k=1}^{i} \frac{(N+1)^2-(i+1)(N+1)}{(n+1)^2+(i+1)(n+1)} \right)^{\frac{1}{2}}
= \sum_{i=1}^{\frac{N-n-1}{2}} \left( \frac{(N+1-i+1)}{(n+1+i+1)} \right)^{i}
= \sum_{i=1}^{\frac{N-n-1}{2}} \left( \frac{2N-i+1}{2n+i+3} \right)^{i}
= \sum_{i=1}^{\frac{N-n-1}{2}} \left( \frac{2N-\frac{n-1}{2}+1}{2n+\frac{n-1}{2}+3} \right)^{i}
= \sum_{i=1}^{\frac{N-n-1}{2}} \left( \frac{3N+n+3}{N+3n+5} \right)^{i}
= \left( \frac{N+3n+5}{2N-2n-2} \right) \left( \left( \frac{3N+n+3}{N+3n+5} \right)^{\frac{N-n}{2}} - 1 \right) - 1.
\]

(83)

6.1 Final Estimates

At this point we change from using \( N \) and \( n \) as the variables to using \( R = N + n + 1 \) and \( D = N - n - 1 \), so that \( N = \frac{R+D}{2} \) and \( n+1 = \frac{R-D}{2} \). The right side of inequality (82) can be rewritten as

\[
Q(D, R) = \frac{R-D}{R+4+D} (1 - \beta^{D+1}) - \frac{2+2D}{R+2+D} \alpha^{D} + (L_0 + R_0) + M(L_1 + R_1)
\]

(84)
where

\[ M = \left( \frac{2R + 2 - D}{2D} \right) \left( \frac{2R + 2 + D}{2R + 2 - D} \right)^{\frac{D+1}{2}} - 1 \]  

(85)

\[ L_0 = \left( \frac{N}{N + 2} - 2 \right) \beta^1 - \left( \frac{N}{N + 1} - 2 \right) \alpha^0 \]

(86)

\[ = \left( 2 - \frac{R + D}{R + 2 + D} \right) - \left( 2 - \frac{R + D}{R + 4 + D} \right) \beta^1 \]  

(87)

\[ R_0 = \left( \frac{n + 1}{N + 2} - 2 \right) \beta^{N-n} - \left( \frac{n + 1}{N + 1} - 2 \right) \alpha^{N-n-1} \]

(88)

\[ = \left( 2 - \frac{R - D}{R + 2 + D} \right) \alpha^D - \left( 2 - \frac{R - D}{R + 4 + D} \right) \beta^{D+1} \]  

(89)

\[ L_1 = \left( \frac{N - 1}{N + 2} - 2 \right) \beta^2 - \left( \frac{N - 1}{N + 1} - 2 \right) \alpha^1 \]

(90)

\[ = \left( 2 - \frac{R - 2 + D}{R + 2 + D} \right) \alpha^D - \left( 2 - \frac{R - 2 + D}{R + 4 + D} \right) \beta^2 \]  

(91)

\[ R_1 = \left( \frac{n + 2}{N + 2} - 2 \right) \beta^{N-n-1} - \left( \frac{n + 2}{N + 1} - 2 \right) \alpha^{N-n-2} \]

(92)

\[ = \left( 2 - \frac{R + 2 - D}{R + 2 + D} \right) \alpha^{D-1} - \left( 2 - \frac{R + 2 - D}{R + 4 + D} \right) \beta^D. \]  

(93)

where \( \alpha = \frac{N}{N+1} = \frac{R+D}{R+2+D} \) and \( \beta = \frac{N+1}{N+2} = \frac{R+2+D}{R+4+D} \).

We state, as a lemma, the final requirement for our proof.

**Lemma 19.** \( Q(D, R) \geq 0 \) for \( 0 \leq D \leq R - 4 \) and \( R \geq 4 \).

We note that the cases \( D = 0 \) and \( D = 1 \) are already covered by equation (68). We also note that \( L_0, L_1, R_0, R_1 \) are non-negative, as is the first term in \( Q(D, R) \). Moreover the negative (second) term does not exceed 1 in absolute value.

It will be convenient to write

\[ W = \frac{R - D}{R + 4 + D}(1 - \beta^{D+1}) - \frac{2 + 2D}{R + 2 + D} \alpha^D. \]  

(94)

It is a remarkable (and frustrating) fact that all of the non-negative terms are needed to make the inequality \( Q(D, R) \) valid in this range. We begin by replacing each of them by somewhat simpler forms. We write

\[ \rho = \frac{2}{2 + R + D}. \]  

(95)

**Lemma 20.**

\[ L_0 = \rho - \left( \frac{\rho}{\rho + 1} \right)^2, \]  

(96)

\[ W = (1 - \rho)^D - \frac{2}{(1 + \rho)^{D+1}} + 1 - \frac{\rho}{\rho + 1}(D + 2), \]  

(97)

\[ L_1 = \left( \frac{\rho^2}{(\rho + 1)^3} \right)(4 - 2\rho - 5\rho^2 - \rho^3), \]  

(98)

\[ R_1 = (1 + D\rho)(1 - \rho)^{D-1} - \left( 1 + (D + 1) \frac{\rho}{\rho + 1} \right) \left( 1 - \frac{\rho}{\rho + 1} \right)^D, \]  

(99)

\[ M = \frac{1}{u} \left( (1 + u)^{D+1/2} - 1 \right) - 1 \]  

(100)
where
\[
\frac{1}{u} = \frac{2R + 2 - D}{2D} = \left(\frac{2}{D} \left(\frac{1}{\rho} - \frac{1}{2}\right) - \frac{3}{2}\right).
\] (101)

The following lemma is obtained by the judicious use of the binomial expansion. We aim to have a common denominator of \(1/(1 + \rho)^D\) in the various summands in the required inequality.

**Lemma 21.** Let
\[
\mathcal{L}_0 = \frac{\rho^2}{(1 + \rho)^{D+1}}(1 + (D - 1)\rho + \frac{1}{2}(D - 1)(D - 2)\rho^2)(3 + \rho)
\] (102)
\[
\mathcal{L}_1 = \frac{\rho^2}{(1 + \rho)^{D+1}}(1 + (D - 2)\rho + \frac{1}{2}(D - 2)(D - 3)\rho^2)(4 - 2\rho - 5\rho^2 - \rho^3),
\] (103)
\[
\mathcal{W} = \frac{-1}{(\rho + 1)^{D+1}}\left(\frac{5}{2}D\rho^2 + \frac{1}{2}D^2\rho^2 + \frac{1}{2}D\rho^3 + \frac{1}{2}D^3\rho^3\right)
\] (104)
\[
\mathcal{R}_1 = \frac{\rho^2}{(1 + \rho)^{D+1}}(D + 2 - \rho(D^2 - 2) - \rho^2(D - 1)(1 + 2D) - \rho^3(D - 1)D)
\] (105)
\[
\mathcal{M} = \frac{D - 1}{2} + \frac{D(D^2 - 1)\rho}{16}(1 + \frac{1}{2}(D + 2)\rho)\left(1 + \frac{\rho}{4}(D - 2)(1 + \frac{1}{2}(D + 2)\rho)\right).
\] (106)

Then for \(D \geq 3\)
\[
\mathcal{L}_0 \geq \mathcal{L}_0, \quad \mathcal{L}_1 \geq \mathcal{L}_1, \quad \mathcal{M} \geq \mathcal{M}, \quad \mathcal{W} \geq \mathcal{W}, \quad \mathcal{R}_1 \geq \mathcal{R}_1.
\] (107)

The details of the proof are given in Appendix II. In view of this lemma it is enough to show that
\[
\mathcal{L}_0 + \mathcal{W} + \mathcal{M}(\mathcal{L}_1 + \mathcal{R}_1) \geq 0,
\] (108)
for \(R \geq 6\) and \(3 \leq D \leq R - 4\).

The next two lemmas can be proved by straightforward calculations (See Appendix III and IV) based on what we know about the ranges of \(\rho\) and \(D\). We note that, for \(D \geq 5\), \(\rho \leq \frac{2}{13}\).

**Lemma 22.**
\[
\mathcal{L}_0 + \mathcal{W} + \mathcal{M}(\mathcal{L}_1 + \mathcal{R}_1) \geq \frac{1}{16}
\] (109)
\[
- \left[ D^8(2\rho^5) + D^7(6\rho^4) + D^6(16\rho^3) + D^5(44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) 
+ D^4(96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) + D^3(576\rho^5 + 152\rho^4 + 320\rho^3) 
+ D^2(2336\rho^2 - 1248\rho) + D(1984\rho) \right].
\]

**Lemma 23.** For \(D \geq 5\),
\[
\mathcal{L}_0 + \mathcal{W} + \mathcal{M}(\mathcal{L}_1 + \mathcal{R}_1) \geq \frac{1}{16}
\] (110)
\[
- \left[ 2D^8\rho^5 + 6D^7\rho^4 + 16D^6\rho^3 \right] - (128\rho^3 - 8\rho^2)D^5 
- (106\rho^2 + 32\rho)D^4 + 320D^3\rho + D^2(2336\rho^2 - 851\rho) \right].
\]

Now we define
\[
W = - 2336\rho + 851 - 320D + (106\rho + 32)D^2 
+ (128\rho^2 - 8\rho)D^3 - 16\rho^2D^4 - 6\rho^3D^5 - 2\rho^4D^6,
\] (111)
so that by Lemma 23
\[ CL_0 + CM(CL_1 + CR_1) \geq \rho D^2 W \]
for \( D \geq 5 \), and, leaving aside the special case where this fails, it is enough to show that \( W \) is non-negative. For the moment we shall assume that \( D \geq 5 \), and return to the special cases excluded by this at the end.

Now we rewrite \( W \) in terms of \( R \) and \( D \), reversing the substitution \( \rho = \frac{2}{R+D+2} \), and simplify to obtain
\[
W = (32D^2 - 320D + 851)R^6 + (112D^3 - 812D^2 + 844D + 2136)R^5 \\
+ (80D^4 - 100D^3 - 534D^2 - 1272D - 7608)R^4 \\
+ (-96D^5 + 700D^4 + 1660D^3 - 5384D^2 - 25456D - 28832)R \\
- 128D^6 + 212D^5 + 1931D^4 - 56D^3 - 15640D^2 - 33952D - 23760.
\]

Next we split the argument into two cases: \( R/D \geq 10 \) and \( R/D < 10 \). For the former, we note that the sum of the \( R^6 \) and \( R \) terms in \( W \) exceeds
\[
11200D^5 - 81200D^4 + 84400D^3 + 213600D^2 - 95D^5 + 700D^4 \\
+ 1660D^3 - 5384D^2 - 25456D - 28832 \\
= 11105D^5 - 80500D^4 + 86060D^3 + 208216D^2 - 25456D - 28832, \tag{114}
\]
which is positive for \( D \geq 5 \). The sum of the \( R^5 \), \( R^2 \) and constant terms in \( R \) and in \( W \) (111) exceeds
\[
320000D^6 - 3200000D^5 + 8510000D^4 + 8000D^6 - 10000D^5 - 53400D^4 \\
- 127200D^3 - 760800D^2 - 128D^6 + 212D^5 + 1931D^4 - 56D^3 \\
- 15640D^2 - 33952D - 23760 \\
\geq 327D^6 - 3210D^5 + 8450D^4 - 128D^3 - 780D^2 - 34D - 24 \\
\geq 550D^4 - 128D^3 - 780D^2 - 35D - 24,
\]
which is again positive in the range \( D \geq 5 \).

Now we consider the case \( R/D < 10 \). We rewrite \( W \) in terms of \( R/D \) and split it into two parts:

\[
P_1 = \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] D^6 \tag{116}
+ \left[ -320 \left( \frac{R}{D} \right)^4 - 812 \left( \frac{R}{D} \right)^3 - 100 \left( \frac{R}{D} \right)^2 + 700 \left( \frac{R}{D} \right) + 212 \right] D^5 \\
+ \left[ 851 \left( \frac{R}{D} \right)^4 + 844 \left( \frac{R}{D} \right)^3 - 534 \left( \frac{R}{D} \right)^2 + 1660 \left( \frac{R}{D} \right) + 1931 \right] D^4.
\]

\[
P_2 = \left[ 2136 \left( \frac{R}{D} \right)^3 - 1272 \left( \frac{R}{D} \right)^2 - 5384 \left( \frac{R}{D} \right) - 56 \right] D^3 \tag{117}
+ \left[ -7608 \left( \frac{R}{D} \right)^2 - 25456 \left( \frac{R}{D} \right) - 15640 \right] D^2 \\
+ \left[ -28832 \left( \frac{R}{D} \right) - 33952 \right] D - 23760.
\]
It is simple to check that $P_1$ is positive for all $R/D < 10$, $R \geq 9$, $D \geq 5$, and that $P_2$ is positive provided in addition $R/D > 3.3$. Thus $W$ is positive provided $3.3 < R/D < 10$, $R \geq 9$, $D \geq 5$. The details are given in Appendix V. Apart from a few values that can be checked by hand, then, it remains to show the result for $R/D < 3.3$.

We next write

$$W = Q_1D^6 + Q_2D^5 + Q_3D^4 + Q_4D^3 + Q_5D^2 + Q_6D - 23760, \quad (118)$$

where

$$Q_1 = 32\left(\frac{R}{D}\right)^4 + 112\left(\frac{R}{D}\right)^3 + 80\left(\frac{R}{D}\right)^2 - 96\left(\frac{R}{D}\right) - 128;$$

$$Q_2 = -320\left(\frac{R}{D}\right)^4 - 812\left(\frac{R}{D}\right)^3 - 100\left(\frac{R}{D}\right)^2 + 700\left(\frac{R}{D}\right) + 212;$$

$$Q_3 = 851\left(\frac{R}{D}\right)^4 + 844\left(\frac{R}{D}\right)^3 - 534\left(\frac{R}{D}\right)^2 + 1660\left(\frac{R}{D}\right) + 1931;$$

$$Q_4 = 2136\left(\frac{R}{D}\right)^3 - 1272\left(\frac{R}{D}\right)^2 - 5384\left(\frac{R}{D}\right) - 56;$$

$$Q_5 = -7608\left(\frac{R}{D}\right)^2 - 25456\left(\frac{R}{D}\right) - 15640;$$

$$Q_6 = -28832\left(\frac{R}{D}\right) - 33952. \quad (119)$$

We note that $Q_1, Q_3, Q_4$ are increasing functions and that $Q_2, Q_5, Q_6$ are decreasing functions of $R/D$. In turn we replace $R/D$ by each of $1.1, 1.2, 1.3, \ldots, 3.2$ in the increasing functions and respectively, by $1.2, 1.3, 1.4 \ldots, 3.3$ in the decreasing ones to give worst case estimates over, respectively, each of the intervals

$$[1.1, 1.2], \ [1.2, 1.3], \ [1.3, 1.4], \ldots, \ [3.2, 3.3]. \quad (120)$$

It can be shown that, in each case, $W$ given by (118) is increasing (and therefore clearly positive) by repeated differentiation and checking for positivity of the derivatives at the endpoint $D = 14$. The details are given in Appendix VI.

Finally we consider the case when $R/D < 1.1$. We rewrite $W$ in terms of $R$ and $y = D/R$ as follows

$$W = 32y^2R^8 - 320yR^7 + (851 + 112y^3 + 80y^4 - 96y^5 - 128y^6)R^6$$

$$+ (-812y^2 - 100y^3 + 700y^4 + 212y^5)R^5$$

$$+ (844y - 534y^2 + 1660y^3 + 1931y^4)R^4$$

$$+ (2136 - 1272y - 5384y^3 - 56y^3)R^3$$

$$+ (-7608 - 25456y - 15640y^2)R^2$$

$$+ (-28832 - 33952y)R - 23760. \quad (121)$$

For each coefficient of a power of $R$, we again consider the worst case, by replacing $y$ by 1 if that coefficient is decreasing in $y$ over the interval $[1/1.1, 1]$, and by $1/1.1$ if that coefficient is to increasing over that interval. We collect the first two terms ($R^7$ and $R^8$) together and note that this is increasing in $y$ for any $R \geq 8$. The result is that

$$W \geq \frac{32}{(1.1)^2} R^8 - \frac{320}{1.1} R^7 + 800R^6 - 136.5R^5$$

$$+ 2892R^4 - 4576R^3 - 48704R^2 - 62784R - 23760. \quad (122)$$

22
It can be shown that this is increasing (and therefore clearly positive) by repeated differentiation
and checking for positivity of the derivatives at the endpoint $1/1.1$. The details are given in
Appendix VII.

There only remain a fairly small finite number of cases for small values of $R$ and $D$. Each
of these have been checked using Maxima. We give the details in Appendix VII for all of the
special cases, $D = 2, 3, 4$. This completes the proof!

**Appendix I: The calculation of $\Upsilon(n, N)$ for Lemma 15**

Here we give the detailed calculation of the formula (71) for $\Upsilon(n, N)$ needed to prove Lemma 15.
Consider the “sum” terms of $\Omega(n, N)$ in (67)

$$\frac{n + 1 + N}{N} \alpha^n \sum_{i=n}^{N} \binom{n + N}{i} \alpha^{-i} - 2\alpha^n \sum_{i=n+1}^{N} \binom{n + N + 1}{i} \alpha^{-i} + \sum_{i=n+1}^{N} \binom{n + N + 1}{i}. \quad (123)$$

We rewrite the first term as

$$\sum_{i=n+1}^{N+1} \frac{n + N + 1}{N + 1} \binom{n + N}{i - 1} \alpha^{N-i} = \sum_{i=n+1}^{N+1} \frac{(N + n + 1)!}{(i - 1)!(N + n + 1 - i)!} \frac{1}{N + 1} \alpha^{N-i}. \quad (124)$$

Consider the “top” term in this sum (ie the $i = N + 1$ term). This is

$$\frac{(N + n + 1)!}{N!} \frac{1}{N + 1} \frac{1}{\alpha}, \quad (125)$$

which coincides (except for sign) with the fourth term in equation (123), and therefore cancels.
We are left with the following problem:

$$\sum_{i=n+1}^{N} \left[ \frac{(N + n + 1)!}{(i - 1)!(N + n + 1 - i)!} \frac{1}{N + 1} \alpha^{N-i} - 2\alpha^n \binom{n + N + 1}{i} \alpha^{-i} + \binom{n + N + 1}{i} \right] + \alpha^{N-n-1} \left[ \left( \binom{n + N + 1}{N} - \binom{n + N + 1}{N + 1} \right) \right] \geq 0. \quad (126)$$

We write the sum term again as

$$\sum_{i=n+1}^{N} \binom{n + 1}{i} \left[ \frac{i}{N + 1} \alpha^{N-i} - 2\alpha^{N-i} + 1 \right] \quad (127)$$

which reduces to

$$\sum_{i=n+1}^{N} \binom{n + 1}{i} \left[ \left( \frac{i}{N + 1} - 2 \right) \alpha^{N-i} + 1 \right]. \quad (128)$$

At this point, $\Omega(n, N)$ can be rewritten as

$$\Omega(n, N) = \sum_{i=n+1}^{N} \binom{n + 1}{i} \left[ \left( \frac{i}{N + 1} - 2 \right) \alpha^{N-i} + 1 \right] + \alpha^{N-n-1} \left[ \left( \binom{n + 1}{N} - \binom{n + 1}{N + 1} \right) \right]. \quad (129)$$
Now we calculate $\Upsilon(n, N)$ as follows:

$$\Upsilon(n, N) = \Omega(n-1, N+1) - \Omega(n, N)$$

$$= \sum_{i=n}^{N+1} \binom{N+n+1}{i} \left[ \left( \frac{i}{N+2} - 2 \right) \beta^{N+1-i} + 1 \right]$$

$$+ \beta^{N-n-1} \left[ \binom{n+N+1}{N+1} - \binom{n+N+1}{N+2} \right]$$

$$- \sum_{i=n+1}^{N} \binom{N+n+1}{i} \left[ \left( \frac{i}{N+1} - 2 \right) \alpha^{N-i} + 1 \right]$$

$$- \alpha^{N-n-1} \left[ \binom{n+N+1}{N} - \binom{n+N+1}{N+1} \right]$$

$$= \sum_{i=0}^{N-n} \binom{N+n+1}{N-i} \left[ \frac{N-i}{N+2} - 2 \right] \beta^{i+1} + 1$$

$$+ \beta^{N-n-1} \left[ \binom{n+N+1}{N+1} - \binom{n+N+1}{N+2} \right]$$

$$- \sum_{i=0}^{N-n-1} \binom{N+n+1}{N-i} \left[ \frac{N-i}{N+1} - 2 \right] \alpha^{i+1} + 1$$

$$- \alpha^{N-n-1} \left[ \binom{n+N+1}{N} - \binom{n+N+1}{N+1} \right]$$

$$= \binom{N+n+1}{N+1} \left( \beta - \beta^{N-n+1} \right) - \binom{N+n+1}{N} \left( 1 - \frac{n+1}{N+1} \right) \alpha^{N-n-1}$$

$$+ \sum_{i=0}^{N-n-1} \binom{N+n+1}{N-i} \left[ \frac{N-i}{N+2} - 2 \right] \beta^{i+1} + 1 - \frac{N-i}{N+1} - 2 \alpha^{i+1} \right]$$

(130)

**Appendix II: Proof of Lemma 21**

In this appendix we prove the inequalities stated in Lemma 21, specifically:

$$\mathcal{L}_0 \geq C \mathcal{L}_0, \quad \mathcal{L}_1 \geq C \mathcal{L}_1, \quad \mathcal{M} \geq C \mathcal{M}, \quad \mathcal{W} \geq C \mathcal{W}, \quad \mathcal{R}_1 \geq C \mathcal{R}_1.$$  

(131)

These, by and large, do not warrant explanation beyond the calculations.

$$\mathcal{L}_0 = \rho - \left( \frac{\rho}{\rho + 1} \right)^2 + 2 \left( \frac{\rho}{\rho + 1} \right)^2$$

$$= \left( \frac{\rho}{\rho + 1} \right)^2 (3 + \rho)$$

$$= \frac{\rho^2}{(\rho + 1)^{D+1}} (\rho + 1)^{D-1} (3 + \rho)$$

$$\geq \frac{\rho^2}{(1 + \rho)^{D+1}} \left( 1 + (D-1) \rho + \frac{1}{2} (D-1)(D-2) \rho^2 \right) (3 + \rho)$$

$$= C \mathcal{L}_0.$$  

(132)
\[
\mathcal{L}_1 = \left(\frac{\rho^2}{(\rho + 1)^3}\right)(4 - 2\rho - 5\rho^2 - \rho^3)
= \frac{\rho^2}{(1 + \rho)^{D+1}}(\rho + 1)^{D-2}(4 - 2\rho - 5\rho^2 - \rho^3)
\geq \frac{\rho^2}{(1 + \rho)^{D+1}}\left(1 + (D - 2)\rho + \frac{1}{2}(D - 2)(D - 3)\rho^2\right)(4 - 2\rho - 5\rho^2 - \rho^3)
= \mathcal{C}\mathcal{L}_1.
\]

For the final inequality, we note first that
\[
\mathcal{L}_1 = (1 + \rho)(1 - \rho)^{D-1} - \left(1 + (D + 1)\frac{\rho}{\rho + 1}\right)\left(1 - \frac{\rho}{\rho + 1}\right)^D
= \frac{1}{(1 + \rho)^{D+1}}\left((1 + \rho D)(1 - \rho^2)^{D-1} - (2\rho + \rho D + 1)\right)
= \frac{1}{(1 + \rho)^{D+1}}\left(\rho^2(1 + 2D + \rho D)(1 - \rho^2)^{D-1} - (1 + 2\rho + \rho D)(1 - (1 - \rho^2)^{D-1})\right)
\geq \frac{1}{(1 + \rho)^{D+1}}\left(\rho^2(1 + 2D + \rho D)(1 - (D - 1)^2) - (1 + 2\rho + \rho D)(D - 1)\rho^2\right)
= \frac{\rho^2}{(1 + \rho)^{D+1}}((D + 2) - \rho(D^2 - 2) - \rho^2(D - 1)(1 + 2D) - \rho^3(D - 1)D)
= \mathcal{C}\mathcal{R}_1.
\]

For the final inequality, we note first that
\[
\frac{1}{R} = \frac{\rho}{2}\left(1 - \frac{(D + 2)\rho}{2}\right)^{-1}
\geq \frac{\rho}{2}\left(1 + \frac{(D + 2)\rho}{2}\right).
\]
Then

\[
\mathcal{M} = \frac{1}{u} (1 + u)^{(D+1)/2} - 1 \\
\geq \frac{1}{u} \left(1 + \frac{D+1}{2}u + \frac{D^2}{8} - 1\right) - 1 \\
= \frac{D - 1}{2} + \frac{D^2 - 1}{u} \\
= \frac{D - 1}{2} + \frac{D(D^2 - 1)}{8R} \left(1 - \frac{D - 2}{2R}\right)^{-1} \\
\geq \frac{D - 1}{2} + \frac{D(D^2 - 1)}{8R} \left(1 + \frac{D - 2}{2R}\right) \\
\geq \frac{D - 1}{2} + \frac{D(D^2 - 1)}{16} \left(1 + \frac{1}{2}(D + 2)\rho\right) \left(1 + \frac{\rho}{4}(D - 2)(1 + \frac{1}{2}(D + 2)\rho)\right) \\
= CM. \quad (137)
\]

**Appendix III: Proof of Lemma 22**

Here we prove the following inequality (Lemma 22).

\[
\mathcal{C}L_0 + CW + CM(\mathcal{C}L_1 + CR_1) \geq \\
- [D^8(2\rho^8) + D^7(6\rho^7) + D^6(16\rho^6) + D^5(44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
+ D^4(96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) + D^3(576\rho^3 + 152\rho^2 + 320\rho) \\
+ D^2(2336\rho^2 - 1248\rho + D(1984\rho))] . \quad (138)
\]

We highlight in bold face terms that are removed because, collectively, they form a positive quantity for the range of values of \(R\) and \(D\) under consideration, namely, \(R \geq 6\) and \(3 \leq D \leq R - 4\).

Expanding the equations in Lemma 21 and dividing by \(\frac{\rho^2}{512(1+\rho)^D}\), we obtain

\[
\mathcal{C}L_0 + CW + CM(\mathcal{C}L_1 + CR_1) \geq \\
- [D^8(\rho^9 + 5\rho^8 + 4\rho^7 + 2\rho^5) \\
+ D^7(-3\rho^9 - 9\rho^8 + 26\rho^7 + 38\rho^6 - 4\rho^5 + 6\rho^4) \\
+ D^6(-9\rho^9 - 65\rho^8 - 110\rho^7 + 72\rho^6 + 126\rho^5 - 48\rho^4 + 16\rho^3) \\
+ D^5(27\rho^9 + 121\rho^8 - 90\rho^7 - 482\rho^6 + 44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
+ D^4(24\rho^9 + 220\rho^8 + 546\rho^7 - 72\rho^6 - 939\rho^5 + 96\rho^4 + 49\rho^3 - 288\rho^2 - 32\rho) \\
+ D^3(-72\rho^9 - 416\rho^8 - 256\rho^7 + 1036\rho^6 + 712\rho^5 - 152\rho^4 + 476\rho^3 + 152\rho^2 + 320\rho) \\
+ D^2(-16\rho^9 - 160\rho^8 - 440\rho^7 + 40\rho^6 - 3632\rho^5 - 1536\rho^3 + 2336\rho^2 - 1248\rho) \\
+ D(48\rho^9 + 304\rho^8 + 320\rho^7 - 592\rho^6 + 784\rho^5 + 6720\rho^4 + 320\rho^3 - 4256\rho^2 + 1984\rho) \\
- 768\rho^5 - 3328\rho^4 + 256\rho^3 + 2048\rho^2 - 1024\rho] .
\]

26
Eliminating the highlighted terms in (139), we have:

\[
\mathcal{CL}_0 + CV + \mathcal{CM}(\mathcal{CL}_1 + CR_1) \geq - \left[ D^8 (\rho^9 + 5 \rho^8 + 4 \rho^7 + 2 \rho^5) + D^7 (9 \rho^9 + 26 \rho^7 + 38 \rho^5 - 4 \rho^4 + 6 \rho^3) + D^6 (-65 \rho^8 - 110 \rho^7 + 72 \rho^6 - 126 \rho^5 - 48 \rho^4 + 16 \rho^3) + D^5 (27 \rho^9 + 121 \rho^8 - 90 \rho^7 - 482 \rho^6 + 44 \rho^5 + 338 \rho^4 - 128 \rho^3 + 8 \rho^2) + D^4 (24 \rho^9 + 220 \rho^8 + 546 \rho^7 - 72 \rho^6 - 939 \rho^5 + 96 \rho^4 + 496 \rho^3 - 288 \rho^2 - 32 \rho) + D^3 (-416 \rho^8 - 256 \rho^7 + 1036 \rho^6 + 712 \rho^5 - 152 \rho^4 + 576 \rho^3 + 152 \rho^2 + 320 \rho) + D^2 (-160 \rho^8 - 440 \rho^7 + 40 \rho^6 - 3632 \rho^5 + 1536 \rho^3 + 2336 \rho^2 - 1248 \rho) + D (48 \rho^9 + 304 \rho^8 + 320 \rho^7 - 592 \rho^6 + 784 \rho^5 + 6720 \rho^4 + 320 \rho^3 - 4256 \rho^2 + 1984 \rho) + 3328 \rho^4 + 256 \rho^3 + 2048 \rho^2 - 1024 \rho \right],
\]

and repeating the procedure several times,

\[
\mathcal{CL}_0 + CV + \mathcal{CM}(\mathcal{CL}_1 + CR_1) \geq - \left[ D^8 (\rho^9 + 5 \rho^8 + 4 \rho^7 + 2 \rho^5) + D^7 (110 \rho^7 + 72 \rho^6 + 126 \rho^5 - 48 \rho^4 + 16 \rho^3) + D^6 (27 \rho^9 + 121 \rho^8 - 90 \rho^7 - 482 \rho^6 + 44 \rho^5 + 338 \rho^4 - 128 \rho^3 + 8 \rho^2) + D^5 (220 \rho^8 + 546 \rho^7 - 72 \rho^6 - 939 \rho^5 + 96 \rho^4 + 496 \rho^3 - 288 \rho^2 - 32 \rho) + D^4 (256 \rho^7 + 1036 \rho^6 + 712 \rho^5 - 152 \rho^4 + 576 \rho^3 + 152 \rho^2 + 320 \rho) + D^3 (440 \rho^7 + 40 \rho^6 - 3632 \rho^5 + 1536 \rho^3 + 2336 \rho^2 - 1248 \rho) + D (48 \rho^9 + 304 \rho^8 + 320 \rho^7 - 592 \rho^6 + 784 \rho^5 + 6720 \rho^4 + 320 \rho^3 - 4256 \rho^2 + 1984 \rho) + 3328 \rho^4 + 256 \rho^3 + 2048 \rho^2 - 1024 \rho \right],
\]

\[
\mathcal{CL}_0 + CV + \mathcal{CM}(\mathcal{CL}_1 + CR_1) \geq - \left[ D^8 (5 \rho^8 + 4 \rho^7 + 2 \rho^5) + D^7 (26 \rho^7 + 38 \rho^6 - 4 \rho^5 + 6 \rho^4) + D^6 (72 \rho^6 + 126 \rho^5 - 48 \rho^4 + 16 \rho^3) + D^5 (27 \rho^9 + 121 \rho^8 - 90 \rho^7 - 482 \rho^6 + 44 \rho^5 + 338 \rho^4 - 128 \rho^3 + 8 \rho^2) + D^4 (220 \rho^8 + 546 \rho^7 - 72 \rho^6 - 939 \rho^5 + 96 \rho^4 + 496 \rho^3 - 288 \rho^2 - 32 \rho) + D^3 (256 \rho^7 + 1036 \rho^6 + 712 \rho^5 - 152 \rho^4 + 576 \rho^3 + 152 \rho^2 + 320 \rho) + D^2 (440 \rho^7 + 40 \rho^6 - 3632 \rho^5 + 1536 \rho^3 + 2336 \rho^2 - 1248 \rho) + D (48 \rho^9 + 304 \rho^8 + 320 \rho^7 - 592 \rho^6 + 784 \rho^5 + 6720 \rho^4 + 320 \rho^3 - 4256 \rho^2 + 1984 \rho) + 3328 \rho^4 + 256 \rho^3 + 2048 \rho^2 - 1024 \rho \right].
\]
\[
\begin{align*}
\mathcal{C}L_0 + CW + CM(CL_1 + CR_1) & \\
\geq - D^5(5\rho^8 + 4\rho^7 + 2\rho^5) & + D^7(26\rho^7 + 38\rho^6 - 4\rho^5 + 6\rho^4) \\
& + D^6(72\rho^6 + 126\rho^5 - 48\rho^4 + 16\rho^3) \\
& + D^5(-482\rho^6 + 44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
& + D^4(420\rho^8 + 546\rho^7 - 72\rho^6 - 939\rho^5 + 96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \\
& + D^3(-256\rho^7 + 1036\rho^6 + 712\rho^5 - 152\rho^4 + 576\rho^3 + 152\rho^2 + 320\rho) \\
& + D^2(+40\rho^5 - 3632\rho^4 - 1536\rho^3 + 2336\rho^2 - 1248\rho) \\
& + D(48\rho^9 + 304\rho^8 + 320\rho^7 - 592\rho^6 + 784\rho^5 + 6720\rho^4 + 320\rho^3 - 4256\rho^2 + 1984\rho) \\
& - 3328\rho^4 + 256\rho^3 + 2048\rho^2 - 1024\rho]
\end{align*}
\]
\[ CL_0 + CW + CM(CL_1 + CR_1) \geq - \left[ D^8(+5\rho^8 + 4\rho^7 + 2\rho^5) \right. \]
\[ + D^7(+26\rho^7 + 38\rho^6 - 4\rho^5 + 6\rho^4) \]
\[ + D^6(+72\rho^6 + 126\rho^5 - 48\rho^4 + 16\rho^3) \]
\[ + D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \]
\[ + D^4(+546\rho^7 - 72\rho^6 + 939\rho^5 + 96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \]
\[ + D^3(-152\rho^4 + 576\rho^3 + 152\rho^2 + 320\rho) \]
\[ + D^2(-1536\rho^3 + 2336\rho^2 - 1248\rho) \]
\[ + D(-592\rho^6 + 784\rho^5 + 672\rho^4 + 320\rho^3 - 4256\rho^2 + 1984\rho) \]
\[ + 256\rho^3 + 2048\rho^2 - 1024\rho \].

\[ CL_0 + CW + CM(CL_1 + CR_1) \geq - \left[ D^8(+5\rho^8 + 4\rho^7 + 2\rho^5) \right. \]
\[ + D^7(+26\rho^7 + 38\rho^6 - 4\rho^5 + 6\rho^4) \]
\[ + D^6(+72\rho^6 + 126\rho^5 - 48\rho^4 + 16\rho^3) \]
\[ + D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \]
\[ + D^4(+546\rho^7 - 72\rho^6 - 939\rho^5 + 96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \]
\[ + D^3(-152\rho^4 + 576\rho^3 + 152\rho^2 + 320\rho) \]
\[ + D^2(-1536\rho^3 + 2336\rho^2 - 1248\rho) \]
\[ + D(+784\rho^5 + 672\rho^4 + 320\rho^3 - 4256\rho^2 + 1984\rho) \]
\[ + 256\rho^3 + 2048\rho^2 - 1024\rho \].
Now we impose the constraint $D > 3$, to obtain

$$\mathcal{C}L_0 + CW + CM(\mathcal{C}L_1 + CR_1)$$

\begin{align*}
\geq & \quad - [D^8(+5\rho^8 + 4\rho^7 + 2\rho^5) \\
& + D^7(+26\rho^7 + 38\rho^6 - 4\rho^5 + 6\rho^4) \\
& + D^6(+72\rho^6 + 126\rho^5 - 48\rho^4 + 16\rho^3) \\
& + D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
& + D^4(+96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \\
& + D^3(+576\rho^3 + 152\rho^2 + 320\rho) \\
& + D^2(-1536\rho^2 + 2336\rho^2 - 1248\rho) \\
& + D(+1984\rho) \\
& + 256\rho^3 + 2048\rho^2 - 1024\rho ] .
\end{align*}

Again eliminating the (positive) highlighted terms we have:

$$\mathcal{C}L_0 + CW + CM(\mathcal{C}L_1 + CR_1)$$

\begin{align*}
\geq & \quad - [D^8(+5\rho^8 + 4\rho^7 + 2\rho^5) \\
& + D^7(+26\rho^7 + 38\rho^6 - 4\rho^5 + 6\rho^4) \\
& + D^6(+72\rho^6 + 126\rho^5 - 48\rho^4 + 16\rho^3) \\
& + D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
& + D^4(+96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \\
& + D^3(+576\rho^3 + 152\rho^2 + 320\rho) \\
& + D^2(+2336\rho^2 - 1248\rho) \\
& + D(+1984\rho) \\
& + 256\rho^3 + 2048\rho^2 - 1024\rho ] .
\end{align*}

When $D > 4$, we obtain

$$\mathcal{C}L_0 + CW + CM(\mathcal{C}L_1 + CR_1)$$

\begin{align*}
\geq & \quad - [D^8(+2\rho^5) \\
& + D^7(-4\rho^5 + 6\rho^4) \\
& + D^6(+16\rho^3) \\
& + D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
& + D^4(+96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \\
& + D^3(+576\rho^3 + 152\rho^2 + 320\rho) \\
& + D^2(+2336\rho^2 - 1248\rho) \\
& + D(+1984\rho) \\
& + 256\rho^3 + 2048\rho^2 - 1024\rho ] .
\end{align*}
\[
\mathcal{CL}_0 + CW + CM(\mathcal{CL}_1 + CR_1) \geq - \left[ D^8(+2\rho^5) \\
+ D^7(-4\rho^5 + 6\rho^4) \\
+ D^6(+16\rho^3) \\
+ D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
+ D^4(+96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \\
+ D^3(+576\rho^3 + 152\rho^2 + 320\rho) \\
+ D^2(+2336\rho^2 - 1248\rho) \\
+ D(+1984\rho) \right].
\]

Therefore,

\[
\mathcal{CL}_0 + CW + CM(\mathcal{CL}_1 + CR_1) \geq - \left[ D^8(+2\rho^5) \\
+ D^7(+6\rho^4) \\
+ D^6(+16\rho^3) \\
+ D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
+ D^4(+96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \\
+ D^3(+576\rho^3 + 152\rho^2 + 320\rho) \\
+ D^2(+2336\rho^2 - 1248\rho) \\
+ D(+1984\rho) \right].
\]

Appendix IV: Proof of Lemma 23

Here we consider the inequality of Lemma 23

\[
\mathcal{CL}_0 + CW + CM(\mathcal{CL}_1 + CR_1) \geq
- \left[ 2D^8\rho^5 + 6D^7\rho^4 + 16D^6\rho^3 \right] - (128\rho^3 - 8\rho^2)D^5
- (106\rho^2 + 32\rho)D^4 + 320D^3\rho + D^2(2336\rho^2 - 851\rho) \right].
\]

which we need to prove true for \( D \geq 5 \). In this case, we note that \( \rho \leq \frac{2}{13} \). We have

\[
\mathcal{CL}_0 + CW + CM(\mathcal{CL}_1 + CR_1) \geq
- \left[ D^8(+2\rho^5) \\
+ D^7(+6\rho^4) \\
+ D^6(+16\rho^3) \\
+ D^5(+44\rho^5 + 338\rho^4 - 128\rho^3 + 8\rho^2) \\
+ D^4(+96\rho^4 + 496\rho^3 - 288\rho^2 - 32\rho) \\
+ D^3(+576\rho^3 + 152\rho^2 + 320\rho) \\
+ D^2(+2336\rho^2 - 851\rho) \right].
\]
so that eliminating the highlighted terms repeatedly,

\[
CL_0 + CW + CM(CL_1 + CR_1) \geq - [D^8(2\rho^5) + D^7(6\rho^4) + D^6(16\rho^3) + D^5(-128\rho^3 + 8\rho^2) + D^4(-155\rho^2 - 32\rho) + D^3(576\rho^3 + 152\rho^2 + 320\rho) + D^2(2336\rho^2 - 851\rho)].
\] (156)

\[
CL_0 + CW + CM(CL_1 + CR_1) \geq - [D^8(2\rho^5) + D^7(6\rho^4) + D^6(16\rho^3) + D^5(-128\rho^3 + 8\rho^2) + D^4(-124\rho^2 - 32\rho) + D^3(576\rho^3 + 320\rho) + D^2(2336\rho^2 - 851\rho)].
\] (157)

Finally we have, (for \(D \geq 5\))

\[
CL_0 + CW + CM(CL_1 + CR_1) \geq - [D^8(2\rho^5) + D^7(6\rho^4) + D^6(16\rho^3) + D^5(-128\rho^3 + 8\rho^2) + D^4(-106\rho^2 - 32\rho) + D^3(320\rho) + D^2(2336\rho^2 - 851\rho)]. \quad \text{Q.E.D}
\] (158)

**Appendix V:**

**Proof of \(P_1 > 0\)**

Here we consider the polynomials \(P_1\) and \(P_2\) described in equations (116) and (117) and prove the assertions concerning them immediately following those equations.

We recall that

\[
[t]P_1 = \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] D^6
\]

\[
+ \left[ -320 \left( \frac{R}{D} \right)^4 - 812 \left( \frac{R}{D} \right)^3 - 100 \left( \frac{R}{D} \right)^2 + 700 \left( \frac{R}{D} \right) + 212 \right] D^5
\]

\[
+ \left[ 851 \left( \frac{R}{D} \right)^4 + 844 \left( \frac{R}{D} \right)^3 - 534 \left( \frac{R}{D} \right)^2 + 1660 \left( \frac{R}{D} \right) + 1931 \right] D^4.
\]

Specifically, we give a detailed proof that \(P_1 > 0\) for \(R/D \geq 3.3\) and \(D \geq 5\). We do this by showing that \(P_1/D^4 \equiv H_1\) and its repeated derivatives with respect to \(D\) are positive at \(D = 5\). The latter is proved by repeated differentiation and evaluation at the endpoint \(R/D = 3.3\) of
the polynomial $H_1|_{D=5}$. In presenting the details we present each of the derivatives with respect
to $D$ and the values of the derivatives with respect to $R/D$ at the endpoint $R/D = 3.3$. The
latter are calculated in the symbolic mathematics package maxima.

$H_1 > 0$ at $D = 5$

$$H_1|_{D=5} = \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] 5^2$$

$$+ \left[ -320 \left( \frac{R}{D} \right)^4 - 812 \left( \frac{R}{D} \right)^3 - 100 \left( \frac{R}{D} \right)^2 + 700 \left( \frac{R}{D} \right) + 212 \right] 5$$

$$+ \left[ 851 \left( \frac{R}{D} \right)^4 + 844 \left( \frac{R}{D} \right)^3 - 534 \left( \frac{R}{D} \right)^2 + 1660 \left( \frac{R}{D} \right) + 1931 \right]$$

$$= \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] 5$$

For $R/D = 3.3$, we obtain $H_1 = 10517.1451$, $H'_1 = 2876.028$, $H''_1 = 359.88$, $H'''_1 = 1543.2$ and $H''''_1 = 1224$. $dH_1/dD > 0$ at $D = 5$

$$\frac{dH_1}{dD}|_{D=5} = \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] 2 \times 5$$

$$+ \left[ -320 \left( \frac{R}{D} \right)^4 - 812 \left( \frac{R}{D} \right)^3 - 100 \left( \frac{R}{D} \right)^2 + 700 \left( \frac{R}{D} \right) + 212 \right]$$

$$= \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] 2$$

For $R/D = 3.3$, we obtain $\frac{dH_1}{dD} = 16765.596$, $(\frac{dH_1}{dD})' = 14422.36$, $(\frac{dH_1}{dD})'' = 7498.4$ and $(\frac{dH_1}{dD})''' = 1848$. $\frac{d^{(2)}H_1}{dD^{(2)}} > 0$ at $D = 5$

$$\frac{d^{(2)}H_1}{dD^{(2)}}|_{D=5} = \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] 2$$

$$= \left[ 32 \left( \frac{R}{D} \right)^4 + 112 \left( \frac{R}{D} \right)^3 + 80 \left( \frac{R}{D} \right)^2 - 96 \left( \frac{R}{D} \right) - 128 \right] 2$$

For $R/D = 3.3$, we obtain $\frac{d^{(2)}H_1}{dD^{(2)}} = 16492.5824$, $(\frac{d^{(2)}H_1}{dD^{(2)}})' = 17381.952$, $(\frac{d^{(2)}H_1}{dD^{(2)}})'' = 15336.32$, $(\frac{d^{(2)}H_1}{dD^{(2)}})''' = 6412.8$ and $(\frac{d^{(2)}H_1}{dD^{(2)}})'''' = 1536$. Given fixed $R/D$, we have proved $P_1$ is positive. Also, we have proved that at $D = 5$, $P_1 > 0$ for $R/D \geq 3.3$. The proof for $P_1 > 0$ is now complete.
Proof of $P_2 > 0$

Now we consider the polynomial $P_2$ defined in equation (117):

$$[t]P_2 = \left[ 2136 \left( \frac{R}{D} \right)^3 - 1272 \left( \frac{R}{D} \right)^2 - 5384 \left( \frac{R}{D} \right) - 56 \right] D^3$$

$$+ \left[ -7608 \left( \frac{R}{D} \right)^2 - 25456 \left( \frac{R}{D} \right) - 15640 \right] D^2$$

$$+ \left[ -28832 \left( \frac{R}{D} \right) - 33952 \right] D - 23760.$$

We need to prove $P_2 > 0$ for $R/D \geq 3.3$ and $D \geq 5$. We use the same technique as for $P_1$.

$P_2 > 0$ at $D = 5$

$$P_2 \big|_{D=5} = \left[ 2136 \left( \frac{R}{D} \right)^3 - 1272 \left( \frac{R}{D} \right)^2 - 5384 \left( \frac{R}{D} \right) - 56 \right] 5^3$$

$$+ \left[ -7608 \left( \frac{R}{D} \right)^2 - 25456 \left( \frac{R}{D} \right) - 15640 \right] 5^2$$

$$+ \left[ -28832 \left( \frac{R}{D} \right) - 33952 \right] 5 - 23760$$

$$= 267,000 \left( \frac{R}{D} \right)^3 - 349200 \left( \frac{R}{D} \right)^2 - 1453560 \left( \frac{R}{D} \right) - 591529 \quad (162)$$

For $R/D = 3.3$, we obtain $P_2 = 404123$, $P'_2 = 4964610$, $P''_2 = 4588200$ and $P'''_2 = 1602000$.

$\frac{dP_2}{dD} > 0$ at $D = 5$

$$\frac{dP_2}{dD} \big|_{D=5} = \left[ 2136 \left( \frac{R}{D} \right)^3 - 1272 \left( \frac{R}{D} \right)^2 - 5384 \left( \frac{R}{D} \right) - 56 \right] 3 \times 5^2$$

$$+ \left[ -7608 \left( \frac{R}{D} \right)^2 - 25456 \left( \frac{R}{D} \right) - 15640 \right] 2 \times 5$$

$$+ \left[ -28832 \left( \frac{R}{D} \right) - 33952 \right]$$

$$= 160200 \left( \frac{R}{D} \right)^3 - 171480 \left( \frac{R}{D} \right)^2 - 687192 \left( \frac{R}{D} \right) - 184360. \quad (163)$$

For $R/D = 3.3$, we obtain $\frac{dP_2}{dD} = 1437596.6$, $(\frac{dP_2}{dD})' = 4101278.808$, $(\frac{dP_2}{dD})'' = 2829000$ and $(\frac{dP_2}{dD})''' = 961200$. 

34
\[
\frac{d^2 P_2}{dD^2} > 0 \text{ at } D = 5
\]

\[
\left. \frac{d^2 H_2}{dD^2} \right|_{D=5} = \left[ 2136 \left( \frac{R}{D} \right)^3 - 1272 \left( \frac{R}{D} \right)^2 - 5384 \left( \frac{R}{D} \right) - 56 \right] 3 \times 2 \times 5 \quad (164)
\]

\[
+ \left[ -7608 \left( \frac{R}{D} \right)^2 - 25456 \left( \frac{R}{D} \right) - 15640 \right] 2
\]

\[
= 64080 \left( \frac{R}{D} \right)^3 - 53736 \left( \frac{R}{D} \right)^2 + 212432 \left( \frac{R}{D} \right) - 32960.
\]

For \( R/D = 3.3 \), we obtain \( \frac{d^2 P_2}{dD^2} = 987592.72 \), \( (\frac{d^2 P_2}{dD^2})' = 1528780 \), \( (\frac{d^2 P_2}{dD^2})'' = 1162032 \), \( (\frac{d^2 P_2}{dD^2})''' = 384480 \).

\[
\frac{d^3 P_2}{dD^3} > 0 \text{ at } D = 5
\]

\[
\left. \frac{d^3 H_2}{dD^3} \right|_{D=5} = \left[ 2136 \left( \frac{R}{D} \right)^3 - 1272 \left( \frac{R}{D} \right)^2 - 5384 \left( \frac{R}{D} \right) - 56 \right] 3 \times 2 \quad (165)
\]

\[
= 6 \left[ 2136 \left( \frac{R}{D} \right)^3 - 1272 \left( \frac{R}{D} \right)^2 + 5384 \left( \frac{R}{D} \right) - 56 \right].
\]

For \( R/D = 3.3 \), we obtain \( \frac{d^3 P_2}{dD^3} = 270516.912 \), \( (\frac{d^3 P_2}{dD^3})' = 44203.2 \), \( (\frac{d^3 P_2}{dD^3})'' = 238492.8 \), and \( (\frac{d^3 P_2}{dD^3})''' = 76896 \).

Given fixed \( R/D \), we have proved \( P_2 \) is positive. Also, we have proved that at \( D = 5 \), \( P_2 > 0 \) for \( R/D \geq 3.3 \). Therefore, the proof for \( P_2 > 0 \) is now complete.

**Appendix VI:**

We show here that \( W \) given by (118) is increasing (and therefore clearly positive), by repeated differentiation with respect to \( D \) and checking for positivity of the derivatives at the endpoint \( D = 14 \). The positivity at this endpoint has been established in maxima.

\[
\frac{dW}{dD} = 6Q_1D^5 + 5Q_2D^4 + 4Q_3D^3 + 3Q_4D^2 + 2Q_5D + Q_6
\]

\[
\frac{d^2W}{dD^2} = 30Q_1D^4 + 20Q_2D^3 + 12Q_3D^2 + 6Q_4D + 2Q_5
\]

\[
\frac{d^3W}{dD^3} = 120Q_1D^3 + 60Q_2D^2 + 24Q_3D + 6Q_4
\]

\[
\frac{d^4W}{dD^4} = 360Q_1D^2 + 120Q_2D + 24Q_3
\]

\[
\frac{d^5W}{dD^5} = 720Q_1D + 120Q_2
\]

\[
\frac{d^6W}{dD^6} = 720Q_1
\]
Appendix VII:

Here we list the derivatives of $W$ with respect to $R$ (see (122)) for positivity at the endpoint $1/1.1$ to establish positivity of $W$ on the interval $[1/1.1, \infty)$.

\[
\frac{dW}{dR^1} = \frac{256}{(1.1)^2} R^7 - \frac{2240}{1.1} R^6 + 4800 R^5 - 682.5 R^4 + 11568 R^3 - 13728 R^2 - 97408 R - 62784
\]

\[
\frac{d^2W}{dR^2} = \frac{1792}{(1.1)^2} R^6 - \frac{13440}{1.1} R^5 + 24000 R^4 - 2730 R^3 + 34704 R^2 - 27456 R - 97408
\]

\[
\frac{d^3W}{dR^3} = \frac{10752}{(1.1)^2} R^5 - \frac{67200}{1.1} R^4 + 96000 R^3 - 8190 R^2 + 69408 R - 27456
\]

\[
\frac{d^4W}{dR^4} = \frac{53760}{(1.1)^2} R^4 - \frac{26880}{1.1} R^3 + 28800 R^2 - 16380 R + 69408
\]

\[
\frac{d^5W}{dR^5} = \frac{215040}{(1.1)^2} R^3 - \frac{806400}{1.1} R^2 + 576000 R - 16380
\]

\[
\frac{d^6W}{dR^6} = \frac{645120}{(1.1)^2} R^2 - \frac{1612800}{1.1} R + 576000
\]

\[
\frac{d^7W}{dR^7} = \frac{1290240}{(1.1)^2} R - \frac{1612800}{1.1}
\]

\[
\frac{d^8W}{dR^8} = \frac{328}{(1.1)^2}
\]

Appendix VIII: Proof for some boundary cases

In this final appendix we consider the remaining values of $D$. So far we have established the result for $D \geq 5$. Here we consider the cases $D = 2, 3, 4$.

Case $D = 2$

We have

\[
\Upsilon(D = 2, N) = \left(\frac{2N - 2}{N + 1}\right) \left(\beta - \beta^4\right) - \left(\frac{2N - 2}{N}\right) \left(1 - \frac{N - 2}{N + 1}\right) \alpha^2
\]

\[
+ \sum_{i=0}^{2} \binom{2N - 2}{N - i} \left[\left(\frac{N - i}{N + 2} - 2\right) \beta^{i+1} - \left(\frac{N - i}{N + 1} - 2\right) \alpha^i\right].
\]

It is sufficient to prove the following expression is larger than or equal to zero:

\[
\Upsilon'(N) = \frac{1}{N^2 + N} \left(\beta - \beta^4\right) - \frac{1}{N^2 + 2N} \left(\frac{3}{N + 1}\right) \alpha^2
\]

\[
- \frac{1}{N^2 - 2N} \left(\frac{N + 4}{N + 2}\right) \beta - \left(\frac{N + 3}{N + 1}\right) \alpha
\]

\[
- \frac{1}{N^2 - 3N + 2} \left(\frac{N + 5}{N + 2}\right) \beta^2 - \left(\frac{N + 4}{N + 1}\right) \alpha^2
\]
Multiplied by \( N(N - 1)(N + 2)^5(N - 2)(N + 1)^3 \), we have

\[
\Upsilon'(N) = N^7 + 7N^6 + 15N^5 - 3N^4 - 60N^3 - 96N^2 - 64N - 16
\]  

(168)

Next we provide a proof that \( \Upsilon'(N) > 0 \) for \( N \geq 2 \). We do this by showing that \( \Upsilon'(N) \) and its repeated derivatives with respect to \( N \) are positive at \( N = 2 \). Here we list the derivatives of \( \Upsilon'(N) \).

\[
\frac{d^1 W}{dR^1} = 7N^6 + 42N^5 + 75N^4 - 12N^3 - 180N^2 - 192N - 64
\]

\[
\frac{d^2 W}{dR^2} = 42N^5 + 210N^4 + 300N^3 - 36N^2 - 360N - 192
\]

\[
\frac{d^3 W}{dR^3} = 210N^4 + 840N^3 + 900N^2 - 72N - 360
\]

\[
\frac{d^4 W}{dR^4} = 840N^3 + 2520N^2 + 1800N - 72
\]

\[
\frac{d^5 W}{dR^5} = 2520N^2 + 5040N + 1800
\]

**For** \( D = 3 \)

Multiplying \( CL_0 + CW + CM(CL_1 + CR_1) \) in (108) by \((2R - 1)(R + 7)^5(R + 5)^4\), we have

\[
CL_0 + CW + CM(CL_1 + CR_1) =
24R^7 + 1192R^6 + 24408R^5 + 269864R^4 + 1750664R^3
+ 6689080R^2 + 13971976R + 12325880 > 0.
\]

(169)

**For** \( D = 4 \)

Multiplying \( CL_0 + CW + CM(CL_1 + CR_1) \) in (108) by \((R + 8)^5(R + 6)^4\), we have

\[
CL_0 + CW + CM(CL_1 + CR_1) =
(10R^8 + 394R^7 + 6464R^6 + 56548R^5 + 278040R^4 + 721104R^3
+ 680224R^2 - 615488R - 1127296) \left( \frac{R + 3}{R - 1} \right)^2
- 10R^8 - 494R^7 - 10760R^6 - 135156R^5 - 1071960R^4
- 5500304R^3 - 17828768R^2 - 33352896R - 27534976.
\]

Since \( \left( \frac{R + 3}{R - 1} \right)^2 > \left( 1 + \frac{10}{(R - 1)} + \frac{30}{(R - 1)^2} \right) \), we have

\[
CL_0 + CW + CM(CL_1 + CR_1) >
(10R^8 + 394R^7 + 6464R^6 + 56548R^5 + 278040R^4 + 721104R^3
+ 680224R^2 - 615488R - 1127296) \left( 1 + \frac{10}{(R - 1)} + \frac{30}{(R - 1)^2} \right)
- 10R^8 - 494R^7 - 10760R^6 - 135156R^5 - 1071960R^4
- 5500304R^3 - 17828768R^2 - 33352896R - 27534976.
\]

(170)

(171)
Multiplied by \((R - 1)^2\), we obtain
\[
CL_0 + CW + CM(CL_1 + CR_1) > \\
44R^8 + 2404R^7 + 53760R^6 + 641392R^5 + 4387776R^4 \\
+ 16773824R^3 + 29575552R^2 - 226560 - 51208192 \\
> 44 \times 1^8 + 2404 \times 1^7 + 53760 \times 1^6 + 641392 \times 1^5 + 4387776 \times 1^4 \\
+ 16773824 \times 1^3 + (29575552 \times 1 - 226560) \times 1 - 51208192 \\
= 0.
\]

This completes all of the special cases of \(D\).

References


