

# Stability of Nonlinear Systems

By

**Guanrong Chen**

Department of Electronic Engineering  
City University of Hong Kong  
Kowloon, Hong Kong SAR, China  
[gchen@ee.cityu.edu.hk](mailto:gchen@ee.cityu.edu.hk)

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## STABILITY OF NONLINEAR SYSTEMS

GUANRONG CHEN  
City University of Hong Kong  
Kowloon, Hong Kong, China

### 1. INTRODUCTION

A *nonlinear system* refers to a set of nonlinear equations (algebraic, difference, differential, integral, functional, or abstract operator equations, or a combination of some of these) used to describe a physical device or process that otherwise cannot be clearly defined by a set of linear equations of any kind. *Dynamical system* is used as a synonym for mathematical or physical system when the describing equations represent evolution of a solution with time and, sometimes, with control inputs and/or other varying parameters as well.

The theory of nonlinear dynamical systems, or *nonlinear control systems* if control inputs are involved, has been greatly advanced since the nineteenth century. Today, nonlinear control systems are used to describe a great variety of scientific and engineering phenomena ranging from social, life, and physical sciences to engineering and technology. This theory has been applied to a broad spectrum of problems in physics, chemistry, mathematics, biology, medicine, economics, and various engineering disciplines.

*Stability theory* plays a central role in system engineering, especially in the field of control systems and automation, with regard to both dynamics and control. Stability of a dynamical system, with or without control and disturbance inputs, is a fundamental requirement for its practical value, particularly in most real-world applications. Roughly speaking, stability means that the system outputs and its internal signals are bounded within admissible limits (the so-called bounded-input/bounded-output stability) or, sometimes more strictly, the system outputs tend to an equilibrium state of interest (the so-called asymptotic stability). Conceptually, there are different kinds of stabilities, among which three basic notions are the main concerns in nonlinear dynamics and control systems: the stability of a system with respect to its equilibria, the orbital stability of a system output trajectory, and the structural stability of a system itself.

The basic concept of stability emerged from the study of an equilibrium state of a mechanical system, dated back to as early as 1644, when E. Torricelli studied the equilibrium of a rigid body under the natural force of gravity. The classical stability theorem of G. Lagrange, formulated in 1788, is perhaps the best known result about stability of conservative mechanical systems, which states that if the potential energy of a conservative system, currently at the position of an isolated equilibrium and perhaps subject to some simple constraints, has a minimum, then this equi-

librium position of the system is stable [23]. The evolution of the fundamental concepts of system and trajectory stabilities then went through a long history, with many fruitful advances and developments, until the celebrated Ph.D. thesis of A. M. Lyapunov, *The General Problem of Motion Stability*, finished in 1892 [21]. This monograph is so fundamental that its ideas and techniques are virtually leading all kinds of basic research and applications regarding stabilities of dynamical systems today. In fact, not only dynamical behavior analysis in modern physics but also controllers design in engineering systems depend on the principles of Lyapunov's stability theory. This article is devoted to a brief description of the basic stability theory, criteria, and methodologies of Lyapunov, as well as a few related important stability concepts, for nonlinear dynamical systems.

### 2. NONLINEAR SYSTEM PRELIMINARIES

#### 2.1. Nonlinear Control Systems

A continuous-time nonlinear control system is generally described by a differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \mathbf{u}), \quad t \in [t_0, \infty) \quad (1)$$

where  $\mathbf{x} = \mathbf{x}(t)$  is the *state* of the system belonging to a (usually bounded) region  $\Omega_{\mathbf{x}} \subset \mathbf{R}^n$ ,  $\mathbf{u}$  is the *control* input vector belonging to another (usually bounded) region  $\Omega_{\mathbf{u}} \subset \mathbf{R}^m$  (often,  $m \leq n$ ), and  $\mathbf{f}$  is a Lipschitz or continuously differentiable nonlinear function, so that the system has a unique solution for each admissible control input and suitable initial condition  $\mathbf{x}(t_0) = \mathbf{x}_0 \in \Omega_{\mathbf{x}}$ . To indicate the time evolution and the dependence on the initial state  $\mathbf{x}_0$ , the *trajectory* (or *orbit*) of a system state  $\mathbf{x}(t)$  is sometimes denoted as  $\varphi_t(\mathbf{x}_0)$ .

In control system (1), the initial time used is  $t_0 \geq 0$ , unless otherwise indicated. The entire space  $\mathbf{R}^n$ , to which the system states belong, is called the *state space*. Associated with the control system (1), there usually is an *observation* or *measurement* equation

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, t; \mathbf{u}) \quad (2)$$

where  $\mathbf{y} = \mathbf{y}(t) \in \mathbf{R}^\ell$  is the *output* of the system,  $1 \leq \ell \leq n$ , and  $\mathbf{g}$  is a continuous or smooth nonlinear function. When both  $n, \ell > 1$ , the system is called a *multiinput/multioutput* (MIMO) system; while if  $n = \ell = 1$ , it is called a *single-input/single-output* (SISO) system. MISO and SIMO systems are similarly defined.

In the discrete-time setting, a nonlinear control system is described by a difference equation of the form

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, k; \mathbf{u}_k) \\ \mathbf{y}_k = \mathbf{g}(\mathbf{x}_k, k; \mathbf{u}_k), \end{cases} \quad k = 0, 1, \dots \quad (3)$$

where all notations are similarly defined. This article usually discusses only the control system (1), or the first equation of (3). In this case, the system state  $\mathbf{x}$  is also considered as the system output for simplicity.

A special case of system (1), with or without control, is said to be *autonomous* if the time variable  $t$  does not appear separately (independently) from the state vector in the system function  $\mathbf{f}$ . For example, with a state feedback control  $\mathbf{u}(t) = \mathbf{h}(\mathbf{x}(t))$ , this often is the situation. In this case, the system is usually written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{4}$$

Otherwise, as (1) stands, the system is said to be *nonautonomous*. The same terminology may be applied in the same way to discrete-time systems, although they may have different characteristics.

An *equilibrium*, or *fixed point*, of system (4), if it exists, is a solution,  $\mathbf{x}^*$ , of the algebraic equation

$$\mathbf{f}(\mathbf{x}^*) = 0 \tag{5}$$

It then follows from (4) and (5) that  $\dot{\mathbf{x}}^* = 0$ , which means that an equilibrium of a system must be a constant state. For the discrete-time case, an equilibrium of system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k), \quad k = 0, 1, \dots \tag{6}$$

is a solution, if it exists, of equation

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*) \tag{7}$$

An equilibrium is *stable* if some nearby trajectories of the system states, starting from various initial states, approach it; it is *unstable* if some nearby trajectories move away from it. The concept of system stability with respect to an equilibrium will be precisely introduced in Section 3.

A control system is *deterministic* if there is a unique consequence to every change of the system parameters or initial states. It is *random* or *stochastic*, if there is more than one possible consequence for a change in its parameters or initial states according to some probability distribution [6]. This article deals only with deterministic systems.

### 2.2. Hyperbolic Equilibria and Their Manifolds

Consider the autonomous system (4). The Jacobian of this system is defined by

$$J(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{8}$$

Clearly, this is a matrix-valued function of time. If the Jacobian is evaluated at a constant state, say,  $\mathbf{x}^*$  or  $\mathbf{x}_0$ , then it becomes a constant matrix determined by  $\mathbf{f}$  and  $\mathbf{x}^*$  or  $\mathbf{x}_0$ .

An equilibrium  $\mathbf{x}^*$  of system (4) is said to be *hyperbolic* if all eigenvalues of the system Jacobian, evaluated at this equilibrium, have nonzero real parts.

For a  $p$ -periodic solution of system (4),  $\tilde{\mathbf{x}}(t)$ , with a fundamental period  $p > 0$ , let  $J(\tilde{\mathbf{x}}(t))$  be its Jacobian evaluated

at  $\tilde{\mathbf{x}}(t)$ . Then this Jacobian is also  $p$ -periodic:

$$J(\tilde{\mathbf{x}}(t+p)) = J(\tilde{\mathbf{x}}(t)) \quad \text{for all } t \in [t_0, \infty)$$

In this case, there always exist a  $p$ -periodic nonsingular matrix  $M(t)$  and a constant matrix  $Q$  such that the fundamental solution matrix associated with the Jacobian  $J(\tilde{\mathbf{x}}(t))$  is given by [4]

$$\Phi(t) = M(t)e^{tQ}$$

Here, the fundamental matrix  $\Phi(t)$  consists of, as its columns,  $n$  linearly independent solution vectors of the linear equation  $\dot{\mathbf{x}} = J(\tilde{\mathbf{x}}(t))\mathbf{x}$ , with  $\mathbf{x}(t_0) = \mathbf{x}_0$ .

In the preceding, the eigenvalues of the constant matrix  $e^{pQ}$  are called the *Floquet multipliers* of the Jacobian. The  $p$ -periodic solution  $\tilde{\mathbf{x}}(t)$  is called a *hyperbolic periodic orbit* of the system if all its corresponding Floquet multipliers have nonzero real parts.

Next, let  $\mathcal{D}$  be a neighborhood of an equilibrium,  $\mathbf{x}^*$ , of the autonomous system (4). A *local stable* and *local unstable manifold* of  $\mathbf{x}^*$  is defined by

$$W_{\text{loc}}^s(\mathbf{x}^*) = \{\mathbf{x} \in \mathcal{D} \mid \varphi_t(\mathbf{x}) \in \mathcal{D} \ \forall t \geq t_0 \text{ and } \varphi_t(\mathbf{x}) \rightarrow \mathbf{x}^* \text{ as } t \rightarrow \infty\} \tag{9}$$

and

$$W_{\text{loc}}^u(\mathbf{x}^*) = \{\mathbf{x} \in \mathcal{D} \mid \varphi_t(\mathbf{x}) \in \mathcal{D} \ \forall t \leq t_0 \text{ and } \varphi_t(\mathbf{x}) \rightarrow \mathbf{x}^* \text{ as } t \rightarrow -\infty\} \tag{10}$$

respectively. Furthermore, a *stable* and *unstable manifold* of  $\mathbf{x}^*$  is defined by

$$W^s(\mathbf{x}^*) = \{\mathbf{x} \in \mathcal{D} \mid \varphi_t(\mathbf{x}) \cap W_{\text{loc}}^s(\mathbf{x}^*) \neq \emptyset\} \tag{11}$$

and

$$W^u(\mathbf{x}^*) = \{\mathbf{x} \in \mathcal{D} \mid \varphi_t(\mathbf{x}) \cap W_{\text{loc}}^u(\mathbf{x}^*) \neq \emptyset\} \tag{12}$$

respectively, where  $\emptyset$  denotes the empty set. For example, the autonomous system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x(1 - x^2) \end{cases}$$

has a hyperbolic equilibrium  $(x^*, y^*) = (0, 0)$ . The local stable and unstable manifolds of this equilibrium are illustrated by Fig. 1a, and the corresponding stable and unstable manifolds are visualized by Fig. 1b.

A hyperbolic equilibrium only has stable and/or unstable manifolds since its associated Jacobian has only stable and/or unstable eigenvalues. The dynamics of an autonomous system in a neighborhood of a hyperbolic equilibrium is quite simple—it has either stable (convergent) or unstable (divergent) properties. Therefore, complex dynamical behaviors such as chaos are seldom associated with isolated hyperbolic equilibria or isolated hyperbolic

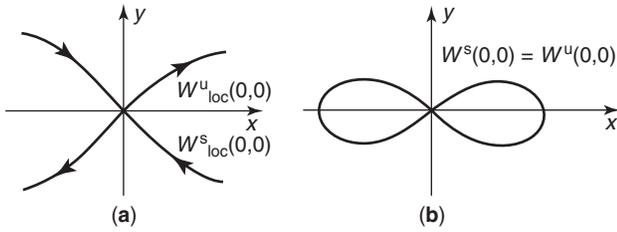


Figure 1. Stable and unstable manifolds.

periodic orbits [7,11,15] (see Theorem 16, below); they generally are confined within the *center manifold*,  $W^c(x^*)$ , where  $\dim(W^s) + \dim(W^c) + \dim(W^u) = n$ .

2.3. Open-Loop and Closed-Loop Systems

Let  $S$  be an MIMO system, which can be linear or nonlinear, continuous-time or discrete-time, deterministic or stochastic, or any well-defined input–output map. Let  $U$  and  $Y$  be the sets (sometimes, spaces) of the admissible input and corresponding output signals, respectively, both defined on the time domain  $\mathcal{D} = [a, b]$ ,  $-\infty \leq a < b \leq \infty$  (for control systems, usually  $a = t_0 = 0$  and  $b = \infty$ ). This simple relation is described by an open-loop map

$$S : \mathbf{u} \rightarrow \mathbf{y} \text{ or } \mathbf{y}(t) = S(\mathbf{u}(t)) \tag{13}$$

and its block diagram is shown in Fig. 2. Actually, every control system described by a differential or difference equation can be viewed as a map in this form. But, in such a situation, the map  $S$  can be implicitly defined only via the equation and initial conditions.

In the control system (1) or (3), if the control inputs are functions of the state vectors,  $\mathbf{u} = \mathbf{h}(\mathbf{x}; t)$ , then the control system can be implemented via a closed-loop configuration. A typical closed-loop system is shown in Fig. 3, where usually  $S_1$  is the plant (described by  $\mathbf{f}$ ) and  $S_2$  is the controller (described by  $\mathbf{h}$ ); yet they can be reversed.

2.4. Norms of Functions and Operators

This article deals only with finite-dimensional systems. For an  $n$ -dimensional vector-valued functions,  $\mathbf{x}(t) = [x_1(t) \dots x_n(t)]^T$  where superscript “T” denotes transpose, let  $\|\cdot\|$  and  $\|\cdot\|_p$  denote its *Euclidean norm* and  *$L_p$  norm*, defined respectively by the “length”

$$\|\mathbf{x}(t)\| = \sqrt{x_1^2(t) + \dots + x_n^2(t)}$$

and

$$\|\mathbf{x}\|_p = \left( \int_a^b \|\mathbf{x}(t)\|^p dt \right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|\mathbf{x}\|_\infty = \text{ess sup}_{\substack{a \leq t \leq b \\ 1 \leq i \leq n}} |x_i(t)|$$

Here, a few remarks are in order:

1. The term “ess sup” means “essential supremum” (i.e., the supremum except perhaps over a set of



Figure 2. The block diagram of an open-loop system.

measure zero). For a piecewise continuous function  $f(t)$ , actually

$$\text{ess sup}_{a \leq t \leq b} |f(t)| = \sup_{a \leq t \leq b} |f(t)|$$

2. The main difference between the “sup” and the “max” is that  $\max |f(t)|$  is attainable but  $\sup |f(t)|$  is not. For example,  $\max_{0 \leq t < \infty} |\sin(t)| = 1$  but  $\sup_{0 \leq t < \infty} |1 - e^{-t}| = 1$ .
3. The difference between the Euclidean norm and the  $L_p$  norms is that the former is a function of time but the latter are all constants.
4. For a finite-dimensional vector  $\mathbf{x}(t)$ , with  $n < \infty$ , all the  $L_p$  norms are equivalent in the sense that for any  $p, q \in [1, \infty]$ , there exist two positive constants  $\alpha$  and  $\beta$  such that

$$\alpha \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq \beta \|\mathbf{x}\|_p$$

For the input–output map in Eq. (13), the so-called operator norm of the map  $S$  is defined to be the maximum gain from all possible inputs over the domain of the map to their corresponding outputs. More precisely, the *operator norm* of the map  $S$  in (13) is defined by

$$\|S\| = \sup_{\substack{\mathbf{u}_1, \mathbf{u}_2 \in U \\ \mathbf{u}_1 \neq \mathbf{u}_2}} \frac{\|\mathbf{y}_1 - \mathbf{y}_2\|_Y}{\|\mathbf{u}_1 - \mathbf{u}_2\|_U} \tag{14}$$

where  $\mathbf{y}_i = S(\mathbf{u}_i) \in Y$ ,  $i = 1, 2$ , and  $\|\cdot\|_U$  and  $\|\cdot\|_Y$  are the norms of the functions defined on the input–output sets (or spaces)  $U$  and  $Y$ , respectively.

3. LYAPUNOV, ORBITAL, AND STRUCTURAL STABILITIES

Three different types of stabilities, namely, the Lyapunov stability of a system with respect to its equilibria, the orbital stability of a system output trajectory, and the structural stability of a system itself, are of fundamental importance in the studies of nonlinear dynamics and control systems.

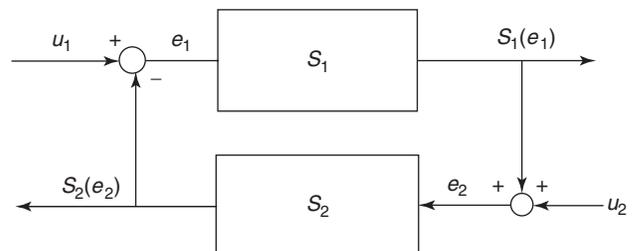


Figure 3. A typical closed-loop control system.

Roughly speaking, the Lyapunov stability of a system with respect to its equilibrium of interest is about the behavior of the system outputs toward the equilibrium state—wandering nearby and around the equilibrium (stability in the sense of Lyapunov) or gradually approaches it (asymptotic stability); the orbital stability of a system output is the resistance of the trajectory to small perturbations; the structural stability of a system is the resistance of the system structure against small perturbations [3,13,16,17,19,20,23,25,26,29,35]. These three basic types of stabilities are introduced in this section, for dynamical systems without explicitly involving control inputs.

Consider the general nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{15}$$

where the control input  $\mathbf{u}(t) = \mathbf{h}(\mathbf{x}(t), t)$ , if it exists [see system (1)], has been combined into the system function  $\mathbf{f}$  for simplicity of discussion. Without loss of generality, assume that the origin  $\mathbf{x} = 0$  is the system equilibrium of interest. Lyapunov stability theory concerns various stabilities of the system orbits with respect to this equilibrium. When another equilibrium is discussed, the new equilibrium is first shifted to zero by a change of variables, and then the transformed system is studied in the same way.

### 3.1. Stability in the Sense of Lyapunov

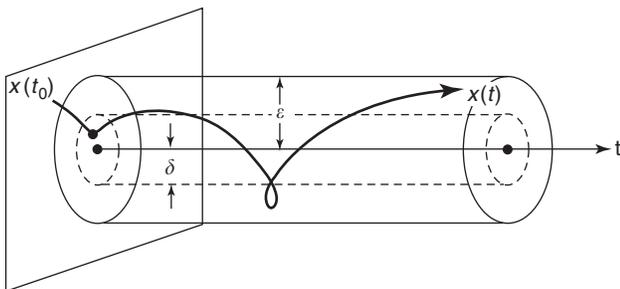
System (15) is said to be *stable in the sense of Lyapunov* with respect to the equilibrium  $\mathbf{x}^* = 0$ , if for any  $\varepsilon > 0$  and any initial time  $t_0 \geq 0$ , there exists a constant,  $\delta = \delta(\varepsilon, t_0) > 0$ , such that

$$\|\mathbf{x}(t_0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \varepsilon \quad \text{for all } t \geq t_0 \tag{16}$$

This stability is illustrated by Fig. 4.

It should be emphasized that the constant  $\delta$  generally depends on both  $\varepsilon$  and  $t_0$ . It is particularly important to point out that, unlike autonomous systems, one cannot simply fix the initial time  $t_0 = 0$  for a nonautonomous system in a general discussion of its stability. For example, consider the following linear time-varying system with a discontinuous coefficient:

$$\dot{x}(t) = \frac{1}{1-t}x(t), \quad x(t_0) = x_0$$



**Figure 4.** Geometric meaning of stability in the sense of Lyapunov.

It has an explicit solution

$$x(t) = x_0 \frac{1-t_0}{1-t}, \quad 0 \leq t_0 \leq t < \infty$$

which is stable in the sense of Lyapunov about the equilibrium  $x^* = 0$  over the entire time domain  $[0, \infty)$  if and only if  $t_0 = 1$ . This shows that the initial time,  $t_0$ , does play an important role in the stability of a nonautonomous system.

The above-defined stability, in the sense of Lyapunov, is said to be *uniform* with respect to the initial time, if the existing constant  $\delta = \delta(\varepsilon)$  is indeed independent of  $t_0$  over the entire time interval  $[0, \infty)$ . According to the discussion above, uniform stability is defined only for nonautonomous systems since it is not needed for autonomous systems (for which it is always uniform with respect to the initial time).

### 3.2. Asymptotic and Exponential Stabilities

System (15) is said to be *asymptotically stable* about its equilibrium  $\mathbf{x}^* = 0$ , if it is stable in the sense of Lyapunov and, furthermore, there exists a constant,  $\delta = \delta(t_0) > 0$ , such that

$$\|\mathbf{x}(t_0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty \tag{17}$$

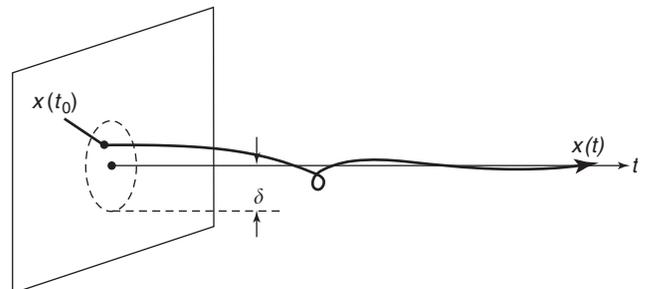
This stability is visualized by Fig. 5.

The asymptotic stability is said to be *uniform* if the existing constant  $\delta$  is independent of  $t_0$  over  $[0, \infty)$ , and is said to be *global* if the convergence,  $\|\mathbf{x}\| \rightarrow 0$ , is independent of the initial state  $\mathbf{x}(t_0)$  over the entire spatial domain on which the system is defined (e.g., when  $\delta = \infty$ ). If, furthermore

$$\|\mathbf{x}(t_0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| \leq ce^{-\sigma t} \tag{18}$$

for two positive constants  $c$  and  $\sigma$ , then the equilibrium is said to be *exponentially stable*. The exponential stability is visualized by Fig. 6.

Clearly, exponential stability implies asymptotic stability, and asymptotic stability implies the stability in the sense of Lyapunov, but the reverse need not be true. For illustration, if a system has output trajectory  $x_1(t) = x_0 \sin(t)$ , then it is stable in the sense of Lyapunov about 0, but is not asymptotically stable; a system with output



**Figure 5.** Geometric meaning of the asymptotic stability.

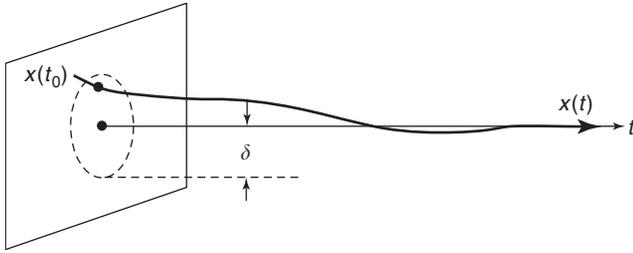


Figure 6. Geometric meaning of the exponential stability.

trajectory  $x_2(t) = x_0(1+t-t_0)^{-1}$  is asymptotically stable (so also is stable in the sense of Lyapunov) if  $t_0 < 1$  but is not exponentially stable about 0; however, a system with  $x_3(t) = x_0e^{-t}$  is exponentially stable (hence, is both asymptotically stable and stable in the sense of Lyapunov).

### 3.3. Orbital Stability

The orbital stability differs from the Lyapunov stabilities in that it concerns with the stability of a system output (or state) trajectory under small external perturbations.

Let  $\varphi_t(\mathbf{x}_0)$  be a  $p$ -periodic solution,  $p > 0$ , of the autonomous system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \quad (19)$$

and let  $\Gamma$  represent the closed orbit of  $\varphi_t(\mathbf{x}_0)$  in the state space, namely,

$$\Gamma = \{\mathbf{y} \mid \mathbf{y} = \varphi_t(\mathbf{x}_0), \quad 0 \leq t < p\}$$

If, for any  $\varepsilon > 0$ , there exists a constant  $\delta = \delta(\varepsilon) > 0$  such that for any  $\mathbf{x}_0$  satisfying

$$d(\mathbf{x}_0, \Gamma) := \inf_{\mathbf{y} \in \Gamma} \|\mathbf{x}_0 - \mathbf{y}\| < \delta$$

the solution of the system,  $\varphi_t(\mathbf{x}_0)$ , satisfies

$$d(\varphi_t(\mathbf{x}_0), \Gamma) < \varepsilon, \quad \text{for all } t \geq t_0$$

then this  $p$ -periodic solution trajectory,  $\varphi_t(\mathbf{x}_0)$ , is said to be *orbitally stable*.

Orbital stability is visualized by Fig. 7. For a simple example, a stable periodic solution, particularly a stable equilibrium of a system is orbitally stable. This is because all nearby trajectories approach it and, as such, it becomes a nearby orbit after a small perturbation and so will move back to its original position (or stay nearby). On the contrary, unstable and semistable (saddle-type of) periodic orbits are orbitally unstable.

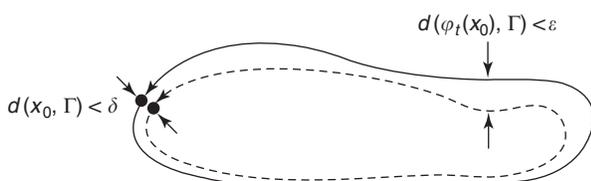


Figure 7. Geometric meaning of the orbital stability.

A more precise concept of orbital stability is given in the sense of Zhukovskij [37].

A solution  $\varphi_t(\mathbf{x}_0)$  of system (19) is said to be stable in the sense of Zhukovskij, if for any  $\varepsilon > 0$ , there exists a  $\delta = \delta(\varepsilon) > 0$  such that for any  $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$ , a ball of radius  $\delta$  centered at  $\mathbf{x}_0$ , there exist two functions,  $\tau_1 = \tau_1(t)$  and  $\tau_2 = \tau_2(t)$ , satisfying

$$\|\mathbf{x}_{\tau_1}(\mathbf{x}_0) - \mathbf{x}_{\tau_2}(\mathbf{x}_0)\| < \varepsilon$$

for all  $t \geq t_0$ , where  $\tau_1$  and  $\tau_2$  are homeomorphisms (i.e., a continuous map whose inverse exists and is also continuous) from  $[0, \infty)$  to  $[0, \infty)$  with  $\tau_1(0) = \tau_2(0) = 0$ .

Furthermore, a Zhukovskij stable solution  $\varphi_t(\mathbf{x}_0)$  of system (19) is said to be asymptotically stable in the sense of Zhukovskij, if for any  $\varepsilon > 0$ , there exists a  $\delta = \delta(\varepsilon) > 0$  such that for any  $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$ , a ball of radius  $\delta$  centered at  $\mathbf{x}_0$ , there exist two functions,  $\tau_1 = \tau_1(t)$  and  $\tau_2 = \tau_2(t)$ , satisfying

$$\|\mathbf{x}_{\tau_1}(\mathbf{x}_0) - \mathbf{x}_{\tau_2}(\mathbf{x}_0)\| \rightarrow 0$$

as  $t \rightarrow \infty$ , where  $\tau_1$  and  $\tau_2$  are homeomorphisms from  $[0, \infty)$  to  $[0, \infty)$  with  $\tau_1(0) = \tau_2(0) = 0$ .

It can be verified that (asymptotic) Zhukovskij stability implies (asymptotic) Lyapunov stability about an equilibrium. However, the converse may not be true. Moreover, these two types of stabilities are equivalent if the orbit  $\varphi_t(\mathbf{x}_0)$  is an equilibrium of the system.

### 3.4. Structural Stability

Two systems are said to be *topologically orbitally equivalent*, if there exists a homeomorphism that transforms the family of trajectories of the first system to that of the second while preserving their motion directions. Roughly, this means that the geometrical pictures of the orbit families of the two systems are similar (no one has extra knots, sharp corners, bifurcating branches, etc.). For instance, systems  $\dot{x} = x$  and  $\dot{x} = 2x$  are topologically orbitally equivalent, but are not so between  $\dot{x} = x$  and  $\dot{x} = \sqrt{x}$ . These three system trajectories are shown in Fig. 8.

Return to the autonomous system (19). If the dynamics of the system in the state space changes radically, for example by the appearance of a new equilibrium or a new periodic orbit, due to small external perturbations, then the system is considered to be structurally unstable.

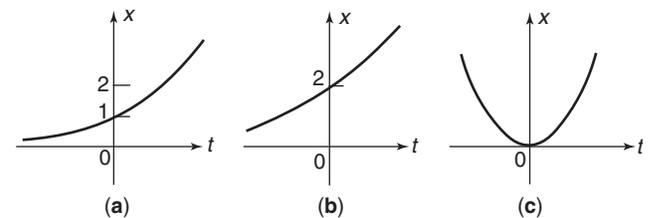


Figure 8. Trajectories of three systems for comparison: (a)  $\dot{x} = x$ ; (b)  $\dot{x} = 2x$ ; (c)  $\dot{x} = \sqrt{x}$ .

To be more precise, consider the following set of functions:

$$\mathcal{S} = \left\{ \mathbf{g}(\mathbf{x}) \mid \|\mathbf{g}(\mathbf{x})\| < \infty, \left\| \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \right\| < \infty \text{ for all } \mathbf{x} \in \mathbf{R}^n \right\}$$

If, for any  $g \in \mathcal{S}$ , there exists an  $\varepsilon > 0$  such that the orbits of the two systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ and } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x})$$

are topologically orbitally equivalent, then the autonomous system (19), namely, the first (unperturbed) system above, is said to be *structurally stable*.

For example,  $\dot{x} = x$  is structurally stable but  $\dot{x} = x^2$  is not, in a neighborhood of the origin. This is because when the second system is slightly perturbed, to become, say,  $\dot{x} = x^2 + \varepsilon$ , where  $\varepsilon > 0$ , then the resulting system has two equilibria,  $x_1^* = \sqrt{\varepsilon}$  and  $x_2^* = -\sqrt{\varepsilon}$ , which has more numbers of equilibria than the original system that possesses only one,  $x^* = 0$ .

#### 4. VARIOUS STABILITY THEOREMS

Consider the general nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{20}$$

where  $\mathbf{f} : \mathcal{D} \times [0, \infty) \rightarrow \mathbf{R}^n$  is continuously differentiable in a neighborhood of the origin,  $\mathcal{D} \subseteq \mathbf{R}^n$ , with a given initial state  $\mathbf{x}_0 \in \mathcal{D}$ . Again, without loss of generality, assume that  $\mathbf{x}^* = 0$  is a system equilibrium of interest.

##### 4.1. Lyapunov Stability Theorems

First, for the autonomous system (19), an important special case of (20), with a continuously differentiable  $\mathbf{f} : \mathcal{D} \rightarrow \mathbf{R}^n$ , the following criterion of stability, called the *first (or indirect) method of Lyapunov*, is very convenient to use.

**Theorem 1 (First Method of Lyapunov: For Continuous-Time Autonomous Systems).** Let  $J = [\partial \mathbf{f} / \partial \mathbf{x}]_{\mathbf{x} = \mathbf{x}^* = 0}$  be the system Jacobian evaluated at the zero equilibrium of system (19). If all the eigenvalues of  $J$  have a negative real part, then the system is asymptotically stable about  $\mathbf{x}^* = 0$ .

First, note that this and the following Lyapunov theorems apply to linear systems as well, for linear systems are merely a special case of nonlinear systems. When  $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ , the linear time-invariant system  $\dot{\mathbf{x}} = A\mathbf{x}$  has the only equilibrium  $\mathbf{x}^* = 0$ . If  $A$  has all eigenvalues with negative real parts, Theorem 1 implies that the system is asymptotically stable about its equilibrium since the system Jacobian is simply  $J = A$ . This is consistent with the familiar linear stability results.

Note also that the region of asymptotic stability given by Theorem 1 is local, which can be quite large for some nonlinear systems but may be very small for some others. However, there is no general criterion for determining the

boundaries of such local stability regions when this and the following Lyapunov methods are applied.

Moreover, it is important to note that this theorem cannot be applied to a *general* nonautonomous system, since for general nonautonomous systems this theorem is neither necessary nor sufficient [36]. A simple counterexample is the following linear time-varying system [17,34]:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 + 1.5 \cos^2(t) & 1 - 1.5 \sin(t) \cos(t) \\ -1 - 1.5 \sin(t) \cos(t) & -1 + 1.5 \sin^2(t) \end{bmatrix} \mathbf{x}(t)$$

This system has eigenvalues  $\lambda_{1,2} = -0.25 \pm j0.25\sqrt{7}$ , both having negative real parts and being independent of the time variable  $t$ . If this theorem is used to judge the system, the conclusion would be that the system is asymptotically stable about its equilibrium 0. However, the solution of this system is

$$\mathbf{x}(t) = \begin{bmatrix} e^{0.5t} \cos(t) & e^{-t} \sin(t) \\ -e^{0.5t} \sin(t) & e^{-t} \cos(t) \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

which is unstable, for any initial conditions with a bounded and nonzero value of  $x_1(t_0)$ , no matter how small this initial value is. This example shows that by using the Lyapunov first method *alone* to determine the stability of a general time-varying system, the conclusion can be wrong.

This type of counterexamples can be easily found [35]. On the one hand, this demonstrates the necessity of other general criteria for asymptotic stability of nonautonomous systems. On the other hand, however, a word of caution is that this types of counterexamples do not completely rule out the possibility of applying the first method of Lyapunov to *some* special nonautonomous systems in case studies. The reason is that there is no theorem saying that “the Lyapunov first method cannot be applied to *all* nonautonomous systems.” Because of the complexity of nonlinear dynamical systems, they often have to be studied class by class, or even case by case. It has been widely experienced that the first method of Lyapunov does work for some, perhaps not too many, *specific* nonautonomous systems in case studies (e.g., in the study of some chaotic systems [7]; see also Theorem 18, below). The point is that one has to be very careful when this method is applied to a particular nonautonomous system; the stability conclusion must be verified by some other means at the same time.

Here, it is emphasized that a rigorous approach for asymptotic stability analysis of general nonautonomous systems is provided by the second method of Lyapunov, for which the following set of class  $\mathcal{K}$  functions are useful:

$$\mathcal{K} = \{g(t) : g(t_0) = 0, \quad g(t) > 0 \text{ if } t > t_0,$$

$$g(t) \text{ is continuous and nondecreasing on } [t_0, \infty)\}$$

**Theorem 2 (Second Method of Lyapunov: For Continuous-Time Nonautonomous Systems).** The system (20) is globally (over the entire domain  $\mathcal{D}$ ), uniformly (with respect to the initial time over the entire time interval  $[t_0, \infty)$ ), and asymptotically stable about its zero equilibrium, if there

exist a scalar-valued function,  $V(\mathbf{x}, t)$ , defined on  $\mathcal{D} \times [t_0, \infty)$ , and three functions  $\alpha(\cdot), \beta(\cdot), \gamma(\cdot) \in \mathcal{K}$ , such that

- (a)  $V(0, t_0) = 0$ .
- (b)  $V(\mathbf{x}, t) > 0$  for all  $\mathbf{x} \neq 0$  in  $\mathcal{D}$  and all  $t \geq t_0$ .
- (c)  $\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$  for all  $t \geq t_0$ .
- (d)  $\dot{V}(\mathbf{x}, t) \leq -\gamma(\|\mathbf{x}\|) < 0$  for all  $t \geq t_0$ .

In Theorem 2, the function  $V$  is called a *Lyapunov function*. The method of constructing a Lyapunov function for stability determination is called the *second* (or *direct method of Lyapunov*).

The geometric meaning of a Lyapunov function used for determining the system stability about the zero equilibrium may be illustrated by Fig. 9. In this figure, assuming that a Lyapunov function,  $V(\mathbf{x})$ , has been found, which has a bowl shape as shown based on conditions (a) and (b). Then, condition (d) is

$$\dot{V}(\mathbf{x}) = \left[ \frac{\partial V}{\partial \mathbf{x}} \right] \dot{\mathbf{x}} < 0 \tag{21}$$

where  $[\partial V / \partial \mathbf{x}]$  is the gradient of  $V$  along the trajectory  $\mathbf{x}$ . It is known, from calculus, that if the inner product of this gradient and the tangent vector  $\dot{\mathbf{x}}$  is constantly negative, as guaranteed by condition (21), then the angle between these two vectors is larger than  $90^\circ$ , so that the surface of  $V(\mathbf{x})$  is monotonically decreasing to zero (this is visualized in Fig. 9). Consequently, the system trajectory  $\mathbf{x}$ , the projection on the domain as shown in the figure, converges to zero as time evolves.

As an example, consider the following nonautonomous system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(\mathbf{x}, t)$$

where  $A$  is a stable constant matrix and  $\mathbf{g}$  is a nonlinear function satisfying  $\mathbf{g}(0, t) = 0$  and  $\|\mathbf{g}(\mathbf{x}, t)\| \leq c\|\mathbf{x}\|$  for a constant  $c > 0$  for all  $t \in [t_0, \infty)$ . Since  $A$  is stable, the following *Lyapunov equation*

$$PA + A^T P + I = 0$$

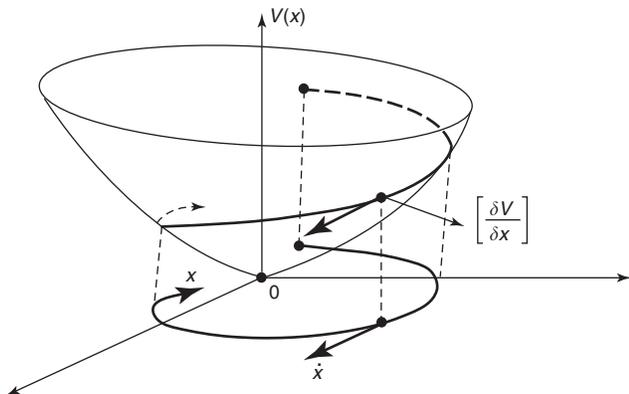


Figure 9. Geometric meaning of the Lyapunov function.

has a unique positive definite and symmetric matrix solution,  $P$ . Using the Lyapunov function  $V(\mathbf{x}, t) = \mathbf{x}^T P \mathbf{x}$ , it can be easily verified that

$$\begin{aligned} \dot{V}(\mathbf{x}, t) &= \mathbf{x}^T [PA + A^T P] \mathbf{x} + 2\mathbf{x}^T P \mathbf{g}(\mathbf{x}, t) \\ &\leq -\mathbf{x}^T \mathbf{x} + 2\lambda_{\max}(P)c\|\mathbf{x}\|^2 \end{aligned}$$

where  $\lambda_{\max}(P)$  is the largest eigenvalue of  $P$ . Therefore, if the constant  $c < 1/(2\lambda_{\max}(P))$  and if the class  $\mathcal{K}$  functions

$$\begin{aligned} \alpha(\zeta) &= \lambda_{\min}(P)\zeta^2, & \beta(\zeta) &= \lambda_{\max}(P)\zeta^2, \\ \gamma(\zeta) &= [1 - 2c\lambda_{\max}(P)]\zeta^2 \end{aligned}$$

are used, then conditions (c) and (d) of Theorem 2 are satisfied. As a result, the above system is globally, uniformly, and asymptotically stable about its zero equilibrium. This example shows that the linear part of a weakly nonlinear nonautonomous system can indeed dominate the stability.

Note that in Theorem 2, the uniform stability is guaranteed by the class  $\mathcal{K}$  functions  $\alpha, \beta, \gamma$  stated in conditions (c) and (d), which is necessary since the solution of a nonautonomous system may sensitively depend on the initial time, as seen from the numerical example discussed in Section 3.1. For autonomous systems, these class  $\mathcal{K}$  functions [hence, condition (c)] are not needed. In this case, Theorem 2 reduces to the following simple form.

**Theorem 3 (Second Method of Lyapunov: For Continuous-Time Autonomous Systems).** The autonomous system (19) is globally (over the entire domain  $\mathcal{D}$ ) and asymptotically stable about its zero equilibrium, if there exists a scalar-valued function  $V(\mathbf{x})$ , defined on  $\mathcal{D}$ , such that

- (a)  $V(0) = 0$ .
- (b)  $V(\mathbf{x}) > 0$  for all  $\mathbf{x} \neq 0$  in  $\mathcal{D}$ .
- (c)  $\dot{V}(\mathbf{x}) < 0$  for all  $\mathbf{x} \neq 0$  in  $\mathcal{D}$ .

Note that if condition (c) in Theorem 3 is replaced by (d)  $\dot{V}(\mathbf{x}) \leq 0$  for all  $\mathbf{x} \in \mathcal{D}$ , then the resulting stability is only in the sense of Lyapunov but may not be asymptotic. For example, consider a simple model of an undamped pendulum of length  $\ell$  described by

$$\begin{cases} \dot{x} = -\frac{g}{\ell} \sin(y) \\ \dot{y} = x \end{cases}$$

where  $x = \dot{\theta}$  is the angular variable defined on  $-\pi < \theta < \pi$ , with the vertical axis as its reference, and  $g$  is the gravity constant. Since the system Jacobian at the zero equilibrium has one pair of purely imaginary eigenvalues,  $\lambda_{1,2} = \pm \sqrt{-g/\ell}$ , Theorem 1 is not applicable. However, if one uses the Lyapunov function

$$V = \frac{g}{\ell} (1 - \cos(y)) + \frac{1}{2} x^2$$

then it can be easily verified that  $\dot{V} = 0$  over the entire domain. Thus, the conclusion is that the undamped

pendulum is stable in the sense of Lyapunov but may not be asymptotically, consistent with the physics of the undamped pendulum.

**Theorem 4 (Krasovskii Theorem: For Continuous-Time Autonomous Systems).** For the autonomous system (19), let  $J(\mathbf{x}) = [\partial \mathbf{f} / \partial \mathbf{x}]$  be its Jacobian evaluated at  $\mathbf{x}(t)$ . A sufficient condition for the system to be asymptotically stable about its zero equilibrium is that there exist two real positive definite and symmetric constant matrices,  $P$  and  $Q$ , such that the matrix

$$J^T(\mathbf{x})P + PJ(\mathbf{x}) + Q$$

is seminegative definite for all  $\mathbf{x} \neq 0$  in a neighborhood  $\mathcal{D}$  of the origin. For this case, a Lyapunov function is given by

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})P\mathbf{f}(\mathbf{x})$$

Furthermore, if  $\mathcal{D} = \mathbf{R}^n$  and  $V(\mathbf{x}) \rightarrow \infty$  as  $\|\mathbf{x}\| \rightarrow \infty$ , then this asymptotic stability is also global.

Similar stability criteria can be established for discrete-time systems [10]. Two main results are summarized as follows.

**Theorem 5 (First Method of Lyapunov: For Discrete-Time “Autonomous” Systems).** Let  $\mathbf{x}^* = 0$  be an equilibrium of the discrete-time “autonomous” system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) \tag{22}$$

where  $\mathbf{f}: \mathcal{D} \rightarrow \mathbf{R}^n$  is continuously differentiable in a neighborhood of the origin,  $\mathcal{D} \subseteq \mathbf{R}^n$ , and let  $J = [\partial \mathbf{f} / \partial \mathbf{x}_k]_{\mathbf{x}_k = \mathbf{x}^* = 0}$  be the Jacobian of the system evaluated at this equilibrium. If all the eigenvalues of  $J$  are strictly less than one in absolute value, then the system is asymptotically stable about its zero equilibrium.

**Theorem 6 (Second Method of Lyapunov: For Discrete-Time “Nonautonomous” Systems).** Let  $\mathbf{x}^* = 0$  be an equilibrium of the “nonautonomous” system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) \tag{23}$$

where  $\mathbf{f}_k: \mathcal{D} \rightarrow \mathbf{R}^n$  is continuously differentiable in a neighborhood of the origin,  $\mathcal{D} \subseteq \mathbf{R}^n$ . Then, system (22) is globally (over the entire domain  $\mathcal{D}$ ) and asymptotically stable about its zero equilibrium, if there exists a scalar-valued function,  $V(\mathbf{x}_k, k)$ , defined on  $\mathcal{D}$  and continuous in  $\mathbf{x}_k$ , such that

- (a)  $V(0, k) = 0$  for all  $k \geq k_0$ .
- (b)  $V(\mathbf{x}_k, k) > 0$  for all  $\mathbf{x}_k \neq 0$  in  $\mathcal{D}$  and for all  $k \geq k_0$ .
- (c)  $\Delta V(\mathbf{x}_k, k) := V(\mathbf{x}_k, k) - V(\mathbf{x}_{k-1}, k-1) < 0$  for all  $\mathbf{x}_k \neq 0$  in  $\mathcal{D}$  and all  $k \geq k_0 + 1$ .
- (d)  $0 < W(\|\mathbf{x}_k\|) < V(\mathbf{x}_k, k)$  for all  $k \geq k_0 + 1$ , where  $W(\tau)$  is a positive continuous function defined on  $\mathcal{D}$ , satisfying  $W(\|\mathbf{x}_{k_0}\|) = 0$  and  $\lim_{\tau \rightarrow \infty} W(\tau) = \infty$  monotonically.

As a special case, for discrete-time “autonomous” systems, Theorem 6 reduces to the following simple form.

**Theorem 7 (Second Method of Lyapunov: For Discrete-Time “Autonomous” Systems).** Let  $\mathbf{x}^* = 0$  be an equilibrium for the “autonomous” system (22). Then the system is globally (over the entire domain  $\mathcal{D}$ ) and asymptotically stable about this zero equilibrium if there exists a scalar-valued function,  $V(\mathbf{x}_k)$ , defined on  $\mathcal{D}$  and continuous in  $\mathbf{x}_k$ , such that

- (a)  $V(0) = 0$ .
- (b)  $V(\mathbf{x}_k) > 0$  for all  $\mathbf{x}_k \neq 0$  in  $\mathcal{D}$ .
- (c)  $\Delta V(\mathbf{x}_k) := V(\mathbf{x}_k) - V(\mathbf{x}_{k-1}) < 0$  for all  $\mathbf{x}_k \neq 0$  in  $\mathcal{D}$ .
- (d)  $V(\mathbf{x}) \rightarrow \infty$  as  $\|\mathbf{x}\| \rightarrow \infty$ .

To this end, it is important to emphasize that all the Lyapunov theorems stated above only offer *sufficient* conditions for asymptotic stability. On the other hand, usually more than one Lyapunov function may be constructed for the same system. For a given system, one choice of a Lyapunov function may yield a less conservative result (e.g., with a larger stability region) than other choices. However, no conclusion regarding stability may be drawn if, for technical reasons, a satisfactory Lyapunov function cannot be found. Nevertheless, there is a necessary condition in theory about the existence of a Lyapunov function [15], as follows.

**Theorem 8 (Massera Inverse Theorem).** Suppose that the autonomous system (19) is asymptotically stable about its equilibrium  $\mathbf{x}^*$  and  $\mathbf{f}$  is continuously differentiable with respect to  $\mathbf{x}$  for all  $t \in [t_0, \infty)$ . Then a Lyapunov function exists for this system.

#### 4.2. Some Instability Theorems

Once again, consider a general autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \tag{24}$$

with an equilibrium  $\mathbf{x}^* = 0$ . To disprove the stability, the following instability theorems may be used.

**Theorem 9 (A Linear Instability Theorem).** In system (24), let  $J = [\partial \mathbf{f} / \partial \mathbf{x}]_{\mathbf{x} = \mathbf{x}^* = 0}$  be the system Jacobian evaluated at  $\mathbf{x}^* = 0$ . If at least one of the eigenvalues of  $J$  has a positive real part, then  $\mathbf{x}^* = 0$  is unstable.

For discrete-time systems, there is a similar result. A *discrete-time “autonomous” system*

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k), \quad k = 0, 1, 2, \dots,$$

is unstable about its equilibrium  $\mathbf{x}^* = 0$  if at least one of the eigenvalues of the system Jacobian is larger than 1 in absolute value.

The following two negative theorems can be easily extended to nonautonomous systems in an obvious way.

**Theorem 10 (A General Instability Theorem).** For system (24), let  $V(\mathbf{x})$  be a positive and continuously differentiable function defined on a neighborhood  $\mathcal{D}$  of the origin, satisfying  $V(0)=0$ . Assume that in any subset, containing the origin, of  $\mathcal{D}$ , there is an  $\tilde{\mathbf{x}}$  such that  $V(\tilde{\mathbf{x}})>0$ . If, moreover

$$\frac{d}{dt} V(\mathbf{x}) > 0 \quad \text{for all } \mathbf{x} \neq 0 \text{ in } \mathcal{D}$$

then the system is unstable about the equilibrium  $\mathbf{x}^* = 0$ .

One example is the system

$$\begin{cases} \dot{x} = y + x(x^2 + y^4) \\ \dot{y} = -x + y(x^2 + y^4) \end{cases}$$

which has equilibrium  $(x^*, y^*) = (0, 0)$ . The system Jacobian at the equilibrium has a pair of imaginary eigenvalues,  $\lambda_{1,2} = \pm \sqrt{-1}$ , so Theorem 1 is not applicable. On the contrary, the Lyapunov function

$$V = \frac{1}{2} (x^2 + y^2)$$

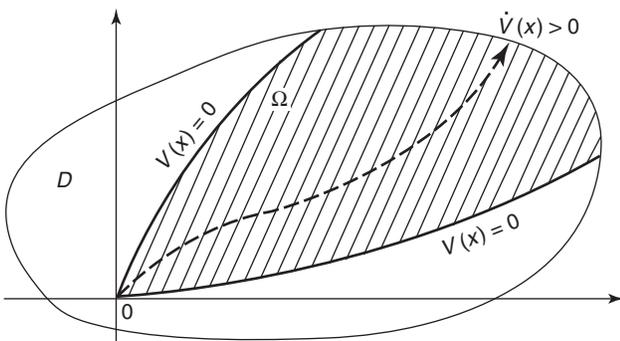
leads to  $\dot{V} = (x^2 + y^2)(x^2 + y^4) > 0$  for all  $(x, y) \neq (0, 0)$ . Therefore, the conclusion is that this system is unstable about its zero equilibrium.

**Theorem 11 (Chetaev Instability Theorem).** For system (24), let  $V(\mathbf{x})$  be a positive and continuously differentiable function defined on  $\mathcal{D}$ , and let  $\Omega$  be a subset, containing the origin, of  $\mathcal{D}$ , (i.e.,  $0 \in \mathcal{D} \cap \Omega$ ). If (a)  $V(\mathbf{x}) > 0$  and  $\dot{V}(\mathbf{x}) > 0$  for all  $\mathbf{x} \neq 0$  in  $\mathcal{D}$  and (b)  $V(\mathbf{x}) = 0$  for all  $\mathbf{x}$  on the boundary of  $\Omega$ , then the system is unstable about the equilibrium  $\mathbf{x}^* = 0$ .

This instability theorem is illustrated by Fig. 10, which graphically shows that if the theorem conditions are satisfied, then there is a gap within any neighborhood of the origin, so that a system trajectory can escape from the neighborhood of the origin along a path in this gap [23].

As an example, consider the system

$$\begin{cases} \dot{x} = x^2 + 2y^5 \\ \dot{y} = xy^2 \end{cases}$$



**Figure 10.** Illustration of the Chetaev theorem.

with the Lyapunov function

$$V = x^2 - y^4$$

which is positive inside the region defined by

$$x = y^2 \quad \text{and} \quad x = -y^2$$

Let  $\mathcal{D}$  be the right half-plane and  $\Omega$  be the shaded area shown in Fig. 11. Clearly,  $V=0$  on the boundary of  $\Omega$ , and  $V>0$  and  $\dot{V} = 2x^3 > 0$  for all  $(x, y) \in \mathcal{D}$ . According to the Chetaev theorem, this system is unstable about its zero equilibrium.

**4.3. LaSalle Invariance Principle**

Consider again the autonomous system (24) with an equilibrium  $\mathbf{x}^* = 0$ . Let  $V(\mathbf{x})$  be a Lyapunov function defined on a neighborhood  $\mathcal{D}$  of the origin. Let also  $\phi_t(\mathbf{x}_0)$  be a bounded solution orbit of the system, with the initial state  $\mathbf{x}_0$  and all its limit states being confined in  $\mathcal{D}$ . Moreover, let

$$E = \{\mathbf{x} \in \mathcal{D} \mid \dot{V}(\mathbf{x}) = 0\} \tag{25}$$

and  $M \subset E$  be the largest invariant subset of  $E$  in the sense that if the initial state  $\mathbf{x}_0 \in M$ , then the entire orbit  $\phi_t(\mathbf{x}_0) \subset M$  for all  $t \geq t_0$ .

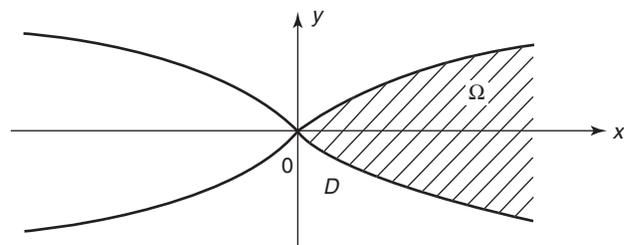
**Theorem 12 (LaSalle Invariance Principle).** Under the assumptions above, for any initial state  $\mathbf{x}_0 \in \mathcal{D}$ , the solution orbit satisfies  $\phi_t(\mathbf{x}_0) \rightarrow M$  as  $t \rightarrow \infty$ .

This invariance principle is consistent with the Lyapunov theorems when they are applicable to a problem [11,19]. Sometimes, when  $\dot{V}=0$  over a subset of the domain of  $V$ , a Lyapunov theorem is not easy to directly apply, but the above LaSalle invariance principle may be convenient to use. For instance, consider the system

$$\begin{cases} \dot{x} = -x + \frac{1}{3}x^3 + y \\ \dot{y} = -x \end{cases}$$

The Lyapunov function  $V = x^2 + y^2$  yields

$$\dot{V} = \frac{1}{2} x^2 \left( \frac{1}{3} x^2 - 1 \right)$$



**Figure 11.** The defining region of a Lyapunov function.

which is negative for  $x^2 < 3$  but is zero for  $x=0$  and  $x^2=3$ , regardless of variable  $y$ . Thus, Lyapunov theorems do not seem to be applicable, at least not directly. However, observe that the set  $E$  defined above has only three straight lines:  $x = -\sqrt{3}$ ,  $x=0$ , and  $x = \sqrt{3}$ , and that all trajectories that intersect the line  $x=0$  will remain on the line only if  $y=0$ . This means that the largest invariant subset  $M$  containing the points with  $x=0$  is the only point  $(0,0)$ . It then follows from the LaSalle invariance principle that starting from any initial state located in a neighborhood of the origin bounded within the two stripes  $x = \pm \sqrt{3}$ , say, located inside the disk

$$\mathcal{D} = \{(x,y) \mid x^2 + y^2 < 3\}$$

the solution orbit will always be attracted to the point  $(0, 0)$ . This means that the system is (locally) asymptotically stable about its zero equilibrium.

4.4. Comparison Principle and Vector Lyapunov Functions

For large-scale and interconnected nonlinear (control) systems, or systems described by differential inequalities rather than differential equations, the stability criteria above may not be directly applicable. In such cases, the comparison principle and vector Lyapunov function methods turn out to be advantageous [18,24,32].

To introduce the comparison principle, consider the general nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{26}$$

where  $\mathbf{f}(0, t) = 0$  is continuous on a neighborhood  $\mathcal{D}$  of the origin,  $t_0 \leq t < \infty$ .

In this case, since  $\mathbf{f}$  is only continuous (but not necessarily satisfying the Lipschitz condition), this differential equation may have more than one solution [12]. Let  $\mathbf{x}_{\max}(t)$  and  $\mathbf{x}_{\min}(t)$  be its maximum and minimum solutions, respectively, in the sense that

$$\mathbf{x}_{\min}(t) \leq \mathbf{x}(t) \leq \mathbf{x}_{\max}(t) \quad \text{componentwise, for all } t \in [t_0, \infty)$$

where  $\mathbf{x}(t)$  is any solution of the equation, and  $\mathbf{x}_{\min}(t_0) = \mathbf{x}(t_0) = \mathbf{x}_{\max}(t_0) = \mathbf{x}_0$ .

**Theorem 13 (The Comparison Principle).** Let  $\mathbf{y}(t)$  be a solution of the following differential inequality:

$$\dot{\mathbf{y}}(t) \leq \mathbf{f}(\mathbf{y}, t) \quad \text{with } \mathbf{y}(t_0) \leq \mathbf{x}_0 \quad \text{componentwise}$$

If  $\mathbf{x}_{\max}(t)$  is the maximum solution of system (26), then

$$\mathbf{y}(t) \leq \mathbf{x}_{\max}(t) \quad \text{componentwise for all } t \in [t_0, \infty)$$

The next theorem is established based on this comparison principle. First, recall that a vector-valued function,  $\mathbf{g}(\mathbf{x}, t) = [g_1(\mathbf{x}, t) \cdots g_n(\mathbf{x}, t)]^T$  is said to be *quasimonotonic*, if

$$\begin{aligned} x_i = \tilde{x}_i \quad \text{and} \quad x_j \geq \tilde{x}_j \quad (j \neq i) \\ \Rightarrow g_i(\mathbf{x}, t) \geq g_i(\tilde{\mathbf{x}}, t), \quad i = 1, \dots, n \end{aligned}$$

**Theorem 14 (Vector Lyapunov Function Theorem).** Let  $\mathbf{v}(\mathbf{x}, t)$  be a vector Lyapunov function associated with the

nonautonomous system (26),  $\mathbf{v}(\mathbf{x}, t) = [V_1(\mathbf{x}, t) \cdots V_n(\mathbf{x}, t)]^T$  in which each  $V_i$  is a continuous Lyapunov function for the system,  $i = 1, \dots, n$ , satisfying  $\|\mathbf{v}(\mathbf{x}, t)\| > 0$  for  $\mathbf{x} \neq 0$ . Assume that

$$\dot{\mathbf{v}}(\mathbf{x}, t) \leq \mathbf{g}(\mathbf{v}(\mathbf{x}, t), t) \quad \text{componentwise}$$

for a continuous and quasimonotonic function  $\mathbf{g}$  defined on  $\mathcal{D}$ . Then

- (a) If the system

$$\dot{\mathbf{y}}(t) = \mathbf{g}(\mathbf{y}, t)$$

is stable in the sense of Lyapunov (or asymptotically stable) about its zero equilibrium  $\mathbf{y}^* = 0$ , then so is the nonautonomous system (26).

- (b) If, moreover,  $\|\mathbf{v}(\mathbf{x}, t)\|$  is monotonically decreasing with respect to  $t$  and the preceding stability (or asymptotic stability) is uniform, then so is the nonautonomous system (26).
- (c) If, furthermore,  $\|\mathbf{v}(\mathbf{x}, t)\| \geq c \|\mathbf{x}\|^\sigma$  for two positive constants  $c$  and  $\sigma$ , and the preceding stability (or asymptotic stability) is exponential, then so is the nonautonomous system (26).

A simple and frequently used comparison function is

$$\mathbf{g}(\mathbf{y}, t) = A\mathbf{y} + \mathbf{h}(\mathbf{y}, t), \quad \lim_{\|\mathbf{y}\| \rightarrow 0} \frac{\|\mathbf{h}(\mathbf{y}, t)\|}{\|\mathbf{y}\|} = 0$$

where  $A$  is a stable  $M$  matrix (Metzler matrix). Here,  $A = [a_{ij}]$  is an  $M$  matrix if

$$a_{ii} < 0 \quad \text{and} \quad a_{ij} \geq 0 \quad (i \neq j), \quad i, j = 1, \dots, n$$

4.5. Orbital and Structural Stability Theorems

**Theorem 15 (Orbital Stability Theorem).** Let  $\tilde{\mathbf{x}}(t)$  be a  $p$ -periodic solution of an autonomous system. Suppose that the system has Floquet multipliers  $\lambda_i$ , with  $\lambda_1 = 0$  and  $|\lambda_i| < 1$  for  $i = 2, \dots, n$ . Then this periodic solution  $\tilde{\mathbf{x}}(t)$  is orbitally stable.

**Theorem 16 (Peixoto Structural Stability Theorem).** Consider a two-dimensional autonomous system. Suppose that  $\mathbf{f}$  is twice differentiable on a compact and connected subset  $\mathcal{D}$  bounded by a simple closed curve  $\Gamma$ , with an outward normal vector  $\bar{\mathbf{n}}$ . Assume that  $\mathbf{f} \cdot \bar{\mathbf{n}} \neq 0$  on  $\Gamma$ . Then the system is structurally stable on  $\mathcal{D}$  if and only if

- (a) All equilibria are hyperbolic.
- (b) All periodic orbits are hyperbolic.
- (c) If  $x$  and  $y$  are hyperbolic saddles (probably  $x=y$ ), then  $W^s(x)$  and  $W^u(y)$  are transversal.

5. LINEAR STABILITY OF NONLINEAR SYSTEMS

The first method of Lyapunov provides a linear stability analysis for nonlinear autonomous systems. In this section, the following general nonautonomous system is

considered:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \quad (27)$$

which is assumed to have an equilibrium  $\mathbf{x}^* = 0$ .

### 5.1. Linear Stability of Nonautonomous Systems

In system (27), Taylor-expanding the function  $\mathbf{f}$  about  $\mathbf{x}^* = 0$  gives

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) = J(t)\mathbf{x} + \mathbf{g}(\mathbf{x}, t) \quad (28)$$

where  $J(t) = [\partial\mathbf{f}/\partial\mathbf{x}]_{\mathbf{x}=0}$  is the Jacobian and  $\mathbf{g}(\mathbf{x}, t)$  is the residual of the expansion, which is assumed to satisfy

$$\|\mathbf{g}(\mathbf{x}, t)\| \leq a\|\mathbf{x}\|^2 \quad \text{for all } t \in [t_0, \infty)$$

where  $a > 0$  is a constant. It is known from the theory of elementary ordinary differential equations [12] that the solution of equation (28) is given by

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0 + \int_{t_0}^t \Phi(t, \tau)\mathbf{g}(\mathbf{x}(\tau), \tau)d\tau \quad (29)$$

where  $\Phi(t, \tau)$  is the fundamental matrix associated with matrix  $J(t)$ .

**Theorem 17 (A General Linear Stability Theorem).** For the nonlinear nonautonomous system (28), if there are two positive constants  $c$  and  $\sigma$  such that

$$\|\Phi(t, \tau)\| \leq ce^{-\sigma(t-\tau)} \quad \text{for all } t_0 \leq \tau \leq t < \infty$$

and if

$$\lim_{\|\mathbf{x}\| \rightarrow 0} \frac{\|\mathbf{g}(\mathbf{x}, t)\|}{\|\mathbf{x}\|} = 0$$

uniformly with respect to  $t \in [t_0, \infty)$ , then there are two positive constants,  $\gamma$  and  $\delta$ , such that

$$\|\mathbf{x}(t)\| \leq c\|\mathbf{x}_0\|e^{-\gamma(t-t_0)}$$

for all  $\|\mathbf{x}_0\| \leq \delta$  and all  $t \in [t_0, \infty)$ .

This result implies that under the theorem conditions, the system is locally, uniformly, and exponentially stable about its equilibrium  $\mathbf{x}^* = 0$ .

In particular, if the system matrix  $J(t) = J$  is a stable constant matrix, then the following simple criterion is convenient to use.

**Theorem 18 (A Special Linear Stability Theorem).** Suppose that in system (28), the matrix  $J(t) = J$  is a stable constant matrix (all its eigenvalues have a negative real part), and  $\mathbf{g}(0, t) = 0$ . Let  $P$  be a positive definite and symmetric matrix solution of the Lyapunov equation

$$PJ + J^T P + Q = 0$$

where  $Q$  is a positive definite and symmetric constant matrix. If

$$\|\mathbf{g}(\mathbf{x}, t)\| \leq a\|\mathbf{x}\|$$

for a constant  $a < \frac{1}{2}\lambda_{\max}(P)$  uniformly on  $[t_0, \infty)$ , where  $\lambda_{\max}(P)$  is the maximum eigenvalue of  $P$ , then system (28) is globally, uniformly, and asymptotically stable about its equilibrium  $\mathbf{x}^* = 0$ .

This actually is the example for illustration of Theorem 2 discussed in Section 4.1.

Again consider system (27) with equilibrium  $\mathbf{x}^* = 0$ .

**Theorem 19 (The Lyapunov Converse Theorem).** Suppose that  $\mathbf{f}$  is continuously differentiable in a neighborhood of the origin, and its Jacobian  $\mathbf{J}(t)$  is bounded and Lipschitz in the neighborhood, uniformly in  $t$ . Then, the system is exponentially stable about its equilibrium if and only if its linearized system  $\dot{\mathbf{x}} = \mathbf{J}(t)\mathbf{x}$  is exponentially stable about the origin.

### 5.2. Linear Stability of Nonlinear Systems with Periodic Linearity

Consider a nonlinear nonautonomous system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) = J(t)\mathbf{x} + \mathbf{g}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \quad (30)$$

where  $\mathbf{g}(0, t) = 0$  and  $J(t)$  is a  $p$ -periodic matrix ( $p > 0$ ):

$$J(t+p) = J(t) \quad \text{for all } t \in [t_0, \infty)$$

**Theorem 20 (Floquet Theorem).** In system (30), assume that  $\mathbf{g}(\mathbf{x}, t)$  and  $\partial\mathbf{g}(\mathbf{x}, t)/\partial\mathbf{x}$  are both continuous in a bounded region  $\mathcal{S}$  containing the origin. Assume, moreover, that

$$\lim_{\|\mathbf{x}\| \rightarrow 0} \frac{\|\mathbf{g}(\mathbf{x}, t)\|}{\|\mathbf{x}\|} = 0$$

uniformly on  $[t_0, \infty)$ . If the system Floquet multipliers satisfy

$$|\lambda_i| < 1, \quad i = 1, \dots, n, \quad \text{for all } t \in [t_0, \infty) \quad (31)$$

then system (30) is globally, uniformly, and asymptotically stable about its equilibrium  $\mathbf{x}^* = 0$ .

Note that if  $\mathbf{g}(\mathbf{x}, t) = 0$  in system (28) or (30), all the linear stability results presented above still hold and are consistent with the familiar results from the linear systems theory.

## 6. TOTAL STABILITY: STABILITY UNDER PERSISTENT PERTURBANCES

Consider a nonautonomous system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{h}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{R}^n \quad (32)$$

where  $\mathbf{f}$  is continuously differentiable, with  $\mathbf{f}(0, t) = 0$ , and  $\mathbf{h}$  is a *persistent perturbation* in the sense that for any  $\varepsilon > 0$ , there are two positive constants,  $\delta_1$  and  $\delta_2$ , such that if  $\|\mathbf{h}(\hat{\mathbf{x}}, t)\| < \delta_1$  for all  $t \in [t_0, \infty)$  and if  $\|\hat{\mathbf{x}}(t_0)\| < \delta_2$ , then  $\|\hat{\mathbf{x}}(t)\| < \varepsilon$ .

The equilibrium  $\mathbf{x}^* = 0$  of the unperturbed system [system (32) with  $\mathbf{h} = 0$  therein] is said to be *totally stable*, if the persistently perturbed system (32) remains to be stable in the sense of Lyapunov.

As the next theorem states, all uniformly and asymptotically stable systems with persistent perturbations are totally stable, namely, a stable orbit starting from a neighborhood of another orbit will stay nearby [13,15].

**Theorem 21 (Malkin Theorem).** If the unperturbed system (32), that is, with  $\mathbf{h} = 0$  therein, is uniformly and asymptotically stable about its equilibrium  $\mathbf{x}^* = 0$ , then it is totally stable, namely, the persistently perturbed system (32) remains to be stable in the sense of Lyapunov.

Next, consider an autonomous system with persistent perturbations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{h}(\mathbf{x}, t), \quad \mathbf{x} \in \mathbf{R}^n \tag{33}$$

**Theorem 22 (Perturbed Orbital Stability Theorem).** If  $\varphi_t(\mathbf{x}_0)$  is an orbitally stable solution of the unperturbed autonomous system (33) (with  $\mathbf{h} = 0$  therein), then it is totally stable, that is, the perturbed system remains to be orbitally stable under persistent perturbations.

**7. ABSOLUTE STABILITY AND FREQUENCY-DOMAIN CRITERIA**

Consider a feedback system in the Lur’e form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{h}(\mathbf{y}) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{34}$$

where  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  are constant matrices, in which  $\mathbf{A}$  is non-singular but  $\mathbf{B}$  and  $\mathbf{C}$  are not necessarily square (yet, probably,  $\mathbf{B} = \mathbf{C} = \mathbf{I}$ ), and  $\mathbf{h}$  is a vector-valued nonlinear function. By taking the Laplace transform with zero initial conditions, and denoting the transform by  $\hat{\mathbf{x}} = \mathcal{L}\{\mathbf{x}\}$ , the state vector is obtained as

$$\hat{\mathbf{x}} = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathcal{L}\{\mathbf{h}(\mathbf{y})\} \tag{35}$$

so that the output is given by

$$\hat{\mathbf{y}} = \mathbf{C}\mathbf{G}(s)\mathcal{L}\{\mathbf{h}(\mathbf{y})\} \tag{36}$$

with the system transfer matrix

$$\mathbf{G}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} \tag{37}$$

This can be implemented via the block diagram shown in Fig. 12, where, for notational convenience, both time- and frequency-domain symbols are mixed.

The Lur’e system shown in Fig. 12 is a closed-loop configuration, where the feedback loop is usually considered as a “controller.” Thus, this system is sometimes written in the following equivalent form:

$$\begin{cases} \dot{\mathbf{x}} + \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = \mathbf{h}(\mathbf{y}) \end{cases} \tag{38}$$

**7.1. SISO Lur’e Systems**

First, single-input/single-output Lur’e systems are discussed, where  $u = h(y)$  and  $y = \mathbf{c}^T\mathbf{x}$  are both scalar-valued functions:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y = \mathbf{c}^T\mathbf{x} \\ u = h(y) \end{cases} \tag{39}$$

Assume that  $h(0) = 0$ , so that  $\mathbf{x}^* = 0$  is an equilibrium of the system.

**7.1.1. The Sector Condition.** The Lur’e system (39) is said to satisfy the *local (global) sector condition* on the nonlinear function  $h(\cdot)$ , if there exist two constants,  $\alpha < \beta$ , such that

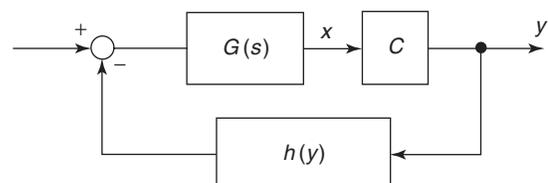
- 1. Local sector condition:

$$\begin{aligned} \alpha y^2(t) \leq y(t)h(y(t)) \leq \beta y^2(t) \quad \text{for all} \\ a \leq y(t) \leq b \quad \text{and } t \in [t_0, \infty) \end{aligned} \tag{40}$$

- 2. Global sector condition:

$$\begin{aligned} \alpha y^2(t) \leq y(t)h(y(t)) \leq \beta y^2(t) \quad \text{for all} \\ -\infty < y(t) < \infty \quad \text{and } t \in [t_0, \infty) \end{aligned} \tag{41}$$

Here,  $[\alpha, \beta]$  is called the *sector* for the nonlinear function  $h(\cdot)$ . Moreover, the system (39) is said to be (globally) *absolutely stable within the sector*  $[\alpha, \beta]$  if the system is (globally) asymptotically stable about its equilibrium  $\mathbf{x}^* = 0$  for any nonlinear function  $h(\cdot)$  satisfying the



**Figure 12.** Configuration of the Lur’e system.

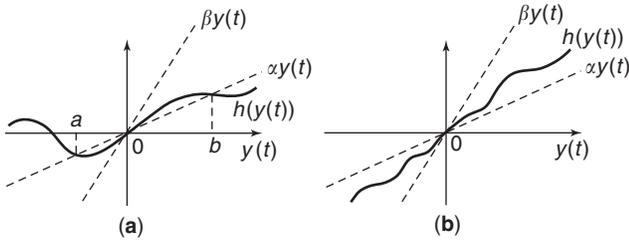


Figure 13. Local and global sector conditions.

(global) sector condition. These local and global sector conditions are visualized by Figs. 13a and 13b, respectively.

7.1.2. The Popov Criterion.

**Theorem 23 (Popov Criterion).** Suppose that the SISO Lur'e's system (39) satisfied the following conditions:

- (a)  $A$  is stable and  $\{A, \mathbf{b}\}$  is controllable.
- (b) The system satisfies the global sector condition with  $\alpha = 0$  therein.
- (c) For any  $\varepsilon > 0$ , there is a constant  $\gamma > 0$  such that

$$\text{Re}\{(1 + j\gamma\omega)G(j\omega)\} + \frac{1}{\beta} \geq \varepsilon \quad \text{for all } \omega \geq 0 \quad (42)$$

where  $G(s)$  is the transfer function defined by (37), and  $\text{Re}\{\cdot\}$  denotes the real part of a complex number (or function). Then, the system is globally asymptotically stable about its equilibrium  $\mathbf{x}^* = 0$  within the sector.

The Popov criterion has the following geometric meaning. Separate the complex function  $G(s)$  into its real and imaginary parts, namely

$$G(j\omega) = G_r(\omega) + jG_i(\omega)$$

and rewrite condition (C) as

$$\frac{1}{\beta} > -G_r(\omega) + \gamma\omega G_i(\omega) \quad \text{for all } \omega \geq 0$$

Then the graphical situation of the Popov criterion shown in Fig. 14 implies the global asymptotic stability of the system about its zero equilibrium.

7.1.3. The Circle Criterion. The Popov criterion has a natural connection to the linear Nyquist criterion

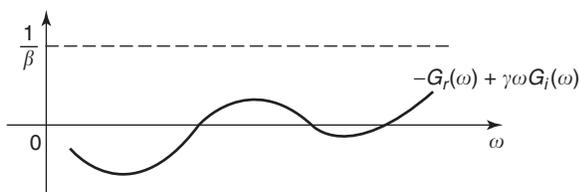


Figure 14. Geometric meaning of the Popov criterion.

[17,26,27,29]. A more direct generalization of the Nyquist criterion to nonlinear systems is the following.

**Theorem 24 (Circle Criterion).** Suppose that the SISO Lur'e system (39) satisfies the following conditions:

- (a)  $A$  has no purely imaginary eigenvalues, and has  $\kappa$  eigenvalues with positive real parts.
- (b) The system satisfies the global sector condition.
- (c) One of the following situation holds:
  - (1)  $0 < \alpha < \beta$ —the Nyquist plot of  $G(j\omega)$  encircles the disk  $D[-(1/\alpha), -(1/\beta)]$  counterclockwise  $\kappa$  times but does not enter it.
  - (2)  $0 = \alpha < \beta$ —the Nyquist plot of  $G(j\omega)$  stays within the open half-plane  $\text{Re}\{s\} > -(1/\beta)$ .
  - (3)  $\alpha < 0 < \beta$ —the Nyquist plot of  $G(j\omega)$  stays within the open disk  $D[-(1/\beta), -(1/\alpha)]$ .
  - (4)  $\alpha < \beta < 0$ —the Nyquist plot of  $-G(j\omega)$  encircles the disk  $D[(1/\alpha), (1/\beta)]$  counterclockwise  $\kappa$  times but does not enter it.

Then, the system is globally asymptotically stable about its equilibrium  $\mathbf{x}^* = 0$ .

Here, the disk  $D[-(1/\alpha), -(1/\beta)]$ , for the case of  $0 < \alpha < \beta$ , is shown in Fig. 15.

7.2. MIMO Lur'e systems

Consider a multiinput/multioutput Lur'e system, as shown in Fig. 12, described by

$$\begin{cases} \mathbf{x}(s) = G(s)\mathbf{u}(s) \\ \mathbf{u}(t) = -\mathbf{h}(\mathbf{y}(t)) \end{cases} \quad (43)$$

with  $G(s)$  as defined in (37). If this system satisfies the following Popov inequality

$$\int_{t_0}^{t_1} \mathbf{y}^T(\tau)\mathbf{x}(\tau)d\tau \geq -\gamma \quad \text{for all } t_1 \geq t_0 \quad (44)$$

for a constant  $\gamma \geq 0$  independent of  $t$ , then it is said to be hyperstable.

The linear part of this MIMO system is described by the transfer matrix  $G(s)$ , which is said to be positive real if

- 1. There are no poles of  $G(s)$  located inside the open half-plane  $\text{Re}\{s\} > 0$ .
- 2. Poles of  $G(s)$  on the imaginary axis are simple, and the residues form a semipositive definite matrix.

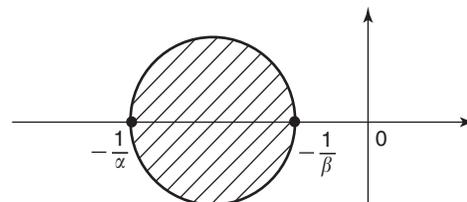


Figure 15. The disk  $D[-(1/\alpha), -(1/\beta)]$ .

3. The matrix  $[G(j\omega) + G^T(j\omega)]$  is a semipositive definite matrix for all real values of  $\omega$  that are not poles of  $G(s)$ .

**Theorem 25 (Hyperstability Theorem).** The MIMO Lur'e system (43) is hyperstable if and only if its transfer matrix  $G(s)$  is positive real.

**7.3. Describing Function Method**

Return to the SISO Lur'e system (39) and consider its periodic output  $y(t)$ . Assume that the nonlinear function  $h(\cdot)$  therein is a time-invariant odd function, and satisfies the property that for  $y(t) = \alpha \sin(\omega t)$ , with real constants  $\omega$  and  $\alpha \neq 0$ , only the first-order harmonic of  $-h(y)$  in its Fourier series expansion is significant. Under this setup, the specially defined function

$$\Psi(\alpha) = -\frac{2\omega}{\alpha\pi} \int_0^{\pi/\omega} h(\alpha \sin(\omega t)) \sin(\omega t) dt \quad (45)$$

is called the *describing function* of the nonlinearity  $-h(\cdot)$ , or of the system [1,26,27].

**Theorem 26 (First-Order Harmonic Balance Approximation).** Under the conditions stated above, if the first-order harmonic balance equations

$$G_r(j\omega)\Psi(\alpha) = 1 \quad \text{and} \quad G_i(j\omega) = 0$$

have solutions  $\omega$  and  $\alpha \neq 0$ , then

$$y^{(1)}(t) = \frac{j\alpha}{2} e^{-j\omega t} - \frac{j\alpha}{2} e^{j\omega t}$$

is the first-order approximation of a possible periodic orbit of the output of system (39); but if the preceding harmonic balance equations have no solution, then the system is unlikely to have any periodic output.

When solving the equation  $G_r(j\omega)\Psi(\alpha) = 1$  graphically, one can sketch two curves in the complex plane:  $G_r(j\omega)$  and  $-1/\Psi(\alpha)$  by increasing gradually  $\omega$  and  $\alpha$ , respectively, to find their crossing points:

1. If the two curves are (almost) tangent, as illustrated by Fig. 16a, then a conclusion drawn from the describing function method will not be satisfactory in general.
2. If the two curves are (almost) transversal, as illustrated by Fig. 16b, then a conclusion drawn from the describing function analysis will generally be reliable.

**Theorem 27 (Graphical Stability Criterion for a Periodic Orbit).** Each intersection point of the two curves above,  $G_r(j\omega)$  and  $-1/\Psi(\alpha)$ , corresponds to a periodic orbit,  $y^{(1)}(t)$ , of the output of system (39). If the points, near the intersection and on one side of the curve  $-1/\Psi(\alpha)$  where  $-\alpha$  is increasing, are not encircled by the curve  $G_r(j\omega)$ , then the corresponding periodic output is stable; otherwise, it is unstable.

**8. BIBO STABILITY**

A relatively simple, and also relatively weak, notion of stability is discussed in this section. This is the *bounded-input/bounded-output (BIBO) stability*, which refers to the property of a system that any bounded input to the system produces a bounded output throughout the system [8,9,17].

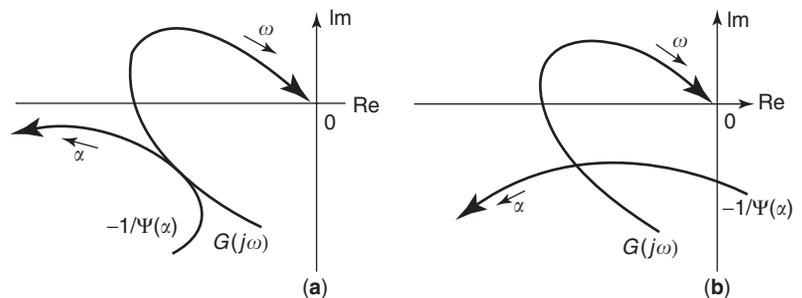
It can be verified that the linear system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$  is BIBO stable if the matrix  $\mathbf{A}$  is asymptotically stable.

Here, the focus is on the input-output map (13), with its configuration shown in Fig. 2.

The system  $S$  is said to be *BIBO stable* from the input set  $U$  to the output set  $Y$ , if for each admissible input  $\mathbf{u} \in U$  and the corresponding output  $\mathbf{y} \in Y$ , there exist two non-negative constants,  $b_i$  and  $b_o$ , such that

$$\|\mathbf{u}\|_U \leq b_i \Rightarrow \|\mathbf{y}\|_Y \leq b_o \quad (46)$$

Since all norms are equivalent for a finite-dimensional vector, it is generally insignificant to distinguish under what kind of norms for the input and output signals the BIBO stability is defined and achieved. Moreover, it is important to note that in the definition above, even if  $b_i$  is small and  $b_o$  is large, the system is still considered to be BIBO-stable. Therefore, this stability may not be very practical for some systems in certain applications.



**Figure 16.** Graphical depiction of describing function analysis.

8.1. Small-Gain Theorem

A convenient criterion for verifying the BIBO stability of a closed-loop control system is the small-gain theorem [8,9,17], which applies to almost all kinds of systems (linear and nonlinear, continuous-time and discrete-time, time-delayed, of any dimensions), as long as the mathematical setup is appropriately formulated to meet the theorem conditions. The main disadvantage of this criterion is its overconservativity.

Return to the typical closed-loop system shown in Fig. 3, where the inputs, outputs, and internal signals are related via the following equations:

$$\begin{cases} S_1(e_1) = e_2 - u_2 \\ S_2(e_2) = u_1 - e_1 \end{cases} \quad (47)$$

It is important to note that the individual BIBO stability of  $S_1$  and  $S_2$  is not sufficient for the BIBO stability of the connected closed-loop system. For instance, in the discrete-time setting of Fig. 3, suppose that  $S_1 \equiv 1$  and  $S_2 \equiv -1$ , with  $u_1(k) \equiv 1$  for all  $k = 0, 1, \dots$ . Then  $S_1$  and  $S_2$  are BIBO-stable individually, but it can be easily verified that  $y_1(k) = k \rightarrow \infty$  as the discrete-time variable  $k$  evolves. Therefore, a stronger condition describing the interaction of  $S_1$  and  $S_2$  is necessary.

**Theorem 28 (Small-Gain Theorem).** If there exist four constants,  $L_1, L_2, M_1, M_2$ , with  $L_1L_2 < 1$ , such that

$$\begin{cases} \|S_1(e_1)\| \leq M_1 + L_1\|e_1\| \\ \|S_2(e_2)\| \leq M_2 + L_2\|e_2\| \end{cases} \quad (48)$$

then

$$\begin{cases} \|e_1\| \leq (1 - L_1L_2)^{-1}(\|u_1\| + L_2\|u_2\| + M_2 + L_2M_1) \\ \|e_2\| \leq (1 - L_1L_2)^{-1}(\|u_2\| + L_1\|u_1\| + M_1 + L_1M_2) \end{cases} \quad (49)$$

where the norms  $\|\cdot\|$  are defined over the spaces that the signals belong. Consequently, (48) and (49) together imply that if the system inputs ( $u_1$  and  $u_2$ ) are bounded then the corresponding outputs [ $S_1(e_1)$  and  $S_2(e_2)$ ] are bounded.

Note that the four constants,  $L_1, L_2, M_1, M_2$ , can be somewhat arbitrary (e.g., either  $L_1$  or  $L_2$  can be large) provided that  $L_1L_2 < 1$ , which is the key condition for the theorem to hold [and is used to obtain  $(1 - L_1L_2)^{-1}$  in the bounds (49)].

In the special case where the input-output spaces  $U$  and  $Y$  are both the  $L_2$  space, a similar criterion based on the system passivity property can be obtained [9,17]. In this case, an inner product between any two vectors in the space is defined by

$$\langle \zeta, \eta \rangle = \int_{t_0}^{\infty} \zeta^T(\tau)\eta(\tau) d\tau$$

**Theorem 29 (Passivity Stability Theorem).** If there exist four constants,  $L_1, L_2, M_1, M_2$ , with  $L_1 + L_2 > 0$ , such that

$$\begin{cases} \langle e_1, S_1(e_1) \rangle \geq L_1\|e_1\|_2^2 + M_1 \\ \langle e_2, S_2(e_2) \rangle \geq L_2\|S_2(e_2)\|_2^2 + M_2 \end{cases} \quad (50)$$

then the closed-loop system (47) is BIBO-stable.

As mentioned, the main disadvantage of this criterion is its overconservativity in providing the sufficient conditions for the BIBO stability. One resolution is to transform the system into the Lur'e structure, and then apply the circle or Popov criterion under the sector condition (if it can be satisfied), which can usually lead to less conservative stability conditions.

8.2. Contraction Mapping Theorem

The small-gain theorem discussed above, by nature, is a kind of contraction mapping theorem. The contraction mapping theorem can be used to determine the BIBO stability property of a system described by a map in various forms, provided that the system (or the map) is appropriately formulated. The following is a typical (global) contraction mapping theorem.

**Theorem 30 (Contraction Mapping Theorem).** If the operator norm of the input-output map  $S$ , defined on  $\mathbf{R}^n$ , satisfies  $\|S\| < 1$ , then the system mapping

$$\mathbf{y}(t) = S(\mathbf{y}(t)) + \mathbf{c}$$

has a unique solution for any constant vector  $\mathbf{c} \in \mathbf{R}^n$ . This solution satisfies

$$\|\mathbf{y}\| \leq (1 - \|S\|)^{-1}\|\mathbf{c}\|$$

In particular, the solution of the equation

$$\mathbf{y}_{k+1} = S(\mathbf{y}_k), \mathbf{y}_0 \in \mathbf{R}^n, k = 0, 1, \dots$$

satisfies

$$\|\mathbf{y}_k\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

9. CONCLUDING REMARKS

This article has offered a brief introduction to the basic theory and methodology of Lyapunov stability, orbital stability, structural stability, and input-output stability for nonlinear dynamical systems. More subtle details for stability analysis of general dynamical systems can be found in the literature [3,5,9,11-13,16,17,19,20,23,26,27,29,30]. In cases where control was explicitly involved, stability and stabilization issues have also been studied [8,17,25,28,33].

Several important classes of nonlinear (control) systems have been omitted in the discussion of various stability issues: some general functional systems such as

systems with time delays [14], measure ordinary differential equations such as systems with impulses [2,31], and some weakly nonlinear systems such as piecewise linear and switching (non)linear systems. Discussion of more advanced nonlinear systems, such as singular nonlinear systems (perhaps with time delays), infinite-dimensional (non)linear systems, spatiotemporal systems described by nonlinear partial differential equations, and nonlinear stochastic (control) systems, are beyond the scope of this elementary expository article.

Finally, for more recent studies of system stability theories, the reader is referred to the reviews edited by Martynuk [22].

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## STANDING WAVE METERS AND NETWORK ANALYZERS

BURKHARD SCHIEK  
ANDREAS GRONEFELD  
Ruhr-Universität Bochum

### 1. MEASUREMENT OF NETWORKS AT MICROWAVE FREQUENCIES

Determining the characteristics of electrical networks (network analysis) is an important measurement tool both for developers of electrical circuits and systems and in production testing. The measurement of network characteristics at microwave frequencies employs different concepts than are used at lower frequencies, where measuring port voltages and currents readily allows determination of impedance ( $Z$ ) or admittance ( $Y$ ) parameters. At higher frequencies, neither high-impedance voltage probes nor low-impedance current measurements are feasible, because parasitic capacitance and inductance of