Pinning Control and Controllability of Complex Networks

Guanrong (Ron) Chen
City University of Hong Kong

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Dedicated to the Memory of

Rudolf E Kalman
(1930-5-19 – 2016-7-3)
Motivational Examples
Example:

C. elegans

In its Neural Network:

Neurons: 300~500  Synapses: 2500~7000
“The worm Caenorhabditis elegans has 297 nerve cells. The neurons switch one another on or off, and, making 2345 connections among themselves. They form a network that stretches through the nematode’s millimeter-long body.”

“How many neurons would you have to commandeer to control the network with complete precision?”

The answer is, on average: 49


Here, control = stimuli
Another Example

“… very few individuals (approximately 5%) within honeybee swarms can guide the group to a new nest site.”


These 5% of bees can be considered as “controlling” or “controlled” agents

**Leader-Followers network**
Now ... mathematically

- Given a network of identical dynamical systems (e.g., ODEs)
- Given a specific control objective (e.g., synchronization)
- Assume: a certain type of controllers (e.g., local linear state-feedback controllers) have been chosen to use
Questions:

Objective: To achieve a certain control goal

Questions:
- How many controllers to use?
- Where to put them? (which nodes to “pin”)

--- “Pinning Control”
Network Model

Linearly coupled network:

\[ \dot{x}_i = f(x_i) + c \sum_{j=1}^{N} \beta_{ij} H x_j \quad x_i \in \mathbb{R}^n \quad i = 1,2,\ldots, N \]

- General assumption: \( f(.) \) is Lipschitz. Here, it is linear:

\[ \dot{x}_i = A x_i + c \sum_{j=1}^{N} \beta_{ij} H x_j \quad x_i \in \mathbb{R}^n \quad i = 1,2,\ldots, N \]

- Coupling strength \( c > 0 \) and \( H \) – input coupling matrix

- Adjacency matrix: \( \begin{bmatrix} \beta_{ij} \end{bmatrix}_{N \times N} \)

If node \( i \) points to node \( j \) (\( j \neq i \)), then \( \beta_{ij} = 1 \); otherwise \( \beta_{ij} = 0 \); and \( \beta_{ii} = 0 \)

Note: For undirected networks, \( \begin{bmatrix} \beta_{ij} \end{bmatrix}_{N \times N} \) is symmetrical; for directed networks, not so
What kind of controllers? How many? Where?

\[
\dot{x}_i = Ax_i + c \sum_{j=1}^{N} \beta_{ij} Hx_j \leftarrow + Bu_i \quad (e.g., \quad u_i = -\Gamma x_i)
\]

\[
\dot{x}_i = Ax_i + c \sum_{j=1}^{N} \beta_{ij} Hx_j + \delta_i Bu_i
\]

\[
\delta_i = \begin{cases} 
1 & \text{if to - control} \\
0 & \text{if not - control} 
\end{cases}
\]

Q: How many \( \delta_i = 1 \)? Which \( i \)? \( (i = 1, 2, \ldots, N) \)
Pinning Control: Our Research Progress


Controllability Theory
MATHEMATICAL DESCRIPTION OF LINEAR DYNAMICAL SYSTEMS*

R. E. KALMAN†

Abstract. There are two different ways of describing dynamical systems: (i) by means of state variables and (ii) by input/output relations. The first method may be regarded as an axiomatization of Newton’s laws of mechanics and is taken to be the basic definition of a system.

It is then shown (in the linear case) that the input/output relations determine only one part of a system, that which is completely observable and completely controllable. Using the theory of controllability and observability, methods are given for calculating irreducible realizations of a given impulse-response matrix. In particular, an explicit procedure is given to determine the minimal number of state variables necessary to realize a given transfer-function matrix. Difficulties arising from the use of reducible realizations are discussed briefly.
State Controllability

Linear Time-Invariant (LTI) system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

- \( x \in \mathbb{R}^n \): state vector
- \( u \in \mathbb{R}^p \): control input
- \( A \in \mathbb{R}^{n \times n} \): state matrix
- \( B \in \mathbb{R}^{n \times p} \): control input matrix

[Concept] **State Controllable**: The system orbit can be driven by an input from any initial state to the origin in finite time

State Controllability Theorems

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

(i) Kalman Rank Criterion

The controllability matrix \( Q \) has full row rank:

\[ Q = [B \ AB \ \cdots \ A^{n-1}B] \]

(ii) Popov-Belevitch-Hautus (PBH) Test

The following relationship holds:

\[ v^T A = \lambda v^T, \quad v^T B \neq 0 \]

\( \lambda \) : eigenvalue of \( A \)

\( v \) : nonzero left eigenvector with \( \lambda \)
Observability

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

**Duality:**

\((A, B, C)\) is observable

If and only if

\((A^T, C^T, B^T)\) is controllable

[Concept] **Observable:**

Input-output pair \((u(t), y(t))\) on \([t_0, t_1]\) uniquely determines the initial state \(x(t_0)\)

What about networks? -- Some earlier attempts

- **Leader-follower multi-agent systems**
  H.G. Tanner, *CDC*, 2004

- **Pinning state-controllability of complex networks**

- **Structural controllability of complex networks**
Structural Controllability

A network of single-input/single-output (SISO) node systems, where the node systems can be of higher-dimensional
Structural Controllability

In the controllability matrix $Q$: $Q = [B \ AB \ \ldots \ A^{n-1}B]$

All 0 are fixed

There is a realization of independent nonzero parameters such that $Q$ has full row-rank

Example 1:

$$Q = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

Realization: All admissible parameters $a \neq 0, \ d \neq 0$

Example 2: Frobenius Canonical Form

$$Q = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$
Examples: Structure matters

C = [B, A ⋅ B, A² ⋅ B]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & 0 \\
0 & 0 & a_{32}a_{21}
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & 0 \\
0 & a_{31} & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & 0 \\
0 & a_{31} & a_{33}a_{31}
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & a_{23}a_{31} \\
0 & a_{31} & a_{32}a_{21}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & a_{21} \end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & a_{21} \end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & a_{21} \end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & a_{21} \end{bmatrix}
\]

\[
\begin{bmatrix}
rank C = 3 = n \\
controllable
\end{bmatrix},
\begin{bmatrix}
rank C = 2 < n = 3 \\
uncontrollable
\end{bmatrix},
\begin{bmatrix}
rank C = 3 = n \\
controllable
\end{bmatrix},
\begin{bmatrix}
rank C = ? \\
controllable
\end{bmatrix}
\]

Structurally controllable

In retrospect: large-scale systems theory

Structural Controllability (and Structural Observability)

Building Blocks

Cactus is the minimum structure that contains no inaccessible nodes and no dilations.
Structural Controllability Theorem

The following two criteria are equivalent:

1. Algebraic:
   The LTI control system \((A,B)\) is structurally controllable

2. Geometric:
   The digraph \(G(A,B)\) is spanned by a cactus

Matching in Directed Networks

- **Matching**: a set of directed edges without common heads and tails
- **Unmatched node**: the tail node of a matching edge

**Maximum matching**: Cannot be extended

**Perfect matching**: All nodes are matched nodes

**← Perfect matching**

**← Maximum but not perfect matching**
Minimum Inputs Theorem

Q: How many?
A: The minimum number of inputs $N_D$ needed is:

Case 1: If there is a perfect matching, then
$$N_D = 1$$

Case 2: If there is no perfect matching, then
$$N_D = \text{number of unmatched nodes}$$

Q: Where to put them?
A: Case 1: Anywhere
Case 2: At unmatched nodes

A network with SISO nodes is **controllable** if and only if

- \((A,H)\) is controllable,
- \((A,C)\) is observable,
- for any \(s \in \sigma(A)\) and \(\alpha \in \Gamma(s)\), \(\alpha L \neq 0\) if \(\alpha \neq 0\),
- for any \(s \notin \sigma(A)\), \(\text{rank}(I - L\gamma, \Delta\eta) = N\), with \(\gamma = C(sI - A)^{-1}H\), \(\eta = C(sI - A)^{-1}B\).
State Controllability

A network of multi-input/multi-output (MIMO) node systems, where the node systems are of higher-dimensional
A Network of Multi-Input/Multi-Output LTI Systems

Node system
\[ \dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i, \quad x_i \in \mathbb{R}^n, \quad y_i \in \mathbb{R}^m, \quad u_i \in \mathbb{R}^p \]

Networked system
\[ \dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} Hy_j, \quad y_i = Cx_i, \quad i = 1, 2, \ldots, N \]

Networked system with external control
\[ \dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} HCx_j + \delta_i Bu_i, \quad i = 1, 2, \ldots, N \]

\[ \delta_i = 1: \text{with external control} \quad \delta_i = 0: \text{without external control} \]

Some notations
- Node system \((A,B,C)\)
- Network structure \(L = [\beta_{ij}] \in \mathbb{R}^{N \times N}\)
- Coupling matrix \(H\)
- External control inputs \(\Delta = \text{diag}(\delta_1, \ldots, \delta_N)\)

A Simple Counter-Example

\[
\begin{align*}
\dot{x}_1 &= Ax_1 + Bu_1 \\
\dot{x}_2 &= Ax_2 + Bu_2 \\
y_1 &= Cx_1 \\
y_2 &= Cx_2
\end{align*}
\]

\[
A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0]
\]

\[
(A, B, C) \text{ is state-controllable}
\]

\[
u_2 = \beta_{12} BCx_1 = \beta_{12} x_1^1 \quad (\beta_{12} \neq 0)
\]

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ \beta_{12} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1
\]

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

It is not state-controllable
More counter-intuitive examples

Network structure

\[
L = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

Node system

\[
A = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix} \\
B = \begin{bmatrix}
1 \\
0 \\
\end{bmatrix} \\
H = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix}
\]

Networked MIMO system

\[
C = \begin{bmatrix}
0 & 1 \\
\end{bmatrix}
\]

structurally controllable

\[(A, B)\) is controllable

state uncontrollable

\[(A, C)\) is observable
More counter-intuitive examples

Network structure

Node system

Networked MIMO system

\[
L = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 1 \\
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
1 \\
0 \\
\end{bmatrix}
\]

\[\beta_{21}\]

\[\beta_{12}\]

\(x^1\)

\(x^2\)

\(x^3\)

\(x_1^1\)

\(x_1^2\)

\(x_2^1\)

\(x_2^2\)

\(u_1\)

structurally controllable

\((A, B)\) is uncontrollable

state controllable

\((A, C)\) is observable

coupling matrix \(H\) is important
A necessary and sufficient condition

\[
\dot{x}_i = A x_i + \sum_{j=1}^{N} \beta_{ij} H C x_j + \sum_{k=1}^{s} \delta_{ik} B u_k, \quad x_i \in \mathbb{R}^n, \quad i = 1, \ldots N
\]
\[
y_l = \sum_{j=1}^{N} m_{lj} D x_j, \quad u_k \in \mathbb{R}^p, \quad k = 1, \ldots s
\]
\[
\Delta = [\delta_{ij}] \in \mathbb{R}^{N \times s}
\]

\[
L = [\beta_{ij}] \in \mathbb{R}^{N \times N}
\]

State Controllable

If and only if

Matrix equations

\[
\Delta^T X B = 0, \quad L^T X H C = X (\lambda I - A), \quad \forall \lambda \in \mathbb{C}
\]

have a unique solution \(X = 0\)

Research Outlook

General networks of linear time-varying (LTV) node-systems

General networks of non-identical node-systems

General temporal networks of LTI or LTV node-systems

Some special types of networks of nonlinear node-systems

......

There are more, of course
Thanks
References

References