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CONTROLLING IN BETWEEN THE LORENZ AND THE CHEN SYSTEMS

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This letter investigates a new chaotic system and its role as a joint function between two complex chaotic systems, the Lorenz and the Chen systems, using a simple variable constant controller. With the gradual tuning of the controller, the controlled system evolves from the canonical Lorenz attractor to the Chen attractor through the new transition chaotic attractor. This evolving procedure reveals the forming mechanisms of all similar and closely related chaotic systems, and demonstrates that a simple control technique can be very useful in generating and analyzing some complex chaotic dynamical phenomena.

Keywords: Chaos; Chen's system; Lorenz system; critical system.

1. Introduction

Chaos as an interesting complex dynamical phenomenon has been extensively studied within the scientific, engineering and mathematical communities for more than three decades. Recently, the traditional trend of analyzing and understanding chaos has evolved to a new phase in investigation: controlling and utilizing chaos. Research in the field of chaos control, synchronization and modeling including not only suppressing chaos when it is harmful, but also chaotification, i.e. generating chaos intentionally when it is useful. These tasks can both be carried out by means of conventional control technology [Chen, 2001; Chen & Dong, 1998; Wang & Chen, 2000].

Lorenz found the first canonical chaotic attractor in 1963, in a simple three-dimensional autonomous system [Sparrow, 1982], which has just been mathematically recently confirmed to exist [Stewart, 2000]. In 1999, Chen found another chaotic attractor [Chen & Ueta, 1999; Ueta & Chen, 2000], also a simple three-dimensional autonomous system, as the *dual* of the Lorenz system, in a sense defined by Vaněček and Čelikovský [1996]: For the

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linear part of the system, $A = [a_{ij}]_{3\times 3}$, the Lorenz system satisfies the condition $a_{12}a_{21} > 0$ while the Chen system satisfies $a_{12}a_{21} < 0$. Very recently, Lü and Chen [2002] found a new critical chaotic system, which satisfies the condition $a_{12}a_{21} = 0$ and represents the transition between the Lorenz and the Chen attractors.

We have provided a somewhat detailed dynamical analysis on this new chaotic system in [Lü *et al.*, 2002a]. Furthermore, we have found that this new chaotic attractor has a compound structure by merging together two simple attractors after performing one mirror operation [Lü *et al.*, 2002b], which is similar to the modified Lorenz system and the Chen system [Elwakil & Kennedy, 2001; Özoğuz *et al.*, 2002; Lü *et al.*, 2002c]. Meanwhile, we have pointed out that the new critical chaotic attractor is a transition between the Lorenz and the Chen attractors [Lü *et al.*, 2002a]. This letter offers further details on the observation of this interesting transition with some analysis on the joint function of this intermediate chaotic attractor.

2. The Joint Function of the New Chaotic Attractor

The aforementioned critical chaotic system is described by

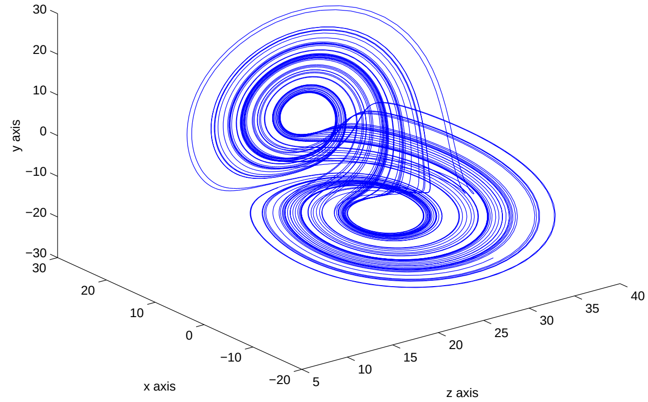
$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

which has a chaotic attractor as shown in Fig. 1(a) when $a = 36$, $b = 3$, $c = 20$.

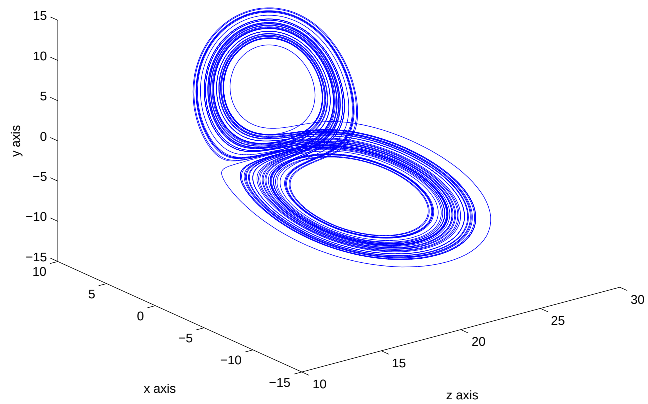
To further investigate the joint function of this new chaotic attractor, a constant control term is added to the third equation:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz + m\,. \end{cases} \tag{2}$$

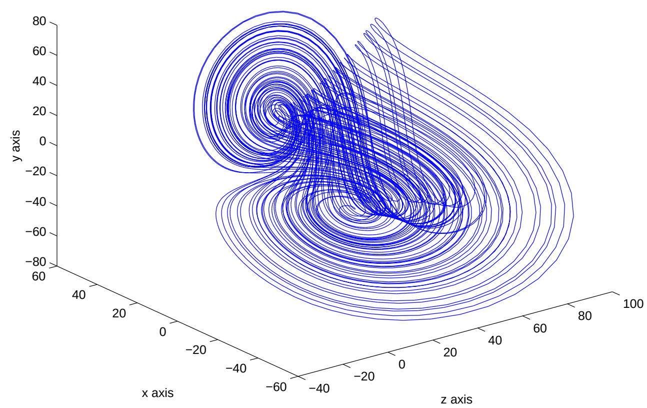
When $m = 40$, it has a similar topological structure to the Lorenz attractor, as shown in Fig. 1(b); while when $m = -300$, it has similar topological structure to the Chen attractor, as shown in Fig. 1(c).



(a)



(b)



(c)

Fig. 1. The new chaotic attractor and its variants similar to the Lorenz attractor and the Chen attractor. (a) $m = 0$, (b) $m = +40$, (c) $m = -300$.

2.1. *Some basic properties of the controlled system*

The controlled system (2) shares several important qualitative properties with the original chaotic system (1). They are further discussed in the following:

2.1.1. *Symmetry and invariance*

At first, it is easy to notice the invariance of the system under the transformation $(x, y, z) \to (-x, -y, z)$, i.e. under the reflection about the z-axis. The symmetry persists for all values of the system parameters a, b, c and m. Also, it is clear that the z-axis itself is an orbit, i.e. if $x = y = 0$ at $t = 0$ then $x = y = 0$ for all $t > 0$. Furthermore, the trajectory on the z-axis tends to the origin as $t \to \infty$, since for such a trajectory, $dx/dt = dy/dt = 0$ and $dz/dt = -bz + m$. Therefore, the controlled system (2) shares the symmetry and invariance with system (1) for various values of m.

2.1.2. *Dissipativity and the existence of attractor*

For system (2), one has

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(a + b - c)\,.$$

Hence, with $a+b > c$, system (2) is dissipative, with an exponential contraction rate:

$$\frac{dV}{dt} = e^{-(a+b-c)}\,.$$

That is, a volume element V_0 is contracted by the flow into a volume element $V_0 e^{-(a+b-c)}$ in time t. This means that each volume containing the system trajectory shrinks to zero as $t \to \infty$ at an exponential rate $-(a + b - c)$. Therefore, all system orbits are ultimately confined into a specific subset of zero volume, and this asymptotic motion settles onto an attractor. Thus, when $a + b > c$, the controlled system (2) has the same dissipativity as the original system (1) for any value of m.

2.2. *Equilibria and bifurcations*

In the following, assume that the parameters a, b and c are all positive. The equilibria of system (2) can be easily found by solving the three equations $\dot{x} = \dot{y} = \dot{z} = 0$, which lead to

$$a(y-x) = 0\,, \quad -xz+cy = 0 \quad \text{and} \quad xy-bz+m = 0\,.$$

It can be easily verified that the equilibria of system (2) are:

(i) when $m \geq bc$, there is only one equilibrium: $S_0(0,\, 0,\, m/b)$;

(ii) when $m < bc$, there are three equilibria:

$$S_0(0,\, 0,\, m/b)$$
$$S_-(-\sqrt{bc-m},\, -\sqrt{bc-m},\, c)$$
$$S_+(\sqrt{bc-m},\, \sqrt{bc-m},\, c)\,.$$

Note that the null solution is not any longer an equilibrium of system (2) if $m \neq 0$. Pitchfork bifurcation of the equilibrium S_0 at $m = bc$ can be observed. The other equilibria, S_- and S_+, are symmetrically placed with respect to the z-axis.

Linearizing the controlled system (2) about the equilibrium S_0 provides an eigenvalue $\lambda_1 = -b$ along with the following characteristic equation:

$$f(\lambda) = \lambda^2 + (a - c)\lambda + \frac{a(m - bc)}{b} = 0\,. \qquad (3)$$

When $m < bc$, the two eigenvalues satisfy $\lambda_2 > 0 > \lambda_3$, so the equilibrium S_0 is a saddle point in three-dimension; when $m > bc$ and $a < c$, the equilibrium S_0 is a saddle; when $m > bc$ and $a > c$, the equilibrium S_0 becomes a sink.

Next, linearizing the system about the other equilibria yields the following characteristic equation:

$$f(\lambda) = \lambda^3+(a+b-c)\lambda^2+(ab-m)\lambda+2a(bc-m) = 0\,. \qquad (4)$$

Obviously, the two equilibria $S_\pm$ have the same stability. The Routh–Hurwitz conditions lead to the conclusion that the real parts of the roots λ are negative if and only if

$$a + b - c > 0\,, \quad 2a(bc - m) > 0\,,$$

and

$$(a + b - c)(ab - m) - 2a(bc - m) > 0\,.$$

Therefore:

(i) if $a+c > b$, then when $m > (ab(3c-a-b)/(a+c-b))$, the equilibria $S_\pm$ are sinks;

(ii) if $a+c < b$, then when $m < (ab(3c-a-b)/(a+c-b))$, the equilibria $S_\pm$ are sinks.

In the following, assume that $a > c$. For the equilibria $S_\pm$, one has $bc - m > 0$ and $ab - m > bc - m > 0$. Note that the coefficients of the cubic polynomial (4) are all positive. Therefore, $f(\lambda) > 0$ for all $\lambda > 0$. Consequently, there is instability ($\mathrm{Re}(\lambda) > 0$) only if there are two complex conjugate zeros of f.

Now, it is clear that when $m = bc$, the three zeros are $\lambda = 0, -(a-c), -b$, and therefore the system has linear stability or marginal stability. The first zero gives $\lambda \sim -(2a(bc - m)/(ab - m))$, as $m \uparrow bc$, so stability is lost in the limit as m approaches bc from below. As m decreases from bc, instability can set in only when $\mathrm{Re}(\lambda) = 0$, i.e. when two zeros are

$\lambda = \pm\omega i$ for some real ω. But the sum of three zeros of the cubic polynomial f is

$$\lambda_1 + \lambda_2 + \lambda_3 = -(a + b - c)\,.$$

Hence, $\lambda_3 = -(a+b-c)$. On the margin of stability, $\lambda = \pm\omega i$, so that, on this margin,

$$0 = f(-(a+b-c)) = ab(3c - a - b) + m(b - a - c)\,,$$

that is,

$$m_h = \frac{ab(3c - a - b)}{a + c - b}\,. \tag{5}$$

In fact, if there is instability then as m decreases from bc the following phenomenon can be observed: λ_1 decreases from zero until it coalesces with λ_2 (when $\lambda_1 = \lambda_2 < 0$); then they become a complex conjugate pair, and eventually their real part increases through zero; while λ_3 remains negative for all $m < bc$. One can thus see that each of the points S_+ and S_-, when being unstable, has one negative eigenvalue and two complex conjugate eigenvalues. So this equilibrium is a saddle-focus.

Hopf bifurcations emerge from the value of $m_h = (ab(3c - a - b)/(a + c - b))$, where the complex conjugate eigenvalues are $\lambda = \pm\sqrt{(2ab(a-c)/(a+c-b))}i$ (with $a + c > b$). When $m > m_h$, S_+ and S_- are both stable sinks. At $m = m_h$, however, they change to two two-dimensional unstable saddles. If $a = 36$ and $b = 3$ are fixed while c and m are varied, then one can observe the continuous Hopf bifurcations, as shown in Fig. 2.

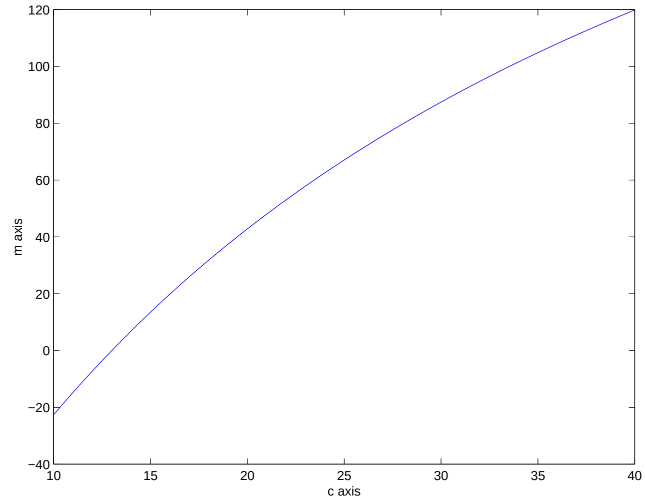


Fig. 2. The continuous Hopf bifurcations of system (2).

Table 1. A summary of the controller parameter range for behaviors of system (2), as determined by both theory and computation.

For $m > 43.8$, the system converges to a point;
For $43.6 < m \le 43.8$, there exists the onset of chaos [see Fig. 3(a)];
For $42.79 < m \le 43.5$, there exist a chaotic attractor and a pair of stable attracting rest points $S_\pm$ [see Fig. 3(b)];
For $25 < m \le 42.79$, the attractor is similar to the Lorenz attractor [see Fig. 1(b)];
For $-30 < m < 25$, the attractor is a transition attractor [see Fig. 1(a)];
For $-785 < m < -30$, the attractor is similar to the Chen attractor [see Fig. 1(c)];
For $-1043 \le m < -785$, there are period-doubling bifurcations [see Figs. 3(c)–3(e)];
For $-900.5 \le m \le -897.8$, there is a periodic window [see Fig. 3(f)];
For $-10^5 < m < -1043$, there is a limit cycle [see Fig. 3(g)];
For $-10^5 \le m \le -10^6$, there is an attractor [see Fig. 3(h)].

2.3. *Dynamical behaviors of the controlled system*

To investigate the joint function of the new chaotic attractor and to clarify the relationship between the Lorenz and the Chen attractors, the dynamical behavior of the controlled system (2) is further studied here.

By varying the variable constant control input m, as listed in Table 1, one can observe different dynamical behaviors of the controlled system.

It can be seen from Table 1 that (i) when m is large enough (e.g. $m \ge 43.8$), the system converges to a point; (ii) when m decreases gradually, system (2) enters into a chaotic region. Especially, the procedure is quite similar to the Lorenz system [Sparrow, 1982], and the attractor is an invariant of the Lorenz attractor; (iii) when m is relatively small, system (2) goes through the transition gradually, generating the joint chaotic attractor; (iv) when m is small enough, system (2) enters into another chaotic region, and the attractor has similar topological structure with the Chen attractor. Here, it is interesting to see that the invariant of the Chen attractor is also produced by period-doubling bifurcations.

Both Table 1 and Fig. 3 show that the new chaotic attractor has a joint function. Indeed, the controlled system (2) represents the transition from one system to another when the key control parameter m is slowly varied. Moreover, the routes to chaos and bifurcations in this system are both similar to that of the original Lorenz and the Chen attractors [Sparrow, 1982; Ueta & Chen, 1999].

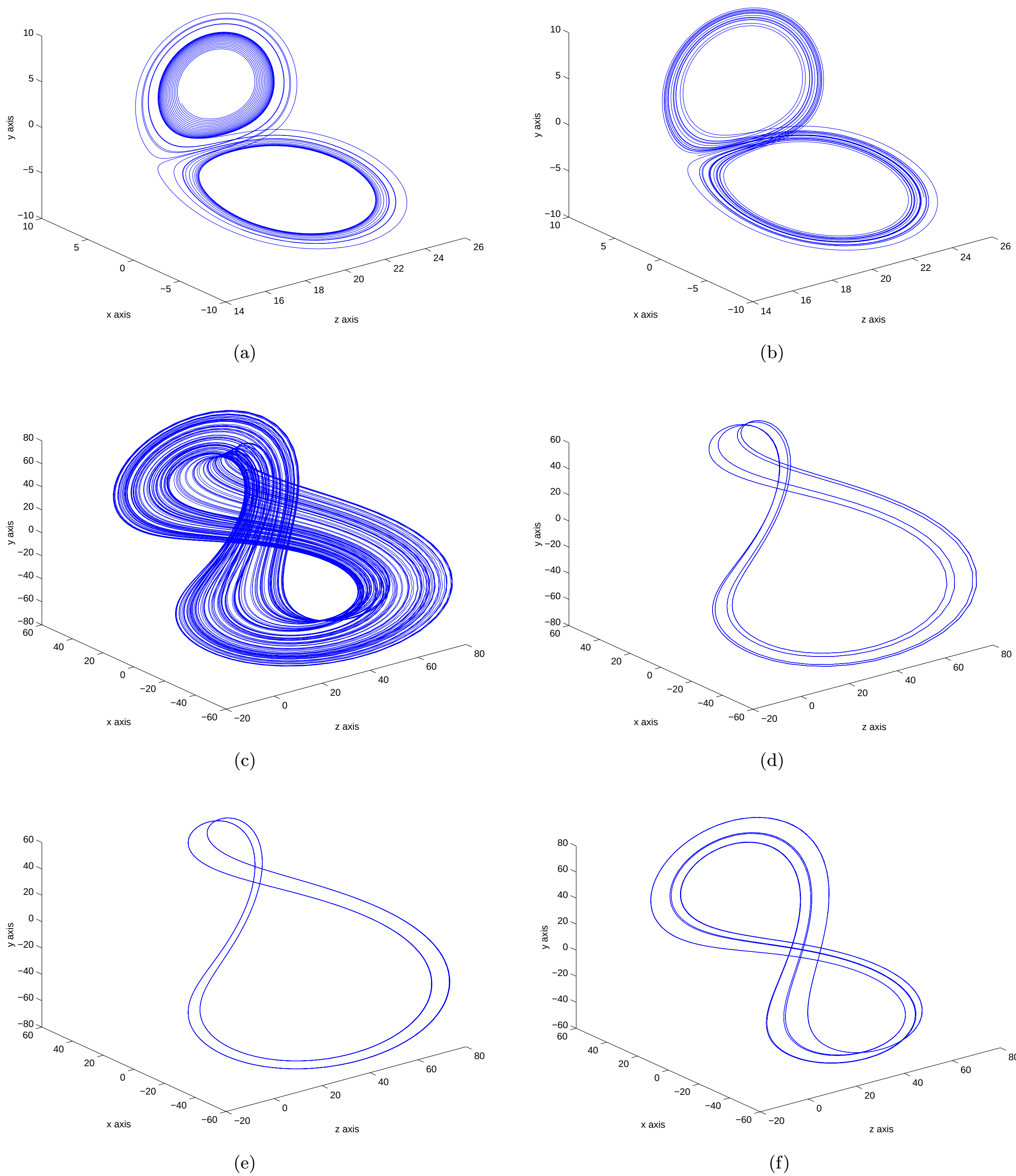
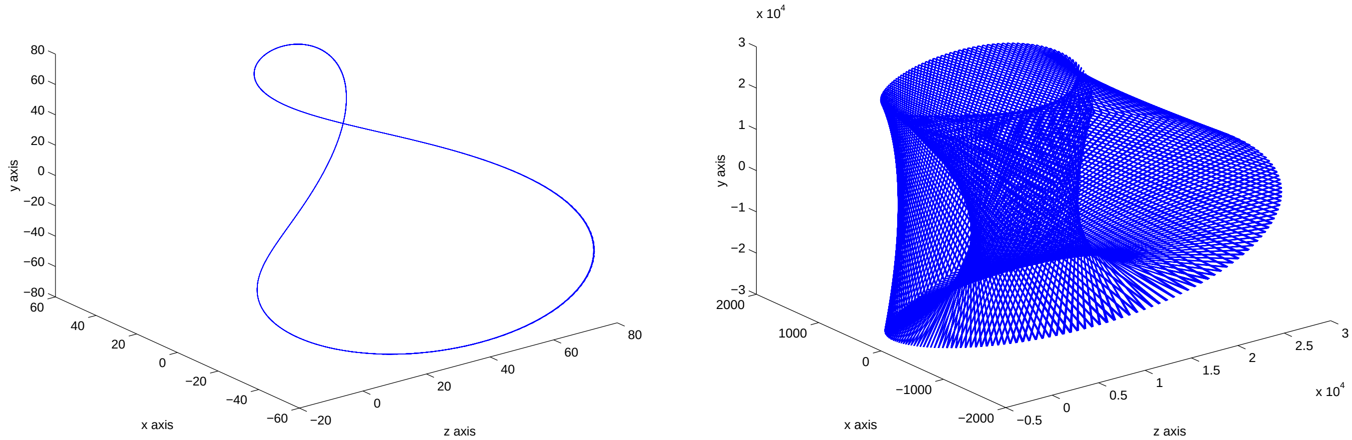


Fig. 3. The phase portraits of the controlled system (2). (a) $m = 43.8$, (b) $m = 43$, (c) $m = -800$, (d) $m = -950$, (e) $m = -10\,000$, (f) $m = -900$, (g) $m = -1100$, (h) $m = -10^6$.



(g)

(h)

Fig. 3. (*Continued*)

3. Conclusions

Recently, a new chaotic attractor connecting the dual Lorenz and Chen attractors is coined. In addition to the previously given analysis of its dynamical behavior and compound structure, this Letter has further studied the joint function of this new attractor and explored the relationship between the Lorenz and the Chen attractors. This new attractor has contributed to a better understanding of all similar and closely related chaotic systems, and therefore deserves further investigation in the near future.

Acknowledgments

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