Introducing the Generalized Lorenz Systems Family: Theory and Applications

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To the Memory of

Father of Chaos

Edward N. Lorenz

(23 May 1917 – 16 April 2008)
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Contents

- Lorenz System
- Chen System
- Generalized Lorenz System
- Hyperbolic Generalized Lorenz System
- Generalized Lorenz Systems Family
- Hyperchaotic Chen System
- Fractional-Order Chen System
- Conclusions
The Lorenz System

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= cx - xz - y \\
\dot{z} &= xy - bz, \\
\end{align*}
\]

\[a = 10, b = \frac{8}{3}, c = 28\]

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Any Extension or Connection?

Lorenz (1963)  

3-D Autonomous with  
1 or 2 Quadratic Terms  

Sprott (1997)  

Rössler (1976)  

(others)
Lorenz System

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= cx - xz - y + u \\
\dot{z} &= xy - bz 
\end{align*}
\]

Chen System

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x - xz + cy \\
\dot{z} &= xy - bz 
\end{align*}
\]

\[u = -ax + (1 - c)y + 0z\]

The Chen System is described by the following system of equations:

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x - xz + cy \\
\dot{z} &= xy - bz,
\end{align*}
\]

with parameters \(a = 35\), \(b = 3\), and \(c = 28\).

Some Comparisons:

Stable Manifolds

Lorenz Attractor

Chen Attractor

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Remark 1:

Equivalence  Lorenz  ↔  Chen?  


  
  Lorenz system and Chen system are not smoothly equivalent
  (i.e., no diffeomorphism between them)

Q: Are Lorenz system and Chen system topologically equivalent
  (i.e., any homeomorphism between them)?
Remark 2: Global Boundedness

Early Attempt: G. R. Chen, W. X. Qin, J. A. Lu, D. M. Li, …, R. Barboza

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Proof

Existence of a Chaotic Attractor

Shilnikov Theorem (1967):

If a 3D autonomous system has two distinct saddle fixed points and there exists a heteroclinic orbit connecting them, and if the eigenvalues of the Jacobin of the system at these fixed points are

\[ \alpha_k, \beta_k \pm j \omega_k \quad (k = 1, 2) \quad \text{satisfying} \quad |\alpha_k| > |\beta_k| > 0 \quad (k = 1, 2) \]

and \( \beta_1 \beta_2 > 0 \) or \( \omega_1 \omega_2 > 0 \) then the system has infinitely many Smale horseshoes and hence has horseshoe chaos.
Proof

Show the existence of a heteroclinic orbit between two saddle-focus fixed points (a constructive approach)

- Start from a series expansion of the heteroclinic orbit
- Substituting it into the characteristic equation of the system
- Force it to satisfy the basic properties as a heteroclinic orbit
- Force it to satisfy the Shilnikov conditions
- Guarantee the uniform convergence of the series expansion

Circuit Implementation

Electronic Attractor

Chen Attractor
Some Applications
Generalized Lorenz System

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & 0 \\
0 & 0 & a_{33}
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

According to the C-V Canonical Form –

**Lorenz System satisfies:** \( a_{12}a_{21} > 0 \)

**Chen System satisfies:** \( a_{12}a_{21} < 0 \)

**Q:** What system satisfies \( a_{12}a_{21} = 0 \)?

\[ \begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= -xz + cy \\
\dot{z} &= xy - bz,
\end{align*} \]

\[ a = 36; \quad b = 3; \quad c = 20 \]

\[ a_{12} a_{21} = 0 \]

A Unified Chaotic System

\[
\begin{align*}
\dot{x} &= (25\alpha + 10)(y - x) \\
\dot{y} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y \\
\dot{z} &= xy - \frac{1}{3}(\alpha + 8)z,
\end{align*}
\]

where \( a \in [0,1] \)

When \( \alpha = 0, \alpha = 1, \alpha = 0.8 \)

it becomes the Lorenz, Chen, or Lü system, respectively.

Experimental Observations

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**Generalized Lorenz Canonical Form**

\[
\begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix} + \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & -1 \\
1 & \tau & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix}, \quad \lambda_1 > 0, \; \lambda_{2,3} < 0,
\]

where

\[
z = [z_1, \; z_2, \; z_3]^T, \; c = [1, \; -1, \; 0] \quad -\lambda_2 > \lambda_1 > -\lambda_3 > 0, \quad \tau \in R
\]

- **Lorenz:** \( 0 < \tau < +\infty \)
- **Lü:** \( \tau = 0 \)
- **Chen:** \( -1 < \tau < 0 \)
- **?:** \( \tau \leq -1 \)


Proof

Show the existence of a heteroclinic orbit between two saddle-focus fixed points (a constructive approach)

- Start from a series expansion of the heteroclinic orbit
- Substituting it into the characteristic equation of the system
- Force it to satisfy the basic properties as a heteroclinic orbit
- Force it to satisfy the Shilnikov conditions
- Guarantee the uniform convergence of the series expansion
Hyperbolic Generalized Lorenz Canonical Form

\[ \dot{x} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} x + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\text{sgn}(\tau + 1) \\ 0 & 1 & 0 \end{bmatrix} x, \]

where \( x = [x_1, x_2, x_3]^T \), \( \lambda_1 > 0, \lambda_{2,3} < 0 \), \( \tau \leq -1 \)

**Lorenz:** Eigenvalues = \( \{0, \pm j\} \)  \quad **HGLC:** Eigenvalues = \( \{0, \pm 1\} \)

The case of $\tau = -1$

Shimizu-Morioka Model (1976):

\[
\begin{aligned}
\frac{dx}{d\theta} &= y \\
\frac{dy}{d\theta} &= x(1 - z) - \lambda y \\
\frac{dz}{d\theta} &= -\alpha z + x^2
\end{aligned}
\]

It is the case of the Generalized Lorenz Canonical Form with $\tau = -1$

\[
\begin{aligned}
x &= (z_1 - z_2) \sqrt[3]{\frac{\lambda_1 - \lambda_2}{(-\lambda_1 \lambda_2)^{3/2}}} \\
y &= (\lambda_1 z_1 - \lambda_2 z_2) \sqrt[5]{\frac{\lambda_1 - \lambda_2}{(-\lambda_1 \lambda_2)^{5/2}}} \\
z &= z_3 - \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} \\
\theta &= t \sqrt{-\lambda_1 \lambda_2},
\end{aligned}
\]

\[\lambda = -\frac{\lambda_1 + \lambda_2}{\sqrt{-\lambda_1 \lambda_2}}, \quad \alpha = \frac{\lambda_3}{\sqrt{-\lambda_1 \lambda_2}}.\]

A Special Case

Liu-Liu-Liu-Liu Model (Xi’an JTU, 2004):

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= bx - kxz \\
\dot{z} &= -cz + hx^2,
\end{align*}
\]

where \( a, b, c, k, h > 0 \)

It is a special case of the Shimizu-Morioka Model (1976). Therefore,

It is a special case of the Generalized Lorenz Canonical Form with \( \tau = -1 \)

# Summary

**Generalized Lorenz Canonical Form (GLCF) and Its Special Realization**

<table>
<thead>
<tr>
<th>GLCF</th>
<th>Special Chaotic Systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau \in (-\infty, -1)$</td>
<td>Hyperbolic Generalized Lorenz System (2002)</td>
</tr>
<tr>
<td>$\tau = -1$</td>
<td>Shimizu-Morioka System (1979)</td>
</tr>
<tr>
<td>$\tau \in (-1, 0)$</td>
<td>$a_{21}a_{12} &lt; 0$ Chen System (1999)</td>
</tr>
<tr>
<td>$\tau = 0$</td>
<td>$a_{21}a_{12} = 0$ Lü System (2002)</td>
</tr>
<tr>
<td>$\tau \in (0, \infty)$</td>
<td>$a_{21}a_{12} &gt; 0$ Lorenz System (1963)</td>
</tr>
</tbody>
</table>

广义 Lorenz 系统族
Transition Between 
Lorenz and Chen Attractors
Controlling Chaotic Chen System To Hyperchaotic

- Using a simple dynamical state-feedback controller
- Using a simple sinusoidal parameter perturbation input
From Chaos
To Hyperchaos

Controlled Generalized Lorenz Canonical Form:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{21} & a_{22} & 0 & 1 \\
0 & 0 & a_{33} & 0 \\
-k & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
u
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
u
\end{bmatrix}
\]

where \( u \) is the controller and \( k \) is the constant control gain to be determined (a very simple dynamical linear state feedback controller).
Controlling the Chen System

Parameters:

\[ a_{11} = -a_{12} = -35, \quad a_{21} = 7, \quad a_{22} = 12, \]
\[ a_{33} = -3, \quad k = 20 \]

(a) \( x \) vs \( z \)

(b) \( y \) vs \( z \)
Bifurcation Analysis

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Visualizing the Bifurcation Process
Visualizing the Bifurcation Process
Visualizing the Bifurcation Process

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Circuit Implementation
(Hyperchaotic Chen system)
G Chen: Generalized Lorenz systems family
G Chen: Generalized Lorenz systems family
**Sinusoidal Control Input**

The Controlled Unified Chaotic Systems:

\[
\begin{align*}
\dot{x} &= (25 - 10a) (y - x) \\
\dot{y} &= (17.5a + 10.5)x - \text{sign}(a) xz + (13.3 - 14a)y \\
\dot{z} &= \text{sign}(a) xy - \frac{8}{3} z
\end{align*}
\]

where \( a = \cos(\omega t) \in [-1,1] \) and \( \text{sign}(u) = \begin{cases} 
1 & u \geq 0 \\
-1 & u < 0 
\end{cases} \)
According to the canonical-form criterion:

\[ a = \cos(\omega t) \in [-1,1] \]

\[ t \in \left[ \frac{1}{\omega} \left( 2n\pi + \frac{\pi}{2} \right), \frac{1}{\omega} \left( 2n\pi + \frac{3\pi}{2} \right) \right] \rightarrow \text{Generalized Lorenz System} \]

\[ t \in \left[ \frac{1}{\omega} \left( 2n\pi - \frac{\pi}{2} \right), \frac{1}{\omega} \left( 2n\pi + \frac{\pi}{2} \right) \right] \rightarrow \text{Generalized Chen System} \]
Bifurcation Analysis
Experimental Results
Observing Hyperchaotic Chen Attractor

Hyperchaotic Chen Attractor

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Fractional-Order Chen System

\[
\begin{aligned}
\frac{d^\sigma x}{dt^\sigma} &= a(y - x) \\
\frac{d^\sigma y}{dt^\sigma} &= (c - a)x - xz + cy \\
\frac{d^\sigma z}{dt^\sigma} &= xy - bz,
\end{aligned}
\]

When \( \sigma = 0.6 \sim 0.7 \) this system is **chaotic** [J. G. Lu and G. Chen (2006): \( \sigma = 0.3 \)]

Fractional-Order Chen System

\[
\frac{d^\alpha x}{dt^\alpha} = a(y - x) + \gamma \cos(u), \\
\frac{d^\alpha y}{dt^\alpha} = (c - a)x - xz + cy, \\
\frac{d^\alpha z}{dt^\alpha} = xy - bz, \\
\frac{du}{dt} = \omega,
\]

When \( a = 35, b = 3, c = 32, \gamma = 35, \alpha = 0.8, \omega = 15 \) this system is hyperchaotic

Some Potential Applications

Chen-system-based hyperchaotic mixer

Related Technology:

- 吕金虎、禹思敏、陈关荣 (专利): 一种四阶网格状多环面混沌电路及其使用方法; 2005年申请，2009年批准；专利号 ZL2005 1 0086638.2.

Concluding Remarks

3-D Autonomous with 1 or 2 Quadratic Terms

Lorenz Done!

Rössler ?

Sprott ?

(Others) ?
with the compliments of

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