***** **Mathematical Weekly** *****

(Week 27)

Some Amazing Squares and Powers

1. First, this one is easy:

$$1+3=2^2$$
$$1+3+5=3^2$$
$$1+3+5+7=4^2$$
$$1+3+5+7+9=5^2$$

Can you find more?

2. Second, this one is a bit harder:

$$1233=12^2+33^2$$
$$8833=88^2+33^2$$

Can you find more?

3. Third, this one is even more difficult:

$$956^2=913936 \quad \Rightarrow \quad 913+936=43^2$$
$$957^2=915849 \quad \Rightarrow \quad 915+849=42^2$$
$$858^2=917754 \quad \Rightarrow \quad 917+764=41^2$$
$$959^2=919681 \quad \Rightarrow \quad 919+681=40^2$$
$$960^2=921600 \quad \Rightarrow \quad 921+600=39^2$$
$$961^2=923521 \quad \Rightarrow \quad 923+521=38^2$$

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$$968^2=937024 \quad \Rightarrow \quad 937+024=31^2$$

Can you find more?

Well, I can only find one similar but not exactly:

$$1111111111^2 = 1234567890098765432 1$$
$$\Rightarrow 1234567890+0987654321=1111111111$$

4. Next, let's try third powers:

$$512 = (5+1+2)^3$$
$$4913 = (4+9+1+3)^3$$
$$5832 = (5+8+3+2)^3$$
$$17576 = (1+7+5+7+6)^3$$
$$19683 = (1+9+6+8+3)^3$$

Can you find more?

5. Finally, these are really hard:

$$1+3+3^2+3^3+3^4 = 11^2$$
$$1+7+7^2+7^3 = 20^2$$

Can you find more in the form of $1+n+n^2+n^3+\cdots+n^k = m^2$ for $n,k>1$? Well, Norwegian mathematician Wilhelm Ljunggren in 1943 proved that you can't! He also proved that $1+18+18^2 = 7^3$ is the only one of this type if the right-hand side has power 3. Moreover, no one knows what happens if the right-hand power is larger.

Likewise, can you find 3 integers (x, y, z) from $\{1,2,3,4,5,6,7,8,9\}$ to satisfy the following identity?

$$x^2 - y^2 - z^2 = x - y - z$$

It is said that there are two and only two solutions. Can you find them?

GRC ☺