Nonlinear Systems

Stability, Dynamics and Control

The topic of nonlinear systems is fundamental to the study of systems engineering. So extensive investigations have been carried out by both the nonlinear control and nonlinear dynamics communities, but the focus can be different — on controllers design and dynamics analysis, respectively. The last two decades have witnessed the gradual merging of control theory and dynamics analysis, but not yet to the extent of controlling nonlinear dynamics such as bifurcations and chaos. This monograph is an attempt to fill that gap while presenting a rather comprehensive coverage of the fundamental nonlinear systems theory in a self-contained and approachable manner.

This introductory treatise is written for self-study and, in particular, as an elementary textbook that can be taught in a one-semester course to advanced undergraduates or entrance level graduates with curricula focusing on nonlinear systems, both on control theory and dynamics analysis.

World Scientific
www.worldscientific.com
13488 hc

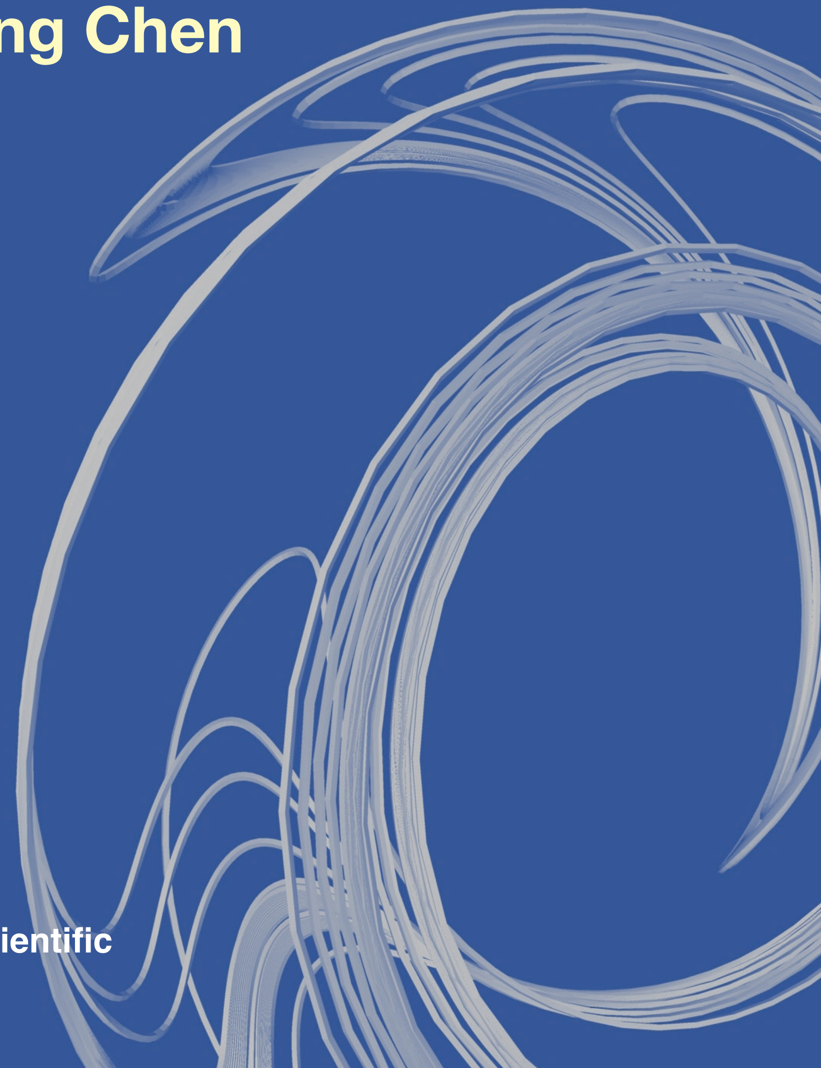
ISBN 978-981-127-898-3
9 789811 278983

Nonlinear Systems
Chen

Nonlinear Systems

Stability, Dynamics and Control

Guanrong Chen



World Scientific

Dedicated to the memory of

Professor Gennady Alexeyevich Leonov (1947–2018)

Preface

Nonlinear systems constitute a fundamental subject for study in systems engineering. The subject has been extensively investigated by both nonlinear control and nonlinear dynamics communities, whereas the focus is usually very different, on controllers design and dynamics analysis, respectively. The last two decades have witnessed gradual merging of control theory and dynamics analysis, but not to the extent of controlling nonlinear dynamics such as bifurcations and chaos. This monograph is an attempt to fill the gap to a certain extent while presenting a rather comprehensive coverage of the nonlinear systems theory in a self-contained and hopefully easily-readable manner.

This introductory treatise is not intended to be a research reference with the state-of-the-art theories and techniques presented, nor as a very comprehensive handbook, given that there are already many available in the market today. It is written for self-study and, in particular, as an elementary textbook that can be taught in a one-semester course at the advanced undergraduate level or entrance level of graduate curricula focusing on nonlinear systems — both control theory and dynamics analysis.

The main contents of the book comprise systems stability (Chapters 2–4), bifurcation and chaos dynamics (Chapter 5) and controllers design (Chapter 6), for both continuous-time and discrete-time settings. In particular, it discusses the special topics on bifurcation control and chaos control at the end of the last chapter.

This monograph is presented in a textbook style, in which most contents are elementary with some classical results and popular examples taken or modified from the existing literature, which might have also appeared in some other introductory textbooks. Since this is not a survey, a long list of related references is not included, yet appreciation to the various original sources are indicated. Throughout the book, to keep its contents at an elementary level, some advanced theories are presented without detailed proofs, merely for the completeness of the relevant discussions. This kind of contents and exercises are marked by * for indication.

To this end, I would like to especially thank Dr. Yi Jiang from the City University of Hong Kong for helping proof-read the entire manuscript, and thank Ms. Lakshmi Narayanan from the World Scientific Publishing Company for her support and assistance.

Guanrong Chen
Hong Kong, 2023

Contents

Chapter 1

Nonlinear Systems: Preliminaries

A *nonlinear system* in the mathematical sense refers to a set of nonlinear equations, which can be algebraic, difference, differential, integral, functional, and operator equations, or a combination of some of them. A nonlinear system is used to describe a physical device or process that otherwise cannot be well defined by a set of linear equations of any kind, although a linear system is considered as a special case of a nonlinear system. *Dynamical system*, on the other hand, is used as a synonym of a mathematical or physical system, where the output behavior evolves with time and sometimes with other varying system parameters as well. The system responses, or behaviors, of a dynamical system is referred to as *system dynamics*.

1.1 A Typical Nonlinear Dynamical Model

A representative mathematical model of nonlinear dynamical systems is the pendulum equation. The study of pendula can be traced back to as early as Christian Huygens who investigated in 1665 the perfect synchrony of two identical pendulum clocks that he invented in 1656. He then reported his findings to the Royal Society of The Netherlands [Huygens (1665)].

A simple and idealized pendulum consists of a volumeless ball and a rigid and massless rod, which is connected to a pivot, as shown in Fig. 1.1. In this figure, ℓ is the length of the rod, m is the mass of the ball, g is the constant of gravity acceleration, $\theta = \theta(t)$ is the angle of the rod with respect to the vertical axis, and $f = f(t)$ is the resistive force applied to the ball. The straight down position finds the ball at rest; but if it is displaced by an angle from the reference axis and then let go, it will swing back and forth on a circular arc within a vertical plane to which it is confined.

For general purpose of mathematical analysis of the pendulum, a basic

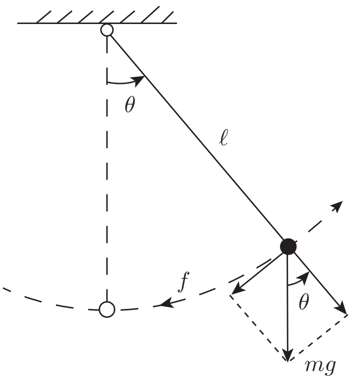


Fig. 1.1 Schematic diagram of a pendulum model.

assumption is that the resistive force f is proportional to the velocity along the arc of the motion trajectory of the volumeless ball, i.e. $f = \kappa\,\dot{s}$, where $\kappa \geq 0$ is a constant and $s = s(t) = \ell\,\theta(t)$ is the ball-traveled arc length measured from the vertical reference axis.

It follows from Newton's second law of motion that

$$m\,\ddot{s} = -\,mg\,\sin(\theta) - \kappa\,\dot{s}\,,$$

or

$$\ddot{\theta} + \frac{\kappa}{m}\,\dot{\theta} + \frac{g}{\ell}\,\sin(\theta) = 0\,. \tag{1.1}$$

This is the idealized and damped pendulum equation, which is nonlinear due to the involvement of the sine function. When $\kappa = 0$, i.e. $f = 0$, it becomes the undamped pendulum equation

$$\ddot{\theta} + \frac{g}{\ell}\,\sin(\theta) = 0\,. \tag{1.2}$$

To this end, by introducing two new variables,

$$x_1 = \theta \qquad \text{and} \qquad x_2 = \dot{\theta}\,,$$

the damped pendulum equation can be rewritten in the following state-space form:

$$\begin{aligned}\dot{x}_1 &= x_2\,,\\ \dot{x}_2 &= -\,\frac{\kappa}{m}\,x_2 - \frac{g}{\ell}\,\sin(x_1)\,.\end{aligned} \tag{1.3}$$

Here, the variables $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are called the *system states*, for they describe the physical states, namely the angular position and angular velocity respectively, of the pendulum.

From elementary physics, it is known that the pendulum state vector $\mathbf{x}(t) = [x_1(t) \;\; x_2(t)]^\top$ is periodic. In general, even in a higher-dimensional case, a state $\mathbf{x}(t)$ of a dynamical system is a *periodic solution* if it is a solution of the system and moreover satisfies $\mathbf{x}(t + t_p) = \mathbf{x}(t)$ for some constant $t_p > 0$. The least value of such t_p is called the (*fundamental*) *period* of the periodic solution, while the solution is said to be t_p-*periodic*.

Although conceptually straightforward and formally simple, this pendulum model has many important and interesting properties. This representative model of nonlinear systems will be frequently referred to, not only within this chapter but also throughout the book.

1.2 Autonomous Systems and Map Iterations

The pendulum model (1.3) is called an *autonomous system*, in which there is no independent (or separated) time variable t, other than a time variable as the system states, anywhere in the model formulation. On the contrary, the following forced pendulum is *nonautonomous*:

$$\begin{aligned} \dot{x}_1 &= x_2 \,, \\ \dot{x}_2 &= -\frac{\kappa}{m}\, x_2 - \frac{g}{\ell}\, \sin(x_1) + h(t)\,, \end{aligned} \tag{1.4}$$

since t exists as a variable in the function $h(t)$, independent of the system states, which in this example is an external force applied to the pendulum. One example of such a force input is $h(t) = a\,\cos(\omega t)$, which will change the angular acceleration of the pendulum, where a and ω are some constants.

The forced pendulum (1.4) has a time variable, t, within the external force term $h(t)$, which may not be shown as the time variables in the system states x_1 and x_2 for brevity. However, if $h(t) = a\,\cos(\theta(t))$, then the forced pendulum is considered to be autonomous because the time variable in the force term becomes the time variable of the system state $x_1(t)$. In the latter case, the external input should be denoted as $h(x_1)$ instead of $h(t)$ in the system equations.

In general, an n-dimensional autonomous system is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \mathbf{p})\,, \qquad \mathbf{x}_0 \in R^n\,, \qquad t \in [t_0, \infty)\,, \tag{1.5}$$

and a nonautonomous system is expressed as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \mathbf{p})\,, \qquad \mathbf{x}_0 \in R^n\,, \qquad t \in [t_0, \infty)\,, \tag{1.6}$$

where $\mathbf{x} = \mathbf{x}(t) = [x_1(t) \;\; \cdots \;\; x_n(t)]^\top$ is the *state vector*, $\mathbf{x}_0$ is the *initial state* with *initial time* $t_0 \geq 0$, $\mathbf{p}$ is a vector of *system parameters*, which can be varied but are independent of time, and

$$\mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}$$

is called the *system function* or *vector field*.

In a well-formulated mathematical model, the system function should satisfy some defining conditions such that the model, for example (1.5) or (1.6), has a unique solution for each initial state $\mathbf{x}_0$ in a region of interest, $\Omega \subseteq R^n$, and for each permissible set of parameters $\mathbf{p}$. According to the elementary theory of ordinary differential equations, this is ensured if the function $\mathbf{f}$ satisfies the *Lipschitz condition*

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| \leq \alpha \, ||\mathbf{x} - \mathbf{y}||$$

for all $\mathbf{x}$ and $\mathbf{y}$ in Ω that satisfy the system equation, and for some constant $\alpha > 0$, called the *Lipschitz constant.* Here and throughout the book, $|| \cdot ||$ denotes the standard Euclidean norm (the "length") of a vector, i.e. the L_2-norm. This general setting, i.e. with the fulfillment of some necessary defining conditions for a given mathematical model, will not be repeatedly mentioned and described later on, for simplicity of presentation.

Sometimes, an n-dimensional continuous-time dynamical system is described by a time-varying map,

$$F_c(t): \quad \mathbf{x} \rightarrow \mathbf{g}\big(\mathbf{x}, t; \mathbf{p}\big), \qquad \mathbf{x}_0 \in R^n, \qquad t \in [t_0, \infty), \tag{1.7}$$

or, by a time-invariant map,

$$F_c: \quad \mathbf{x} \rightarrow \mathbf{g}\big(\mathbf{x}; \mathbf{p}\big), \qquad \mathbf{x}_0 \in R^n, \qquad t \in [t_0, \infty). \tag{1.8}$$

These two maps, in the continuous-time case, take a function to a function; so by nature they are *operators*, which however will not be further studied in this book.

For the discrete-time case, with similar notation, a nonlinear dynamical system is either described by a time-varying difference equation,

$$\mathbf{x}_{k+1} = \mathbf{f}_k\big(\mathbf{x}_k; \mathbf{p}\big), \qquad \mathbf{x}_0 \in R^n, \qquad k = 0, 1, \ldots, \tag{1.9}$$

where the subscript of $\mathbf{f}_k$ signifies the dependence of $\mathbf{f}$ on the discrete time variable k, or described by a time-invariant difference equation,

$$\mathbf{x}_{k+1} = \mathbf{f}\big(\mathbf{x}_k; \mathbf{p}\big), \qquad \mathbf{x}_0 \in R^n, \qquad k = 0, 1, \ldots. \tag{1.10}$$

Also, it may be described either by a time-varying map,

$$F_d(k): \quad \mathbf{x}_k \rightarrow \mathbf{g}_k\big(\mathbf{x}_k; \mathbf{p}\big), \qquad \mathbf{x}_0 \in R^n, \qquad k = 0, 1, \ldots, \tag{1.11}$$

or by a time-invariant map,

$$F_d: \quad \mathbf{x}_k \to \mathbf{g}\big(\mathbf{x}_k; \mathbf{p}\big)\,, \qquad \mathbf{x}_0 \in R^n\,, \qquad k = 0, 1, \ldots\,. \tag{1.12}$$

These discrete-time maps, particularly the time-invariant ones, are very important and convenient for the study of system dynamics; they will be further discussed in the book later on.

For a system given by a time-invariant difference equation, repeatedly iterating the system function $\mathbf{f}$ leads to

$$\mathbf{x}_k = \underbrace{\mathbf{f} \circ \cdots \circ \mathbf{f}}_{k \text{ times}}\big(\mathbf{x}_0\big) := \mathbf{f}^k\big(\mathbf{x}_0\big)\,, \tag{1.13}$$

where "$\circ$" denotes the composition operation of two functions or maps. Similarly, for a map F_d, repeatedly iterating it backwards yields

$$\mathbf{x}_k = F_d\big(\mathbf{x}_{k-1}\big) = F_d\big(F_d(\mathbf{x}_{k-2})\big) = \cdots := F_d^k\big(\mathbf{x}_0\big)\,, \tag{1.14}$$

where $k = 0, 1, 2, \ldots$.

Example 1.1. For the map

$$f(x) = p\,x(1-x)\,, \qquad p \in R\,.$$

one has

$$f^2(x) = f\big(p\,x(1-x)\big) = p\,[p\,x(1-x)]\big(1 - [p\,x(1-x)]\big)\,,$$

where the last equality is obtained by substituting each x in the previous step with $[px(1-x)]$. For a large number n of iterations, it quickly becomes very messy and actually impossible to write out the final explicit formula of the composite map $f^n(x)$.

If a function or map $\mathbf{f}$ is invertible, with inverse $\mathbf{f}^{-1}$, then one has $\mathbf{f}^{-2}(\mathbf{x}) = \big(\mathbf{f}^{-1}\big)^2(\mathbf{x})$, and $\mathbf{f}^{-n}(\mathbf{x}) = \big(\mathbf{f}^{-1}\big)^n(\mathbf{x})$, and so on. With the convention that $\mathbf{f}^0$ denotes the identity map, namely $\mathbf{f}^0(\mathbf{x}) := \mathbf{x}$, a general composition formula for an invertible $\mathbf{f}$ can be obtained:

$$\mathbf{f}^n(\mathbf{x}) = \mathbf{f} \circ \mathbf{f}^{n-1}(\mathbf{x}) = \mathbf{f}\big(\mathbf{f}^{n-1}(\mathbf{x})\big)\,, \quad n = 0, \pm 1, \pm 2, \ldots\,. \tag{1.15}$$

The derivative of a composite map can be obtained via the chain rule. For instance, in the 1-dimensional case,

$$\big(f^n\big)'\big(x_0\big) = f'\big(x_{n-1}\big) \cdots f'\big(x_0\big)\,. \tag{1.16}$$

This formula is convenient to use, because one does not need to explicitly compute $f^n(x)$, or $(f^n)'(x)$. Moreover, using the chain rule, one has

$$\big(f^{-1}\big)' = \frac{1}{f'\big(x_{-1}\big)}\,, \qquad x_{-1} := f^{-1}(x)\,. \tag{1.17}$$

Example 1.2. For $f(x) = x(1-x)$ with $x_0 = 1/2$ and $n = 3$, one has

$$f'(x) = 1 - 2x\,, \quad x_1 = f(x_0) = 1/4\,, \quad \text{and} \quad x_2 = f(x_1) = 3/16\,,$$

so that

$$\begin{aligned}\big(f^3\big)'(1/2) &= f'(3/16)\, f'(1/4)\, f'(1/2) \\ &= \big(1 - 2(3/16)\big)\big(1 - 2(1/4)\big)\big(1 - 2(1/2)\big) \\ &= 0\,.\end{aligned}$$

Finally, consider a function or map $\mathbf{f}$ given by either (1.13) or (1.14).

Definition 1.1. For a positive integer n, a point $\mathbf{x}^*$ is called a *periodic point of period* n, or an n*-periodic point*, of $\mathbf{f}$, if it satisfies

$$\mathbf{f}^n\big(\mathbf{x}^*\big) = \mathbf{x}^* \quad \text{but} \quad \mathbf{f}^k\big(\mathbf{x}^*\big) \neq \mathbf{x}^* \quad \text{for} \quad 0 < k < n\,. \tag{1.18}$$

If $\mathbf{x}^*$ is of period one ($n = 1$), then it is also called a *fixed point*, or an *equilibrium point*, which satisfies

$$\mathbf{f}\big(\mathbf{x}^*\big) = \mathbf{x}^*\,. \tag{1.19}$$

Moreover, a point $\mathbf{x}^*$ is said to be *eventually periodic of period* n if there is an integer $m > 0$ such that

$$\mathbf{f}^m\big(\mathbf{x}^*\big) \quad \text{is a periodic point} \quad \text{and} \quad \mathbf{f}^{m+n}\big(\mathbf{x}^*\big) = \mathbf{f}^m\big(\mathbf{x}^*\big)\,. \tag{1.20}$$

Consequently,

$$\mathbf{f}^{n+q}\big(\mathbf{x}^*\big) = \mathbf{f}^q\big(\mathbf{x}^*\big) \qquad \text{for all} \;\; q \geq m\,.$$

This justifies the name "eventually".

Example 1.3. The map $f(x) = x^3 - x$ has three fixed points: $x_1^* = 0$ and $x_{1,2}^* = \pm\sqrt{2}$, which are solutions of the equation $f(x^*) = x^*$. It has two eventually fixed points of period one: $x_{1,2}^* = \pm 1$, since their first iterates go to the fixed point 0.

Definition 1.2. For a continuous-time function or map, $\mathbf{f}$, with a fixed point $\mathbf{x}^*$, the *forward orbit of* $\mathbf{x}^*$ is

$$\Omega^+\big(\mathbf{x}^*\big) := \big\{\, \mathbf{f}^k\big(\mathbf{x}^*\big) :\; k \geq 0 \,\big\}\,.$$

If $\mathbf{f}$ is invertible, then the *backward orbit of* $\mathbf{x}^*$ is

$$\Omega^-\big(\mathbf{x}^*\big) := \big\{\, \mathbf{f}^k\big(\mathbf{x}^*\big) :\; k \leq 0 \,\big\}\,.$$

The *whole orbit of* $\mathbf{x}^*$, thus, is

$$\Omega\big(\mathbf{x}^*\big) = \Omega^+\big(\mathbf{x}^*\big) \cup \Omega^-\big(\mathbf{x}^*\big) = \big\{\, \mathbf{f}^k\big(\mathbf{x}^*\big) :\; k = 0, \pm 1, \pm 2, \dots \,\big\}\,.$$

Definition 1.3. For a continuous-time function or map, $\mathbf{f}$, a set $\mathcal{S} \subset R^n$ is said to be *forward invariant under* $\mathbf{f}$, if $\mathbf{f}^k(\mathbf{x}) \in \mathcal{S}$ for all $\mathbf{x} \in \mathcal{S}$ and for all $k = 0, 1, 2, \ldots$. Furthermore, for an invertible $\mathbf{f}$, a set $\mathcal{S} \subset R^n$ is said to be *backward invariant under* $\mathbf{f}$, if $\mathbf{f}^k(\mathbf{x}) \in \mathcal{S}$ for all $\mathbf{x} \in \mathcal{S}$ and for all $k = 0, -1, -2, \ldots$.

1.3 Dynamical Analysis on Phase Planes

In this section, a general 2-dimensional nonlinear autonomous system is considered:

$$\begin{aligned} \dot{x} &= f\big(x, y\big)\,, \\ \dot{y} &= g\big(x, y\big)\,. \end{aligned} \tag{1.21}$$

In this system, the two functions f and g together describe the *vector field* of the system. Here and in the following, for 2- or 3-dimensional systems, the state variables will be denoted as x, y, and z, instead of x_1, x_2, and x_3, for notational convenience.

1.3.1 *Phase Plane of a Planar System*

The path traveled by a solution of the continuous-time planar system (1.21), starting from an initial state (x_0, y_0), is a *solution trajectory* or *orbit* of the system, and is sometimes denoted as $\varphi_t(x_0, y_0)$.

For autonomous systems, the x–$\dot{x}$ coordinate plane is called the *phase plane* of the system. In general, even if $y \neq \dot{x}$, the x–y coordinate plane is called the (*generalized*) *phase plane*. In the higher-dimensional case, it is called the *phase space* of the underlying dynamical system. Moreover, the orbit family of an autonomous system, corresponding to all possible initial conditions, is called a *solution flow* in the phase space. The graphical layout of the solution flow provides a *phase portrait* of the system dynamics in the phase space, as depicted by Fig. 1.2.

The phase portrait of the damped and undamped pendulum systems, (1.1) and (1.2), are shown in Fig. 1.3.

Examining the phase portraits shown in Fig. 1.3, a natural question arises: how can one determine the motion direction of the orbit flow in the phase plane as the time evolves? Clearly, a computer graphic demonstration can provide a fairly complete answer to this question. However, a quick sketch to show the *qualitative behavior* of the system dynamics is still quite possible, as illustrated by the following two examples.

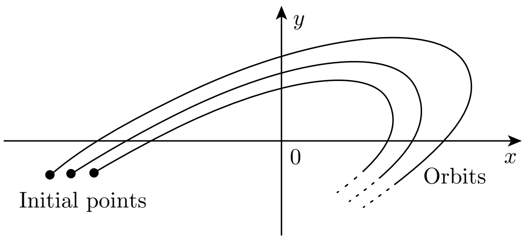
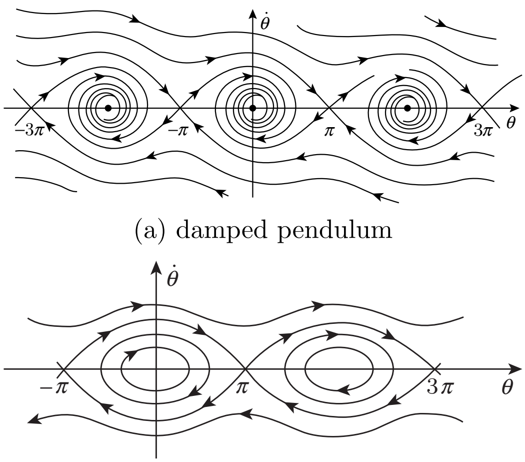


Fig. 1.2 Phase portrait on the phase plane of a dynamical system.



(b) undamped pendulum

Fig. 1.3 Phase portraits of the damped and undamped pendula.

Example 1.4. Consider the simple linear harmonic oscillator

$$\ddot{\theta} + \theta = 0\,.$$

By defining

$$x = \theta \qquad \text{and} \qquad y = \dot{\theta}\,,$$

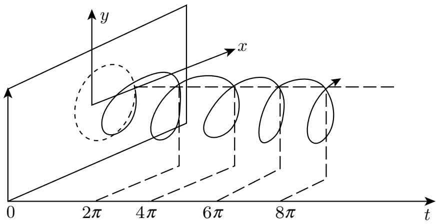
this harmonic equation becomes

$$\dot{x} = y\,,$$
$$\dot{y} = -\,x\,.$$

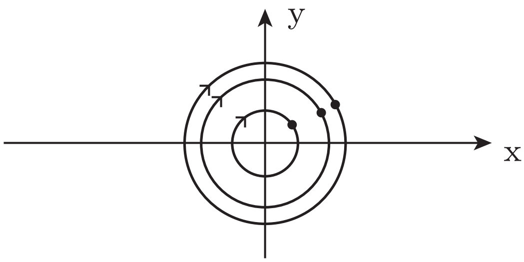
With initial conditions $x(0) = 1$ and $y(0) = 0$, this equation has solution

$$x(t) = \cos(t) \qquad \text{and} \qquad y(t) = -\sin(t)\,.$$

The solution trajectory in the x–y–t space and the corresponding orbit on the x–y phase plane are sketched in Fig. 1.4, together with some other solutions starting from different initial conditions. This shows clearly the direction of motion of the phase portrait.



(a) phase portrait in the x–y–t space



(b) phase portrait on the x–y phase plane

Fig. 1.4 Phase portraits of the simple harmonic equation.

Example 1.5. Consider the normalized undamped pendulum equation

$$\ddot{\theta} + \sin(\theta) = 0\,.$$

By defining

$$x = \theta \qquad \text{and} \qquad y = \dot{\theta}\,,$$

this pendulum equation can be written as

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\sin(x)\,. \end{aligned}$$

With initial conditions $x(0) = 1$ and $y(0) = 0$, it has solution

$$x(t) = \theta(t) = 2 \sin^{-1}\big(\tanh(t)\big) \qquad \text{and} \qquad y(t) = \dot{\theta}(t)\,.$$

The phase portrait of this undamped pendulum, along with some other solutions starting from different initial conditions, is sketched on the x–y phase plane shown in Fig. 1.5. This sketch also clearly indicates the direction of motion of the solution flow. The shape of a solution trajectory of this undamped pendulum in the x–y–t space can also be sketched, which could be quite complex however, depending on the initial conditions.

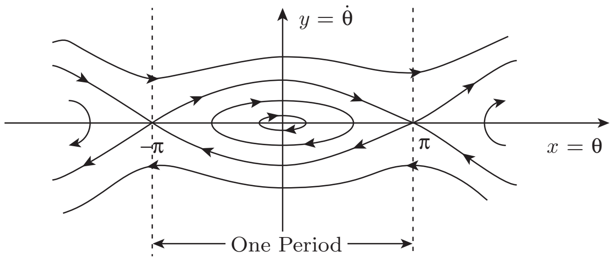


Fig. 1.5 Phase portrait of the undamped pendulum equation.

Example 1.6. Another way to understand the phase portrait of the general undamped pendulum

$$\dot{x} = y\,,$$
$$\dot{y} = -\,\frac{g}{\ell}\,\sin(x)$$

is to examine its total (kinetic and potential) energy

$$E = \frac{y^2}{2} + \frac{g}{\ell}\int_0^x \sin(\sigma)\,d\sigma = \frac{y^2}{2} + \frac{g}{\ell}\,\big(1 - \cos(x)\big)\,.$$

Figure 1.6 shows the potential energy plot, $P(x)$, versus $x = \theta$, and the corresponding phase portrait on the x–y phase plane of the damped pendulum. It is clear that the lowest level of total energy is $E = 0$, which corresponds to the angular positions $x = \theta = \pm 2n\pi$, $n = 0, 1, \ldots$. As the total energy increases, the pendulum swings up or down, with an increasing or decreasing angular speed, $|y| = |\dot{\theta}|$, provided that E is within its limit indicated by E_2. Within each period of oscillation, the total energy $E =$ constant, according to the conservation law of energy, for this idealized undamped pendulum.

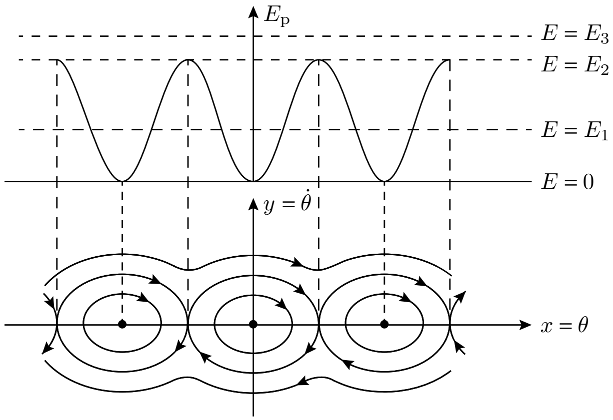


Fig. 1.6 Phase portrait of the undamped pendulum versus its total energy.

1.3.2 *Analysis on Phase Planes*

This subsection addresses the following question: why is it important to study autonomous systems and their phase portraits?

The answer to this question is provided by the following several theorems, which together summarize a few important and useful properties of autonomous systems in the study of nonlinear dynamics. Although these theorems are stated and proven for planar systems in this subsection, they generally hold for higher-dimensional autonomous systems as well.

Theorem 1.1. *A nonautonomous system can be equivalently reformulated as an autonomous one.*

Proof. Consider a general nonautonomous system,

$$\dot{\mathbf{x}} = \mathbf{f}\big(\mathbf{x}, t\big)\,, \qquad \mathbf{x}_0 \in R^n\,.$$

Let the independent time variable t be a new variable by defining $x_{n+1}(t) = t$ for this separated variable t of the system. Then, $\dot{x}_{n+1} = 1$. Consequently, the original system can be equivalently reformulated by augmenting it as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}\big(\mathbf{x}, x_{n+1}\big) \\ 1 \end{bmatrix},$$

which is an autonomous system. □

Obviously the price to pay for this conversion, from a nonautonomous system to an autonomous one, is the increase of dimension. In dynamical systems analysis, this usually is acceptable since the increase is only by one, which is not a big deal for higher-dimensional systems. Nevertheless, this shows that, without loss of generality, one may only discuss autonomous systems in nonlinear dynamical analysis especially in higher-dimensional cases.

However, it is important to note that, in a nonlinear control system of the form

$$\dot{\mathbf{x}} = \mathbf{f}\big(\mathbf{x}, \mathbf{u}(t)\big) ,$$

which will be studied in detail later in the book, the controller $\mathbf{u}(t)$ is a time function and is yet to be designed, which it is not a system variable. In this case, one should not (cannot) convert the control system to be autonomous using this technique; otherwise, the controller loses its physical meaning and can never be designed for the intended control tasks. This issue will be revisited later within the context of feedback controllers design.

Theorem 1.2. *If $\mathbf{x}(t)$ is a solution of the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then so is the trajectory $\mathbf{x}(t+a)$, for any real constant a. Moreover, these two solutions are the same, except that they may pass the same point on the phase plane at two different time instants.*

The last statement of the theorem describes the inherent time-invariant property of autonomous systems.

Proof. Because $\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t))$, for any real constant τ, one has

$$\frac{d}{dt}\mathbf{x}(t+a)\Big|_{t=\tau} = \frac{d}{ds}\mathbf{x}(s)\Big|_{s=\tau+a} = \mathbf{f}(\mathbf{x}(s))\Big|_{s=\tau+a} = \mathbf{f}(\mathbf{x}(t+a))\Big|_{t=\tau} .$$

Since this holds for all real τ, it implies that $\mathbf{x}(t+a)$ is a solution of the equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Moreover, the value assumed by $\mathbf{x}(t)$ at time instant $t = t^*$ is the same as that assumed by $\mathbf{x}(t+a)$ at time instant $t = t^* - a$. Hence, these two solutions are identical, in the sense that they have the same trajectory if they are both plotted on the same phase plane. □

Example 1.7. The autonomous system $\dot{x}(t) = x(t)$ has a solution $x(t) = e^t$. It is easy to verify that e^{t+a} is also a solution of this system for any real constant a. These two solutions are the same, in the sense that they have the same trajectory if they are plotted on the x–$\dot{x}$ phase plane, except that they pass the same point at two different time instants; for instance, the first one passes the point $(x, \dot{x}) = (1, 1)$ at $t = 0$ but the second, at $t = -a$.

However, a nonautonomous system may not have such a property.

Example 1.8. The nonautonomous system $\dot{x}(t) = e^t$ has a solution $x(t) = e^t$. But e^{t+a} is not its solution if $a \neq 0$.

Note that, if one applies Theorem 1.1 to Example 1.8 and let $y(t) = t$, then

$$\begin{aligned} \dot{x}(t) &= e^{y(t)}\,, \\ \dot{y}(t) &= 1\,, \end{aligned} \tag{a}$$

which has solution

$$\begin{aligned} x(t) &= e^{t}\,, \\ y(t) &= t\,. \end{aligned} \tag{b}$$

Theorem 1.1 states that (b) is a solution of (a), which does not mean that (b) is a solution of the original equation $\dot{x}(t) = e^t$. In fact, only the first part of (b), i.e. $x(t) = e^t$, is a solution of the original equation, and the second part of (b) is merely used to convert the given nonautonomous system to be an autonomous one.

Theorem 1.3. *Suppose that a given autonomous system* $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ *has a unique solution starting from an initial state* $\mathbf{x}(t_0) = \mathbf{x}_0$. *Then, there will not be any other (different) orbit of the same system that also passes through this same point* $\mathbf{x}_0$ *on the phase plane at any time.*

Before giving a proof to this result, two remarks are in order. First, this theorem implies that the solution flow of an autonomous system has simple geometry, as depicted in Fig. 1.7, where different orbits starting from different initial states do not cross each other. Second, in the phase portrait of the (damped or undamped) pendulum (see Fig. 1.3, it may seem that there are more than one orbit passing through the points $(-\pi, 0)$ and $(\pi, 0)$ etc. However, those orbits are periodic orbits, so the principal solution of the pendulum corresponds to those curves located between the two vertical lines passing through the two end points $x = -\pi$ and $x = \pi$, respectively. Thus, within each 2π period, actually no self-crossing exists. It will be seen later that all such seemingly self-crossing occur only at those special points called *stable node* (*sink*), *unstable node* (*source*), or *saddle node* (see Fig. 1.8), where the orbits either spiral into a sink, spiral out from a source, or spiral in and out from a saddle node in different directions.

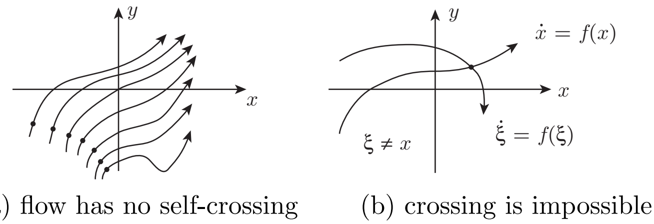


Fig. 1.7 Simple phase portrait of an autonomous system.

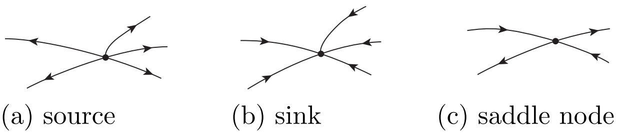


Fig. 1.8 Simple phase portrait of an autonomous system.

Proof. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solutions of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, satisfying

$$\mathbf{x}_1(t_1) = \mathbf{x}_0 \quad \text{and} \quad \mathbf{x}_2(t_2) = \mathbf{x}_0\,, \quad t_1 \neq t_2\,.$$

By Theorem 1.2, one has

$$\widetilde{\mathbf{x}}_2(t) := \mathbf{x}_2\big(t - (t_1 - t_2)\big),$$

which is the same solution of the given autonomous system, namely,

$$\mathbf{x}_2(t) = \widetilde{\mathbf{x}}_2(t)\,. \tag{a}$$

This solution satisfies

$$\widetilde{\mathbf{x}}_2(t_1) := \mathbf{x}_2\big(t_1 - (t_1 - t_2)\big) = \mathbf{x}_2(t_2) = \mathbf{x}_0\,.$$

Therefore, by the uniqueness of the solution, $\mathbf{x}_1$ and $\widetilde{\mathbf{x}}_2$ are the same:

$$\mathbf{x}_1(t) = \widetilde{\mathbf{x}}_2(t)\,, \tag{b}$$

since they are both equal to $\mathbf{x}_0$ at the same initial time t_1. Thus, (a) and (b) together imply that $\mathbf{x}_1$ and $\mathbf{x}_2$ are identical. □

Note that a nonautonomous system may not have such a property.

Example 1.9. Consider the nonautonomous system

$$\dot{x}(t) = \cos(t)\,.$$

This system has the following solutions, among others:

$$x_1(t) = \sin(t) \qquad \text{and} \qquad x_2(t) = 1 + \sin(t)\,.$$

These two solutions are different, for if they are plotted on the phase plane, they show two different trajectories:

$$\begin{aligned}\dot{x}_1(t) &= \cos(t) = \pm\sqrt{1-\sin^2(t)} = \pm\sqrt{1-x_1^2}\,,\\ \dot{x}_2(t) &= \cos(t) = \pm\sqrt{1-\sin^2(t)} = \pm\sqrt{1-[1+\sin(t)-1]^2}\\ &= \pm\sqrt{1-(x_2-1)^2}\,.\end{aligned}$$

These two trajectories cross over at a point, $(x_1, x_2) = (1/2, 1/2)$, as can be seen from Fig. 1.9.

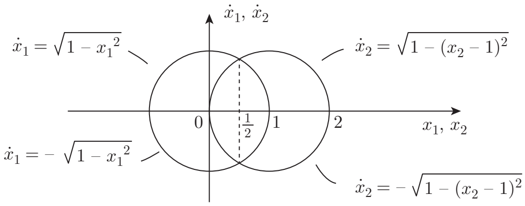


Fig. 1.9 Two crossing trajectories of a nonautonomous system.

Theorem 1.4. *A closed orbit of the autonomous system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *on the phase plane corresponds to a periodic solution of the system.*

Proof. For a τ-periodic solution, $\mathbf{x}(t)$, one has $\mathbf{x}(t_0+\tau) = \mathbf{x}(t_0)$ for any $t_0 \in R$, which means that the trajectory of $\mathbf{x}(t)$ is closed.

On the contrary, suppose that the orbit of $\mathbf{x}(t)$ is closed. Let $\mathbf{x}_0$ be a point in the closed orbit. Then, $\mathbf{x}_0 = \mathbf{x}(t_0)$ for some t_0, and the trajectory of $\mathbf{x}(t)$ will return to $\mathbf{x}_0$ after some time, say $\tau \geq 0$; that is, $\mathbf{x}(t_0+\tau) = \mathbf{x}_0 = \mathbf{x}(t_0)$. Since $\mathbf{x}_0$ is arbitrary, and so is t_0, this implies that $\mathbf{x}(t+\tau) = \mathbf{x}(t)$ for all t, meaning that $\mathbf{x}(t)$ is periodic with period τ. □

Yet, a nonautonomous system may not have such a property.

Example 1.10. The nonautonomous system

$$\begin{aligned}\dot{x} &= 2\,t\,y\,,\\ \dot{y} &= -2\,t\,x\,,\end{aligned}$$

has solution

$$x(t) = \alpha \cos(t^2) + \beta \sin(t^2)\,,$$
$$y(t) = -\,\alpha \sin(t^2) + \beta \cos(t^2)\,,$$

for some constants α and β determined by initial conditions. This solution is not periodic, but it is a circle (a closed orbit) on the x–y phase plane.

As mentioned at the beginning of this subsection, the above four theorems hold for general higher-dimensional autonomous systems. Since these properties are simple, elegant and easy to use, which a nonautonomous system may not have, it is very natural to focus a general study of complex dynamics on autonomous systems in various forms with any dimensions. This motivates the following investigations.

1.4 Qualitative Behaviors of Dynamical Systems

In this section, consider a general 2-dimensional autonomous system,

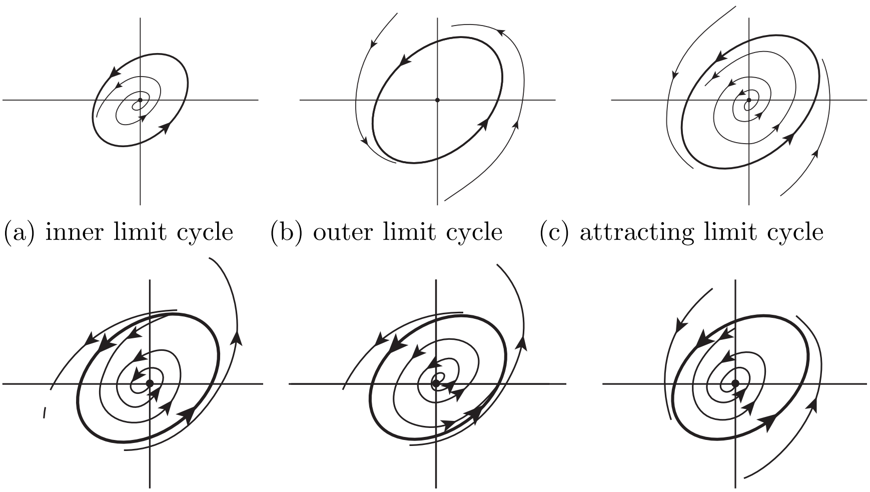
$$\begin{aligned} \dot{x} &= f(x, y)\,, \\ \dot{y} &= g(x, y)\,. \end{aligned} \tag{1.22}$$

Let Γ be a periodic solution of the system which, as discussed above, has a closed orbit on the x–y phase plane.

Definition 1.4. Γ is said to be an *inner* (*outer*) *limit cycle* of system (1.22) if, in an arbitrarily small neighborhood of the inner (*outer*) region of Γ, there is always (part of) a nonperiodic solution orbit of the system. Γ is called a *limit cycle*, if it is both inner and outer limit cycles.

Simply put, a limit cycle is a periodic orbit of the system that corresponds to a closed orbit on the phase plane and possesses certain (attracting or repelling) limiting properties. Figure 1.10 shows some typical limit cycles for the 2-dimensional system (1.22), where the attracting limit cycle is said to be *stable*, while the repelling one, *unstable*.

Example 1.11. The simple harmonic oscillator discussed in Example 1.4 has no limit cycles. The solution flow of the system constitutes a ring of periodic orbits, called *periodic ring*, as shown in Fig. 1.4. Similarly, the undamped pendulum has no limit cycles, as shown in Fig. 1.3.



(a) inner limit cycle (b) outer limit cycle (c) attracting limit cycle

(d) repelling limit cycle (e) saddle limit cycle (f) saddle limit cycle

Fig. 1.10 Periodic orbits and limit cycles.

This example shows that, although a limit cycle is a periodic orbit, not all periodic orbits are limit cycles, not even inner or outer limit cycles.

Example 1.12. A typical example of a stable limit cycle is the periodic solution of the Rayleigh oscillator, described by

$$\ddot{x} + x = p\left(\dot{x} - \dot{x}^3\right), \qquad p > 0, \tag{1.23}$$

which was formulated in the 1920s to describe oscillations in some electrical and mechanical systems. This limit cycle is shown in Fig. 1.11 for some different values of p. These phase portraits are usually obtained either numerically or experimentally, because they do not have simple analytic formulas.

Example 1.13. Another typical example of a stable limit cycle is the periodic solution of the van der Pol oscillator, described by

$$\ddot{x} + x = p\left(1 - x^2\right)\dot{x}, \qquad p > 0, \tag{1.24}$$

which was formulated around 1920 to describe oscillations in a triode circuit. This limit cycle is shown in Fig. 1.12, which is usually obtained either numerically or experimentally, because it does not have a simple analytic formula.

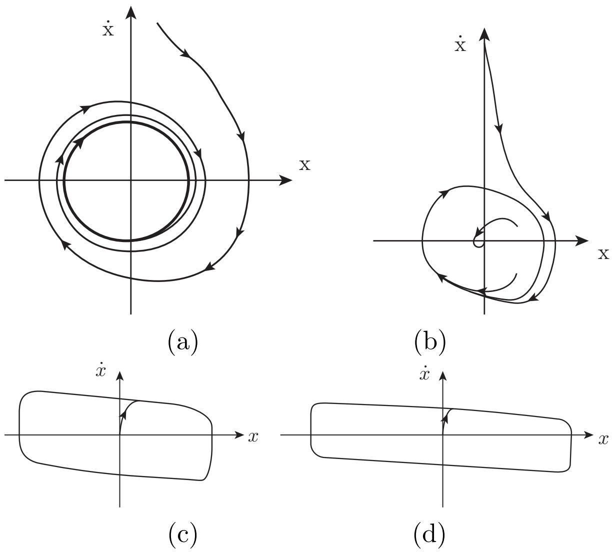


Fig. 1.11 Phase portrait of the Rayleigh oscillator. (a) $p = 0.01$; (b) $p = 0.1$; (c) $p = 1.0$; (d) $p = 10.0$.

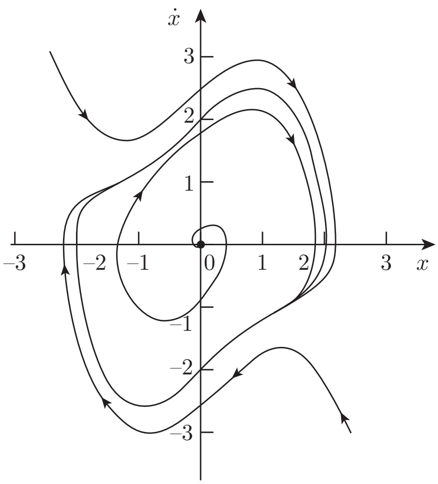


Fig. 1.12 Phase portrait of the van der Pol oscillator.

1.4.1 *Qualitative Analysis of Linear Dynamics*

For illustration, consider a 2-dimensional linear autonomous (i.e. time-invariant) system,

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t)\,, \qquad \mathbf{x}(0) = \mathbf{x}_0\,, \tag{1.25}$$

where A is a given 2×2 constant matrix and, for simplicity, the initial time is set to $t_0 = 0$.

Obviously, this system has a unique fixed point $\mathbf{x}^* = 0$ and has a unique solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$. Decompose its solution as

$$\mathbf{x}(t) = e^{tA}\,\mathbf{x}_0 = M\,e^{tJ}M^{-1}\mathbf{x}_0\,, \tag{1.26}$$

where $M = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$ with $\mathbf{v}_1$ and $\mathbf{v}_2$ being two linearly independent real eigenvectors associated with the two eigenvalues of A, and J is in the Jordan canonical form, which is one of the following three possible forms:

$$\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}, \qquad \begin{bmatrix}\lambda & \kappa \\ 0 & \lambda\end{bmatrix}, \qquad \begin{bmatrix}\alpha & -\beta \\ \beta & \alpha\end{bmatrix},$$

with $\lambda_1, \lambda_2, \lambda, \alpha$, and β being real constants, and $\kappa = 0$ or 1. Note that for the third case, its eigenvalues are complex conjugates: $\lambda_{1,2} = \alpha \pm j\,\beta$, where $j = \sqrt{-1}$.

Thus, there are three cases to study, according to the three different canonical forms of the Jordan matrix J shown above.

Case (i). The two constants λ_1 and λ_2 are different, but both real and nonzero.

In this case, λ_1 and λ_2 are the eigenvalues of matrix A, associated with two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Let

$$\mathbf{z} = M^{-1}\mathbf{x}\,,$$

where $M = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$. Then, the given system is transformed to

$$\dot{\mathbf{z}} = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\mathbf{z}\,, \qquad \text{with} \quad \mathbf{z}_0 = M^{-1}\mathbf{x}_0 := \begin{bmatrix} z_{10} \\ z_{20}\end{bmatrix}.$$

Its solution is

$$z_1(t) = z_{10}\,e^{t\lambda_1} \qquad \text{and} \qquad z_2(t) = z_{20}\,e^{t\lambda_2}\,,$$

which are related by

$$z_2(t) = c\,z_1^{\lambda_2/\lambda_1}(t)\,, \qquad \text{with} \quad c = z_{20}\big(z_{10}\big)^{-\lambda_2/\lambda_1}.$$

To show the phase portraits of the solution flow, there are three situations to consider: (a) $\lambda_2 < \lambda_1 < 0$; (b) $\lambda_2 < 0 < \lambda_1$; (c) $0 < \lambda_2 < \lambda_1$.

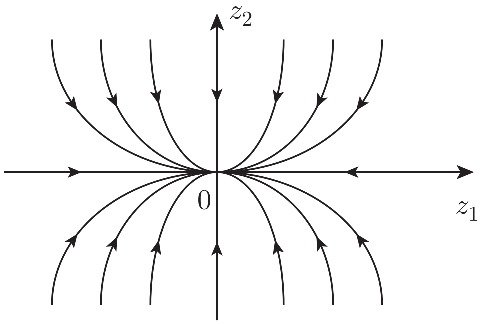


Fig. 1.13 Phase portrait of the transformed case (a): $\lambda_2 < \lambda_1 < 0$.

Only situation (a) is discussed in detail here. In this case, the two eigenvalues are both negative, so that $e^{t\lambda_1} \to 0$ and $e^{t\lambda_2} \to 0$ as $t \to \infty$, but the latter tends to zero faster. The corresponding phase portrait is shown in Fig. 1.13, where the fixed point (the origin) is a stable node.

Now, return to the original state, $\mathbf{x} = M\mathbf{z}$. The original phase portrait is somewhat twisted, as shown in Fig. 1.14. Figures 1.13 and 1.14 are topologically equivalent, hence can be considered to be the same qualitatively. A more precise meaning of topological equivalence will be given later in (1.29). Roughly speaking, it means that their dynamical behaviors are qualitatively similar.

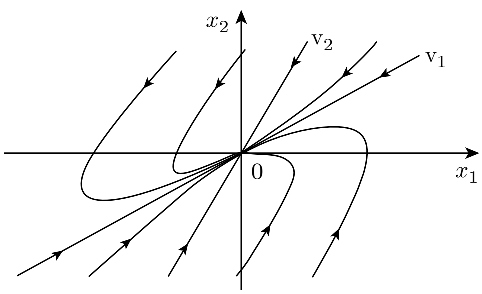


Fig. 1.14 Phase portrait of the original case (a): $\lambda_2 < \lambda_1 < 0$.

The other two situations, (b) and (c), can be analyzed in the same way, where case (b) shows a saddle node and case (c), an unstable node. This is left as an exercise to sketch.

Case (ii). The two constants λ_1 and λ_2 are nonzero complex conjugates: $\lambda_{1,2} = \alpha \pm j\,\beta$, where $j = \sqrt{-1}$. Let

$$\mathbf{z} = M^{-1}\mathbf{x}\,,$$

and transform the given system to

$$\dot{\mathbf{z}} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \mathbf{z}\,, \qquad \text{with} \quad \mathbf{z}_0 = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}.$$

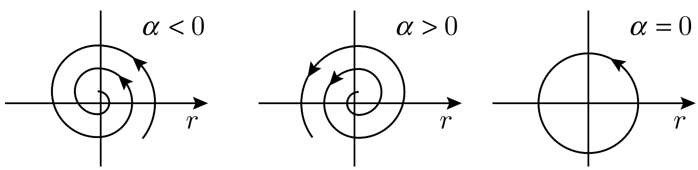
In polar coordinates,

$$r = \sqrt{z_1^2 + z_2^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)\,,$$

which has solution

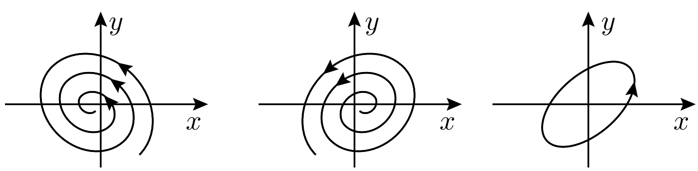
$$r(t) = r_0\, e^{\alpha\, t} \qquad \text{and} \qquad \theta(t) = \theta_0 + \beta\, t\,,$$

where $r_0 = (z_{10}^2 + z_{20}^2)^{1/2}$ and $\theta_0 = \tan^{-1}\big(z_{20}/z_{10}\big)$. This solution trajectory is visualized by Fig. 1.15, where the fixed point (the origin) in case (a) is called a *stable node*, in case (b), an *unstable focus*, and in case (c), a *center*. On the original x–y phase plane, the phase portrait has a twisted shape, as shown in Fig. 1.16.



(a) stable focus (b) unstable focus (c) center

Fig. 1.15 Phase portrait of the transformed Case (ii): $\lambda_{1,2} = \alpha \pm j\,\beta$.



(a) stable focus (b) unstable focus (c) center

Fig. 1.16 Phase portrait of the original Case (ii): $\lambda_{1,2} = \alpha \pm j\,\beta$.

Case (iii). The two constants λ_1 and λ_2 are nonzero multiple real values: $\lambda_1 = \lambda_2 := \lambda$. Let

$$\mathbf{z} = M^{-1}\mathbf{x}\,,$$

and transform the given system to

$$\dot{\mathbf{z}} = \begin{bmatrix} \lambda & \kappa \\ 0 & \lambda \end{bmatrix} \mathbf{z}\,, \qquad \text{with} \quad \mathbf{z}_0 = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}.$$

Its solution is

$$z_1(t) = e^{\lambda t}\big(z_{10} + \kappa\, z_{20} t\big) \qquad \text{and} \qquad z_2(t) = z_{20}\, e^{\lambda t}\,,$$

which are related via

$$z_1(t) = z_2(t)\left[\frac{z_{10}}{z_{20}} + \frac{\kappa}{\lambda}\ln\left(\frac{z_2(t)}{z_{20}}\right)\right].$$

Its phase portrait is shown in Fig. 1.17, and its corresponding phase portrait on the x–y phase plane is similar; in particular, the linear coordinates that transform M do not change the shape of any straight line on the two phase planes.

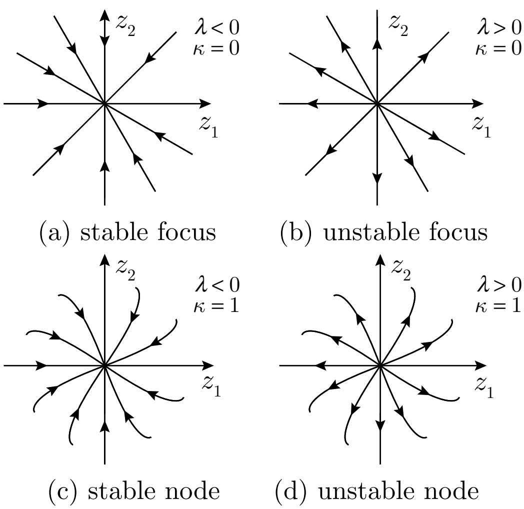


Fig. 1.17 Phase portrait of Case (iii): $\lambda_1 = \lambda_2 \neq 0$.

Case (iv). One, or both, of $\lambda_{1,2}$ is zero.

In this degenerate case, the matrix A in $\dot{\mathbf{x}} = A\mathbf{x}$ has a nontrivial null space, of dimension 1 or 2 respectively, so that any vector in the null space of A is a fixed point. As a result, the system has a *fixed* or *equilibrium subspace.* Specifically, these two situations are as follows:

(a) $\lambda_1 = 0$ but $\lambda_2 \neq 0$

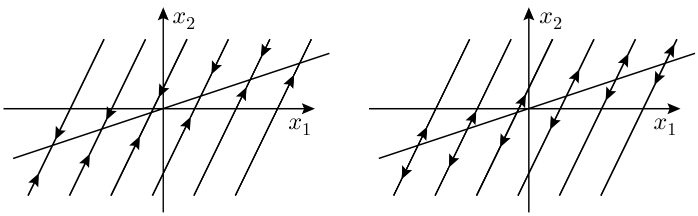
In this case, the system can be transformed to

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{z}\,, \qquad \text{with} \quad \mathbf{z}_0 = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix},$$

which has solution

$$z_1(t) = z_{10} \qquad \text{and} \qquad z_2(t) = z_{20}\, e^{\lambda_2 t}\,.$$

The phase portrait of $\mathbf{x} = M\mathbf{z}$ is shown in Fig. 1.18, where (a) shows a *stable equilibrium subspace* and (b), an *unstable subspace.*



(a) a stable equilibrium subspace (b) an unstable equilibrium subspace

Fig. 1.18 Phase portrait of Case (iv) (a): $\lambda_1 = 0$ but $\lambda_2 \neq 0$.

(b) $\lambda_1 = \lambda_2 = 0$

In this case, the system is transformed to

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z}\,, \qquad \text{with} \quad \mathbf{z}_0 = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix},$$

which has solution

$$z_1(t) = z_{10} + z_{20}\, t \qquad \text{and} \qquad z_2(t) = z_{20}\,.$$

The phase portrait of $\mathbf{x} = M\mathbf{z}$ is shown in Fig. 1.19, which is a *saddle equilibrium subspace.*

1.4.2 ***Qualitative Analysis of Nonlinear Dynamics***

This subsection is devoted to some qualitative analysis of dynamical behaviors of a general nonlinear autonomous system in a neighborhood of a

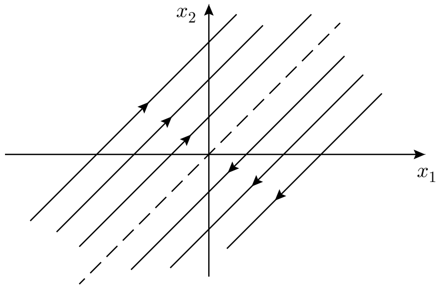


Fig. 1.19 Phase portrait of Case (iv) (b): $\lambda_1 = \lambda_2 = 0$.

fixed point (equilibrium point) of the system. Therefore, unlike the linear systems discussed above, all results derived here are *local.*

Consider a general nonlinear autonomous system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}_0 \in R^n, \tag{1.27}$$

where it is assumed that $\mathbf{f} \in C^1$, i.e. it is continuously differentiable with respect to its arguments, and that the system has a fixed point, $\mathbf{x}^*$.

Taylor-expanding $\mathbf{f}(\mathbf{x})$ at $\mathbf{x}^*$ yields

$$\dot{\mathbf{x}} = \mathbf{f}\big(\mathbf{x}^*\big) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}^*} \big(\mathbf{x} - \mathbf{x}^*\big) + \mathbf{e}(\mathbf{x}) = J\,\big(\mathbf{x} - \mathbf{x}^*\big) + \mathbf{e}(\mathbf{x}),$$

where

$$J = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}^*} = \begin{bmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_n \end{bmatrix}_{\mathbf{x}=\mathbf{x}^*}$$

is the Jacobian, and $\mathbf{e}(\mathbf{x}) = o(||\mathbf{x}||)$ represents the residual of all higher-order terms, which satisfies

$$\lim_{||\mathbf{x}|| \to \infty} \frac{||\mathbf{e}(\mathbf{x})||}{||\mathbf{x}||} = 0.$$

Letting $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ leads to

$$\dot{\mathbf{y}} = J\,\mathbf{y} + \mathbf{e}(\mathbf{y}),$$

where $\mathbf{e}(\mathbf{y}) = o(||\mathbf{y}||)$. In a small neighborhood of $\mathbf{x}^*$, $||\mathbf{x} - \mathbf{x}^*||$ is small, so $o(||\mathbf{y}||) \approx 0$. Thus, the nonlinear autonomous system (1.27) and its linearized system $\dot{\mathbf{x}} = J\,(\mathbf{x} - \mathbf{x}^*)$ have the same dynamical behaviors; particularly, the latter in a small neighborhood of $\mathbf{x}^*$ is the same as $\dot{\mathbf{y}} = J\,\mathbf{y}$ in a small neighborhood of 0. In other words, between $\mathbf{x}$ and $\mathbf{y}$, the following are comparable:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{versus} \qquad \dot{\mathbf{y}} = J\,\mathbf{y} \tag{1.28}$$

$$\mathbf{x}^* \text{ is } \begin{cases} \text{stable node} \\ \text{unstable node} \\ \text{stable focus} \\ \text{unstable focus} \\ \text{saddle node} \end{cases} \iff \quad \mathbf{y} = 0 \text{ is } \begin{cases} \text{stable node} \\ \text{unstable node} \\ \text{stable focus} \\ \text{unstable focus} \\ \text{saddle node} \end{cases} \tag{1.29}$$

In this sense, the local dynamical behaviors of the two systems in (1.28) are said to be *qualitatively the same*, or *topologically equivalent*. A precise mathematical definition is given as follows.

Definition 1.5. Two time-invariant system functions, $f : X \to Y$ and $g : X^* \to Y^*$, where X, Y, X^*, and Y^* are (open sets of) metric spaces, are said to be *topologically equivalent*, if there is a homeomorphism, $h : Y \to Y^*$, such that $h^{-1} : X^* \to X$ and

$$g(x) = h^{-1} \circ f \circ h(x)\,, \qquad x \in X\,,$$

where $\circ$ is the composite operation of two maps.

This definition is illustrated in Fig. 1.20. Here, a *homeomorphism* is an invertible continuous function whose inverse is also continuous. For instance, for $X = Y = R$, the two functions $f(x) = 2x^3$ and $g(x) = 8x^3$ are topologically equivalent. This is because one can find a homeomorphisim, $h(x) = (x)^{1/3}$, which yields

$$h^{-1} \circ f \circ h(x) = \left(2\left((x)^{1/3}\right)^3\right)^3 = 8x^3 = g(x)\,.$$

A homeomorphism preserves the system dynamics as seen from the one-one correspondence (1.29). When both X and Y are Euclidean spaces, the homeomorphism h may be viewed as a nonsingular coordinates transform.

For discrete-time systems (maps), such topological equivalence is also called the *topological conjugacy*, and the two maps are said to be *topologically conjugate* if they satisfy the relationships shown in Fig. 1.20, where h is a homeomorphism.

Theorem 1.5. *If f and g are topologically conjugate, then*

(i) *the orbits of f are mapped to the orbits of g under h;*
(ii) *if x^* is a fixed point of f, then the eigenvalues of $f'(x^*)$ are mapped to the eigenvalues of $g'(x^*)$.*

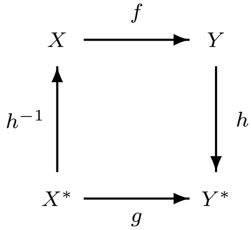


Fig. 1.20 Two topologically equivalent functions or topologically conjugate maps.

Proof. First, note that the orbit of x^* under iterates of map f is

$$\Omega(x^*) = \left\{ \ldots, f^{-k}(x^*), \ldots, f^{-1}(x^*), x^*, f(x^*), \ldots, f^k(x^*) \right\}.$$

Since $f = h^{-1} \circ g \circ h$, for any given $k > 0$ one has

$$\begin{aligned} f^k(x^*) &= \left(h^{-1} \circ g \circ h\right) \circ \cdots \circ \left(h^{-1} \circ g \circ h\right)(x^*) \\ &= h^{-1} \circ g^k \circ h(x^*). \end{aligned}$$

On the other hand, since $f^{-1} = h^{-1} \circ g^{-1} \circ h$, for any given $k > 0$ one has

$$h \circ f^{-k}(x^*) = g^{-k} \circ h(x^*).$$

A comparison of the above two equalities shows that the orbit of x^* under iterates of f is mapped by h to the orbit of $h(x^*)$ under iterates of map g. This proves part (i).

The conclusion of part (ii) follows from a direct calculation:

$$\frac{df}{dx}\bigg|_{x=x^*} = \frac{dh^{-1}}{dx}\bigg|_{x=x^*} \cdot \frac{dg}{dx}\bigg|_{x=h(x^*)} \cdot \frac{dh}{dx}\bigg|_{x=x^*},$$

noting that similar matrices have the same eigenvalues. □

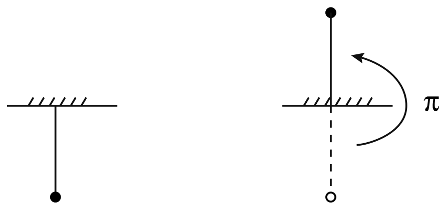
Example 1.14. The damped pendulum system (1.1), namely,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{\kappa}{m}\, y - \frac{g}{\ell}\, \sin(x), \end{aligned}$$

has two fixed points:

$$\left(x^*, y^*\right) = \left(\theta^*, \dot{\theta}^*\right) = (0, 0) \quad \text{and} \quad \left(x^*, y^*\right) = \left(\theta^*, \dot{\theta}^*\right) = (\pi, 0).$$

It is known from the pendulum physics (see Fig. 1.21) that the first fixed point is stable while the second, unstable.



(a) stable fixed point (b) unstable fixed point

Fig. 1.21 Two fixed points of the damped pendulum.

According to the above analysis, the Jacobian of the damped pendulum system is

$$J = \begin{bmatrix} 0 & 1 \\ -g\,\ell^{-1}\cos(x) & -\kappa/m \end{bmatrix}.$$

There are two cases to consider at the two fixed points:

(a) $x^* = \theta^* = 0$

In this case, the two eigenvalues of J are

$$\lambda_{1,2} = -\frac{\kappa}{2m} \pm \frac{1}{2}\sqrt{\left(\kappa/m\right)^2 - 4\left(g/\ell\right)}\ ,$$

implying that the fixed point is stable since $\mathrm{Re}\{\lambda_{1,2}\} < 0$.

(b) $x^* = \theta^* = \pi$

In this case, the two eigenvalues of J are

$$\lambda_{1,2} = -\frac{\kappa}{2m} \pm \frac{1}{2}\sqrt{\left(\kappa/m\right)^2 + 4\left(g/\ell\right)}\ ,$$

where $\mathrm{Re}\{\lambda_1\} > 0$ and $\mathrm{Re}\{\lambda_2\} < 0$, which implies that the fixed point is a saddle node and, hence, is unstable in one direction on the plane shown in Fig. 1.21, along which the pendulum can swing back and forth.

Clearly, the mathematical analysis given here is consistent with the physics of the damped pendulum as discussed before.

Example 1.15. For easy explanation of concept, consider a composite 2- and 1-dimensional autonomous system,

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}\,.$$

At the fixed point $(0,0,0)$, this system has eigenvalues $-1 \pm 2j$ and 1, implying that the origin is a saddle node, as illustrated by the phase portrait in Fig. 1.22.

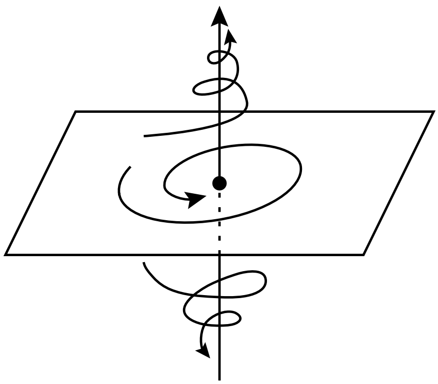


Fig. 1.22 3-dimensional saddle node.

Example 1.16. Consider a simplified coupled-neuron model,

$$\dot{x} = -\alpha x + h(\beta - y)\,,$$
$$\dot{y} = -\alpha y + h(\beta - x)\,,$$

where $\alpha > 0$ and $\beta > 0$ are constants, and $h(u)$ is a continuous function satisfying $h(-u) = -h(u)$ with $h'(u)$ being two-sided monotonically decreasing as $u \to \pm\infty$. One typical case is the sigmoidal function

$$h(u) = \frac{2}{1 - e^{-au}} - 1\,, \qquad a > 0\,.$$

In this coupled-neuron model, with the general function h as described, one has the following:

(i) there is a fixed point at $x^* = y^* := \lambda$;
(ii) if

$$h'(\beta - \lambda) = -\left.\frac{dh(\beta - y)}{dy}\right|_{y=\lambda} < \alpha\,,$$

then this fixed point is unique and is a stable node;
(iii) if $h'(\beta - \lambda) > \alpha$, then there are two other fixed points, at

$$\big(\mu, \alpha^{-1}h(\beta - \mu)\big) \qquad \text{and} \qquad \big(\alpha^{-1}h(\beta - \mu), \mu\big)$$

respectively, for the same value of μ; they are stable nodes; but the one at (λ, λ) becomes a saddle point in this case.

Now it is noted that a fixed point at $x^* = y^* = \lambda$ is equivalent to showing that $-\alpha\lambda + h(\beta - \lambda) = 0$, or that the straight line $z = \alpha x$ and the curve $z = h(\beta - x)$ has a crossing point on the x–z plane. This is obvious from the geometry depicted in Fig. 1.23, since h is continuous.

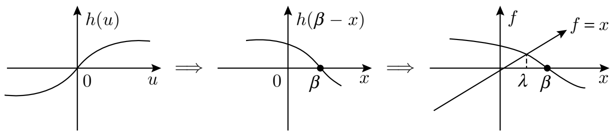


Fig. 1.23 Existence of a fixed point in the coupled-neuron model.

Then it is noted that λ is the unique root of equation

$$f(\lambda) := -\alpha\lambda + h(\beta - \lambda) = 0$$

being equivalent to showing that the function $f(\lambda)$ is strictly monotonic, so that $f(\lambda) = 0$ has only one root. To verify this, observe that

$$f'(\lambda) = -\,\alpha + h'(\beta - \lambda) < 0\,.$$

where $h'(\beta - \lambda) = -d\,h(\beta - \lambda)/d\lambda < \alpha$ by assumption. This implies that $f(\lambda)$ is decreasing. Moreover,

$$h(-u) = -\,h(u) \quad\Longrightarrow\quad -h'(-u) = -\,h'(u) \quad\Longrightarrow\quad h'(-u) = h'(u)\,.$$

Since $h'(u)$ is two-sided monotonically decreasing as $u \to \pm\infty$, so are $h'(-u)$ and

$$f'(\lambda) = -\,\alpha + h'(\beta - \lambda)\,.$$

Therefore, $f(\lambda)$ is strictly monotonic, so $f(\lambda) = 0$ has only one root.

To determine the stability of this root, by examining the Jacobian

$$J\,\Big|_{x=y=\lambda} = \begin{bmatrix} -\,\alpha & h'(\beta - \lambda) \\ h'(\beta - \lambda) & -\,\alpha \end{bmatrix},$$

one can see that its eigenvalues

$$s_1 = -\,\alpha - h'(\beta - \lambda) \qquad \text{and} \qquad s_2 = -\,\alpha + h'(\beta - \lambda)$$

satisfy $s_1 < s_2 < 0$, since $h'(\beta - \lambda) < \alpha$. Hence, $x = y = \lambda$ is a stable node.

Finally, consider the following equation:

$$f(x) = -\,\alpha x + h(\beta - x) = 0$$

on the x–z plane. If $h'(\beta - \lambda) > \alpha$, then it can be verified that the curve $z = h(\beta - x)$ and the straight line $z = \alpha x$ have three crossing points, as shown in Fig. 1.24.

In the above, it has already been shown that at least one crossing point is at $x = \lambda$, where $h'(\beta - \lambda) > \alpha$. It can be further verified that there must

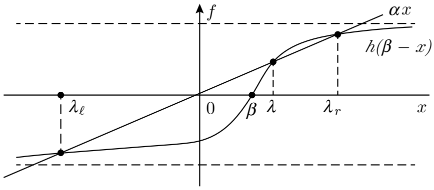


Fig. 1.24 Three crossing points between the curve and the straight line.

be two more crossing points, one at $\lambda_r > \lambda$ and the other at $\lambda_\ell < \lambda$, as depicted in Fig. 1.24.

Indeed, for $x > \lambda$, since $h'(\beta - x)$ is monotonically decreasing as discussed above, one has

$$f'(x) = -\alpha + h'(\beta - x) > -\alpha + h'(\beta - \lambda) > 0\,,$$

so that

$$h'(\beta - x) > \alpha > 0\,, \qquad \text{for all} \quad x > \alpha\,.$$

Since $h'(\beta - x)$ is two-sided monotonically decreasing, $h'(\beta - x) \to -\infty$ as $x \to \infty$, there must be a point, λ_r, such that $h'(\beta - \lambda_r) = \alpha$. This implies that the two curves $h(\beta - x)$ and αx have a crossing point $\lambda_r > \lambda$. The existence of another crossing point, $\lambda_\ell < \lambda$, can be similarly verified.

Next, to find the two new fixed points of the system, one can set

$$-\alpha x + h(\beta - y) = 0$$

to obtain

$$x_1^* = \alpha^{-1} h(\beta - \mu)\,,$$
$$y_1^* = \mu\,,$$

and set

$$-\alpha y + h(\beta - x) = 0$$

to obtain

$$x_2^* = \mu\,,$$
$$y_2^* = \alpha^{-1} h(\beta - \mu)\,,$$

where μ is a real value. These solutions have the same Jacobian, and the eigenvalues of the Jacobian are

$$s_1 = -\alpha - \sqrt{h'(\beta - x) h'(\beta - y)}\,,$$
$$s_2 = -\alpha + \sqrt{h'(\beta - x) h'(\beta - y)}\,,$$

which satisfy $s_1 < s_2 < 0$ at the above two crossing points, and satisfy $s_1 < 0 < s_2$ at $x^* = y^* = \lambda$, where the latter is a saddle node.

Now, return to the general nonlinear autonomous system (1.27).

Definition 1.6. The fixed point $\mathbf{x}^*$ of the autonomous system (1.27) is said to be *hyperbolic*, if all the eigenvalues of the system Jacobian J at this fixed point have nonzero real parts.

The importance of hyperbolic fixed points of a nonlinear autonomous system can be appreciated by the following fundamental result on the local dynamics of the autonomous system.

Theorem 1.6 (Grobman–Hartman Theorem for Systems). *Let* $\mathbf{x}^*$ *be a hyperbolic fixed point of the nonlinear autonomous system* (1.27). *Then, the dynamical properties of this system is qualitatively the same as that of its linearized system, in a (small) neighborhood of* $\mathbf{x}^*$.

Here, the equivalence of the dynamics of the two systems is local, and this theorem is not applicable to a nonautonomous system in general.

Proof. See [Robinson (1995)]: p. 158. □

For discrete-time systems, there is another version of the theorem for maps.

Theorem 1.7 (Grobman–Hartman Theorem for Maps). *Let* $\mathbf{x}^*$ *be a hyperbolic fixed point of the continuously differentiable map* $\mathbf{f} : R^n \to R^n$. *Then, the dynamical properties of this map is qualitatively the same as that of its linearized map* $[D\mathbf{f}(\mathbf{x}^*)] : R^n \to R^n$, *in a (small) neighborhood of* $\mathbf{x}^*$.

Proof. Without loss of generality, assume that $\mathbf{x}^* = 0$. Let $A = [D\mathbf{f}(0)]$, and decompose its state space according to the stable eigenvalues, denoted E^s, and unstable eigenvalues, denoted E^u, respectively. Then, $R^n = E^s \oplus E^u$. Denote $A_s = A\big|_{E^s}$ and $A_u = A\big|_{E^u}$, defined and restricted respectively on the two eigenspaces. By choosing appropriate coordinates, it can be assumed that $||A_s|| < 1$ and $||A_u^{-1}|| < 1 < ||A_u||$. Moreover, denote $\mu = \max\{||A_s||, ||A_u^{-1}||\} < 1$.

In a (small) neighborhood of the fixed point $\mathbf{x}^* = 0$, consider the expansion of the map $\mathbf{f}(\mathbf{x}) = [D\mathbf{f}(0)]\mathbf{x} + \mathbf{g}(\mathbf{x})$, with the higher-order terms satisfying $\mathbf{g}(0) = 0$ and $[D\mathbf{g}(0)] = 0$. Thus, for any small $\delta > 0$ there is a (small) neighborhood of the fixed point, in which $\sup_{\mathbf{x}\in R^n}\{||\mathbf{g}(\mathbf{x})|| + ||\,[D\mathbf{g}(\mathbf{x})]\,||\} < \delta$. This guarantees the existence of $\mathbf{f}^{-1}$, which is also continuously differentiable.

Now, the proof is carried out by verifying the topologically conjugate relationship shown in Fig. 1.20.

The objective is to find a homeomorphism $\mathbf{h} : R^n \to R^n$ in the form of $\mathbf{h} = I + \mathbf{k}$, where I is the identity map and $\mathbf{k} : R^n \to R^n$ is a continuous map, such that

$$\mathbf{h} \circ (A + \mathbf{g}) = A \circ \mathbf{h} .$$

It can be verified that the topologically conjugate relationship is equivalent to either

$$\mathbf{k} = -\mathbf{g} \circ (A + \mathbf{g})^{-1} + A \circ \mathbf{k} \circ (A + \mathbf{g})^{-1}$$

or

$$\mathbf{k} = A^{-1} \circ \mathbf{g} + A^{-1} \circ \mathbf{k} \circ (A + \mathbf{g}) .$$

Now, in the small neighborhood of the fixed point $\mathbf{x}^* = 0$, define a map $F(\mathbf{k}, \mathbf{g}) = F_s(\mathbf{k}, \mathbf{g}) + F_u(\mathbf{k}, \mathbf{g})$, according to the above topologically conjugate relationship, as follows:

$$F_s(\mathbf{k}, \mathbf{g}) = -\mathbf{g}_s \circ (A + \mathbf{g})^{-1} + A_s \circ \mathbf{k}_s \circ (A + \mathbf{g})^{-1}$$

and

$$F_u(\mathbf{k}, \mathbf{g}) = A_u^{-1} \circ \mathbf{g}_u + A_u^{-1} \circ \mathbf{k}_u \circ (A + \mathbf{g}) .$$

It can be verified that $F(\mathbf{k}, \mathbf{g})$ is continuous in $\mathbf{g}$, satisfying $F(0, 0) = 0$, and

$$||F(\mathbf{k}, \mathbf{g})|| \leq ||\mathbf{g}|| + \mu||\mathbf{k}|| \quad \text{and} \quad ||F(\mathbf{k}, \mathbf{g}) - F(\mathbf{k}', \mathbf{g})|| \leq \mu||\mathbf{k} - \mathbf{k}'|| ,$$

for all continuous $\mathbf{k}, \mathbf{k}' \in R^n$. Therefore, $F(\mathbf{k}, \mathbf{g})$ is also continuous in $\mathbf{k}$ and furthermore $F(\cdot, \mathbf{g})$ is a uniform contraction mapping. Consequently, $F(\mathbf{k}', \mathbf{g}) = \mathbf{g}$ for some $\mathbf{k}'$ if and only if there is a $\mathbf{k}$ such that $\mathbf{k}' = \mathbf{k}(\mathbf{g})$. It follows that $\mathbf{h}(\mathbf{g}) = I + \mathbf{k}(\mathbf{g})$ and $\mathbf{h} \circ (A + \mathbf{g}) = A \circ \mathbf{h}$.

It remains to show that this $\mathbf{h} = \mathbf{h}(\mathbf{g})$ is a homeomorphism.

Consider $(A + \mathbf{g}) \circ \mathbf{r} = \mathbf{r} \circ A$ with $\mathbf{r} = I + \mathbf{k}'$. Similarly to the above, it can be verified that for each $\mathbf{g}$, correspondingly there exists a unique $\mathbf{r} = \mathbf{r}(\mathbf{g})$, perhaps for a smaller $\delta > 0$. It follows from the conjugate relationships for both $\mathbf{h}$ and $\mathbf{r}$ that

$$\begin{aligned} \mathbf{h} \circ \mathbf{r} &= \big[A^{-1} \circ \mathbf{h} \circ (A + \mathbf{g})\big] \circ \mathbf{r} \\ &= A^{-1} \circ \mathbf{h} \circ \big[(A + \mathbf{g}) \circ \mathbf{r}\big] \\ &= A^{-1} \circ \mathbf{h} \circ \mathbf{r} \circ A . \end{aligned}$$

On the other hand,

$$\mathbf{h} \circ \mathbf{r} = I + \mathbf{k}' + \mathbf{k} \circ [I + \mathbf{k}'] := I + \mathbf{s} .$$

Hence,

$$\mathbf{s} = A^{-1} \circ \mathbf{s} \circ A \qquad \text{if and only if} \qquad \mathbf{s} = A \circ \mathbf{s} \circ A^{-1} .$$

This implies that $F(\mathbf{s}, 0) = \mathbf{s}$. Since the fixed point is unique and since $F(0,0) = 0$, one has $\mathbf{s} = 0$; therefore, $\mathbf{h} \circ \mathbf{r} = I$. Similarly, it can be shown that $\mathbf{r} \circ \mathbf{h} = I$. Thus, $\mathbf{h}$ is a homeomorphism. □

The following example provides a visual illustration of Theorem 1.6.

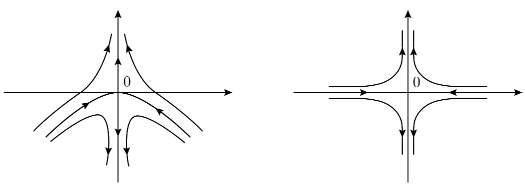
Example 1.17. Consider a nonlinear system,

$$\dot{x} = -x ,$$
$$\dot{y} = x^2 + y .$$

Its linearized system at $(0,0)$ is

$$\dot{x} = -x ,$$
$$\dot{y} = y .$$

Their phase portraits are shown in Fig. 1.25.



(a) nonlinear system (b) linearized system

Fig. 1.25 Illustration of the Grobman–Hartman Theorem.

As can be seen, the two phase portraits are not exactly the same, but they are qualitatively the same, namely topologically equivalent, in the sense that one can be obtained from the other by smoothly bending the flow of the solution curves.

Exercises

1.1 For the following two linear systems, sketch by hand their phase portraits and classify their fixed points:

$$\dot{x} = -3x + 4y\,, \qquad \dot{y} = -2x + 3y\,,$$

and

$$\dot{x} = 4x - 3y\,, \qquad \dot{y} = 8x - 6y\,.$$

1.2 Consider the Duffing oscillator equation

$$\ddot{x}(t) + a\,\dot{x}(t) + b\,x(t) + c\,x^3(t) = \gamma\,\cos(\omega t)\,, \tag{1.30}$$

where a, b, c are constants and $\gamma\,\cos(\omega t)$ is an external force input. By defining $y(t) = \dot{x}(t)$, rewrite this equation in a state-space form. Use a computer to plot its phase portraits for the following cases: $a = 0.4$, $b = -1.1$, $c = 1.0$, $\omega = 1.8$, and (1) $\gamma = 0.620$, (2) $\gamma = 1.498$, (3) $\gamma = 1.800$, (4) $\gamma = 2.100$, (5) $\gamma = 2.300$, (6) $\gamma = 7.000$. Indicate the directions of the orbit flows.

1.3 Consider the Chua circuit, shown in Fig. 1.26, which consists of one inductor (L), two capacitors (C_1, C_2), one linear resistor (R), and one piecewise-linear resistor (g).

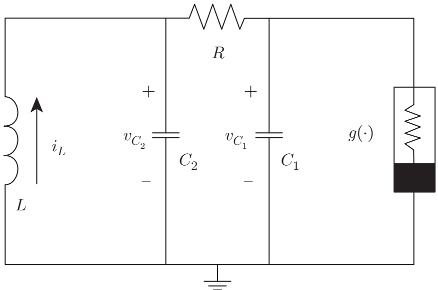


Fig. 1.26 The Chua circuit.

The dynamical equations of the circuit are

$$\begin{aligned} C_1\,\dot{v}_{C_1} &= R^{-1}\big(v_{C_2} - v_{C_1}\big) - g\big(v_{C_1}\big)\,, \\ C_2\,\dot{v}_{C_2} &= R^{-1}\big(v_{C_1} - v_{C_2}\big) + i_L\,, \\ L\,\dot{i}_L &= -v_{C_2}\,, \end{aligned} \tag{1.31}$$

where i_L is the current through the inductor L, v_{C_1} and v_{C_2} are the voltages across C_1 and C_2 respectively, and

$$g(v_{C_1}) = m_0 v_{C_1} + \frac{1}{2}(m_1 - m_0)\left(|v_{C_1} + 1| - |v_{C_1} - 1|\right),$$

with $m_0 < 0$ and $m_1 < 0$ being some appropriately chosen constants. This piecewise linear function is shown in Fig. 1.27.

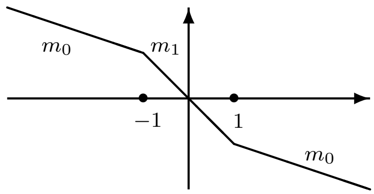


Fig. 1.27 The piecewise linear resistance in the Chua circuit.

Verify that, by defining $p = C_2/C_1 > 0$ and $q = C_2R^2/L > 0$, with a change of variables, $x(\tau) = v_{C_1}(t)$, $y(\tau) = V_{C_2}(t)$, $z(\tau) = R\,i_L(t)$, and $\tau = t/(GC_2)$, the above circuit equations can be reformulated into the state-space form, as

$$\begin{aligned}\dot{x} &= p\left(-x + y - f(x)\right),\\ \dot{y} &= x - y + z\,,\\ \dot{z} &= -q\,y\,,\end{aligned} \tag{1.32}$$

where $f(x) = R\,g(v_{C_1})$.

For $p = 10.0$, $q = 14.87$, $m_0 = -0.68$, $m_1 = -1.27$, with initial conditions $(-0.1, -0.1, -0.1)$, use a computer to plot the circuit orbit portrait in the x–y–z space; or, show the portrait projections on the three principal planes: (a) the x–y plane, z–x the plane, and (c) the z–y plane.

1.4 Consider the following nonlinear system:

$$\begin{aligned}\dot{x} &= y + \kappa\,x\left(x^2 + y^2\right),\\ \dot{y} &= -x + \kappa\,y\left(x^2 + y^2\right).\end{aligned}$$

Show that $(0,0)$ is the only fixed point, and find under what condition on the constant κ, this fixed point is a stable or unstable focus. [Hint: Polar coordinates may be convenient to use.]

1.5 For the following two nonlinear systems, determine the types and the stabilities of their fixed points:

$$\ddot{y} + y + y^3 = 0\,,$$

and

$$\begin{aligned}\dot{x} &= -x + xy\,,\\ \dot{y} &= y - xy\,.\end{aligned}$$

1.6 For each of the following systems, find the fixed points and determine their types and stabilities:

(a)

$$\begin{aligned}\dot{x} &= y\cos(x)\,,\\ \dot{y} &= \sin(x)\,;\end{aligned}$$

(b)

$$\begin{aligned}\dot{x} &= (x-y)\left(x^2+y^2-1\right),\\ \dot{y} &= (x+y)\left(x^2+y^2-1\right);\end{aligned}$$

(c)

$$\begin{aligned}\dot{x} &= 1 - x\,y^{-1}\,,\\ \dot{y} &= -x\,y^{-1}\left(1 - x\,y^{-1}\right);\end{aligned}$$

(d)

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= -x - \frac{1}{3}\,x^3 - y\,.\end{aligned}$$

1.7 Let f and g be two topologically equivalent maps in metric spaces X and Y, and h be the homeomorphism satisfying $g = h^{-1} \circ f \circ h$. Verify that $h\big(g^k(x)\big) = f^k\big(h(x)\big)$ for any integer $k \geq 0$.

1.8 Verify that the following two maps are not topologically equivalent in any neighborhood of the origin: $f(x) = x$ and $g(x) = x^2$.