Discrete-Time Signals and Systems

Chapter Intended Learning Outcomes:

(i) Understanding deterministic and random discrete-time signals and ability to generate them

(ii) Ability to recognize the discrete-time system properties, namely, memorylessness, stability, causality, linearity and time-invariance

(iii) Understanding discrete-time convolution and ability to perform its computation

(iv) Understanding the relationship between difference equations and discrete-time signals and systems

Discrete-Time Signal

- Discrete-time signal can be generated using a computing software such as MATLAB
- It can also be obtained from sampling continuous-time signals in real world

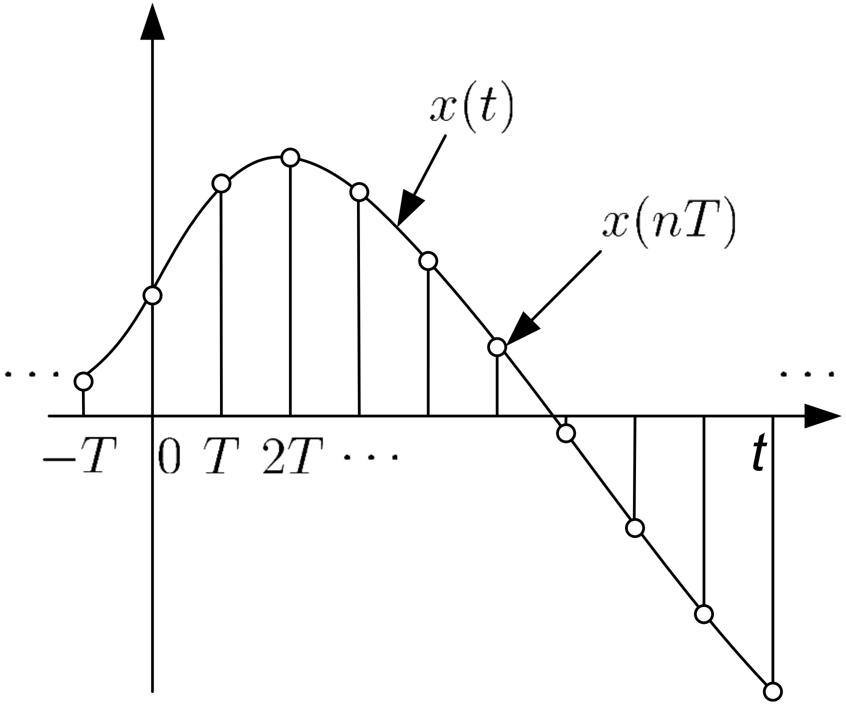


Fig.3.1: Discrete-time signal obtained from analog signal

- The discrete-time signal $x(nT)$ is equal to $x(t)$ only at the sampling interval of $t = nT$, $n = \cdots -1, 0, 1, 2, \cdots$

$$x[n] = x(t)|_{t=nT} = x(nT), \quad n = \cdots -1, 0, 1, 2, \cdots \tag{3.1}$$

where T is called the sampling period

- $x[n]$ is a sequence of numbers, $\cdots x[-1], x[0], x[1], x[2], \cdots$, with n being the time index

Basic Sequences

- Unit Sample (or Impulse)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{3.2}$$

It is similar to the continuous-time unit impulse $\delta(t)$ which is defined in (2.10)-(2.12)

$\delta[n]$ is simpler than $\delta(t)$ because it is well defined for all n while $\delta(t)$ is not defined at $t=0$

- Unit Step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{3.3}$$

It is similar to to the continuous-time $u(t)$ of (2.14)

$u[n]$ is well defined for all n but $u(t)$ is not defined $t=0$.

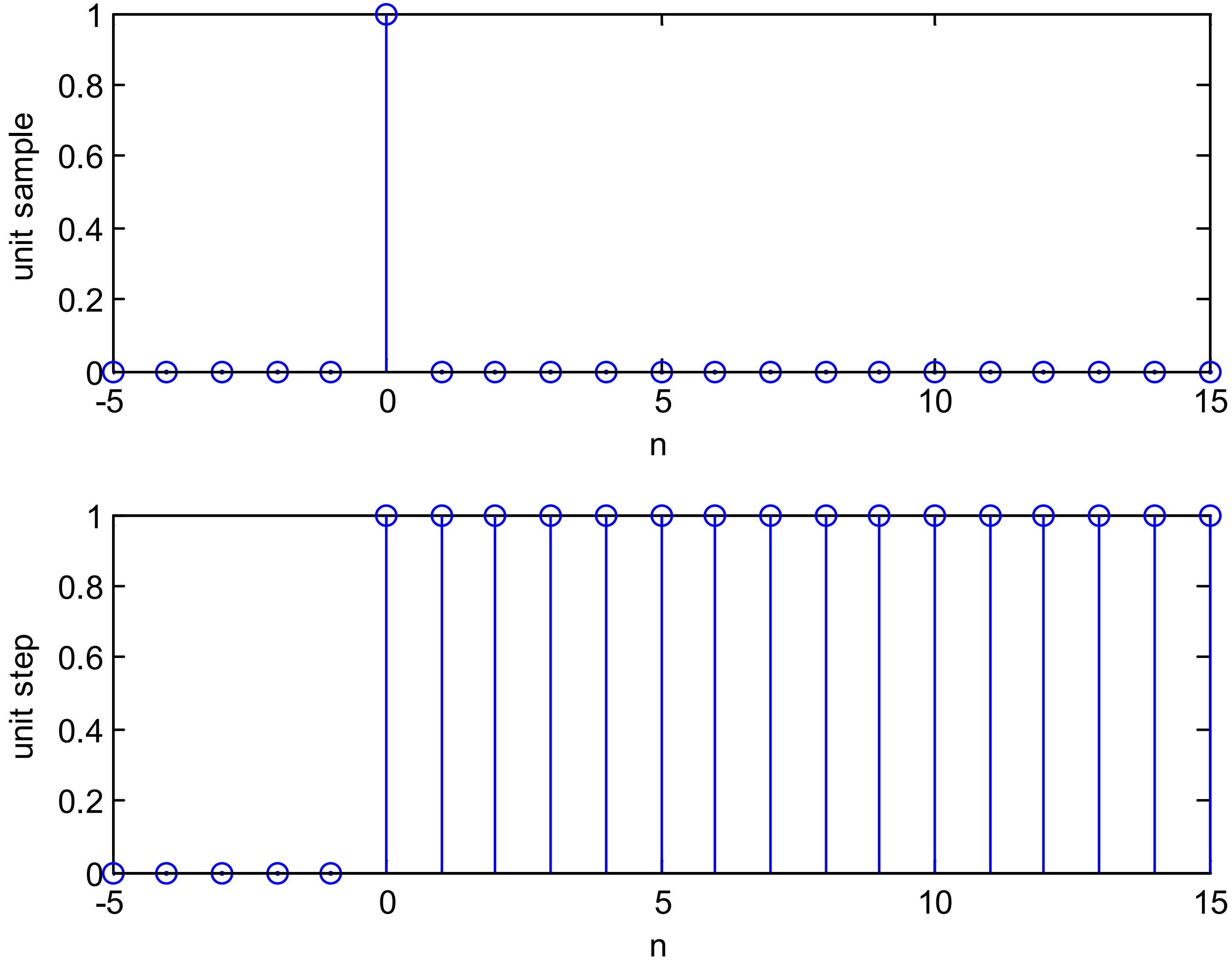


Fig.3.2:Unit sample $\delta[n]$ and unit step $u[n]$

Analogous to (2.13), $\delta[n]$ is an important function because it serves as the building block of any discrete-time signal $x[n]$:

$$\begin{aligned} x[n] &= \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots \\ &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \end{aligned} \tag{3.4}$$

For example, $u[n]$ can be expressed in terms of $\delta[n]$ as:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \tag{3.5}$$

Conversely, we can use $u[n]$ to represent $\delta[n]$:

$$\delta[n] = u[n] - u[n-1] \tag{3.6}$$

Introduction to MATLAB

- MATLAB stands for ”Matrix Laboratory”
- Interactive matrix-based software for numerical and symbolic computation in scientific and engineering applications
- Its user interface is relatively simple to use, e.g., we can use the `help` command to understand the usage and syntax of each MATLAB function
- Together with the availability of numerous toolboxes, there are many useful and powerful commands for various disciplines
- MathWorks offers MATLAB to C conversion utility
- Similar packages include Maple and Mathematica

Discrete-Time Signal Generation using MATLAB

A deterministic discrete-time signal $x[n]$ satisfies a generating model with known functional form:

$$x[n] = f(\boldsymbol{\psi}, n) \qquad (3.7)$$

where $f(\cdot)$ is a function of parameter vector $\boldsymbol{\psi}$ and time index n. That is, given $f(\cdot)$ and $\boldsymbol{\psi}$, $x[n]$ can be produced

e.g., the time-shifted unit sample $\delta[n-n_0]$ and unit step function $u[n-n_0]$, where the parameter is n_0

e.g., for an exponential function α^{n-n_0}, we have $\boldsymbol{\psi} = [\alpha \quad n_0]$ where α is the decay factor and n_0 is the time shift

e.g., for a sinusoid $A\cos(\omega n + \theta)$, we have $\boldsymbol{\psi} = [A \quad \omega \quad \theta]$

Example 3.1

Use MATLAB to generate a discrete-time sinusoid of the form:

$$x[n] = A\cos(\omega n + \theta), \quad n = 0, 1, \cdots, N-1$$

with $A = 1$, $\omega = 0.3$, $\theta = 1$ and $N = 21$, which has a duration of 21 samples

We can generate $x[n]$ by using the following MATLAB code:

```
N=21;                     %number of samples is 21
A=1;                      %tone amplitude is 1
w=0.3;                    %frequency is 0.3
p=1;                      %phase is 1
for n=1:N
x(n)=A*cos(w*(n-1)+p); %time index should be >0
end
```

Note that `x` is a vector and its index should be at least 1.

Alternatively, we can also use:

```
N=21;                    %number of samples is 21
A=1;                     %tone amplitude is 1
w=0.3;                   %frequency is 0.3
p=1;                     %phase is 1
n=0:N-1;                 %define time index vector
x=A.*cos(w.*n+p);        %first time index is also 1
```

Both give

```
x =
Columns 1 through 7
0.5403 0.2675 -0.0292 -0.3233 -0.5885 -0.8011 -0.9422
Columns 8 through 14
-0.9991 -0.9668 -0.8481 -0.6536 -0.4008 -0.1122 0.1865
Columns 15 through 21
0.4685 0.7087 0.8855 0.9833 0.9932 0.9144 0.7539
```

Which approach is better? Why?

To plot $x[n]$, we can either use the commands `stem(x)` and `plot(x)`

If the time index is not specified, the default start time is $n=1$

Nevertheless, it is easy to include the time index vector in the plotting command

e.g., Using `stem` to plot $x[n]$ with the correct time index:

```
n=0:N-1;                 %n is vector of time index
stem(n,x)                %plot x versus n
```

Similarly, `plot(n,x)` can be employed to show $x[n]$

The MATLAB programs for this example are provided as `ex3_1.m` and `ex3_1_2.m`

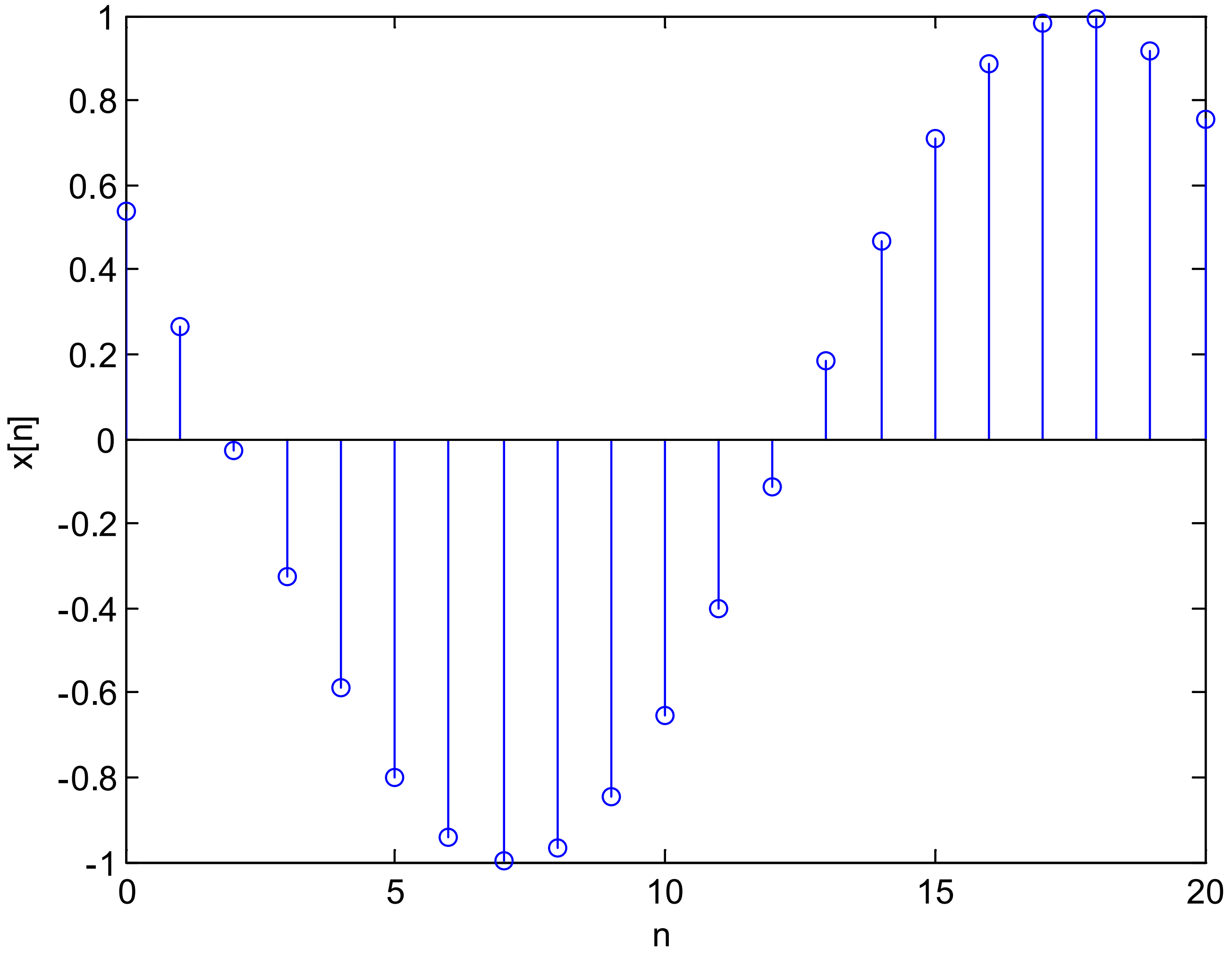


Fig.3.3: Plot of discrete-time sinusoid using stem

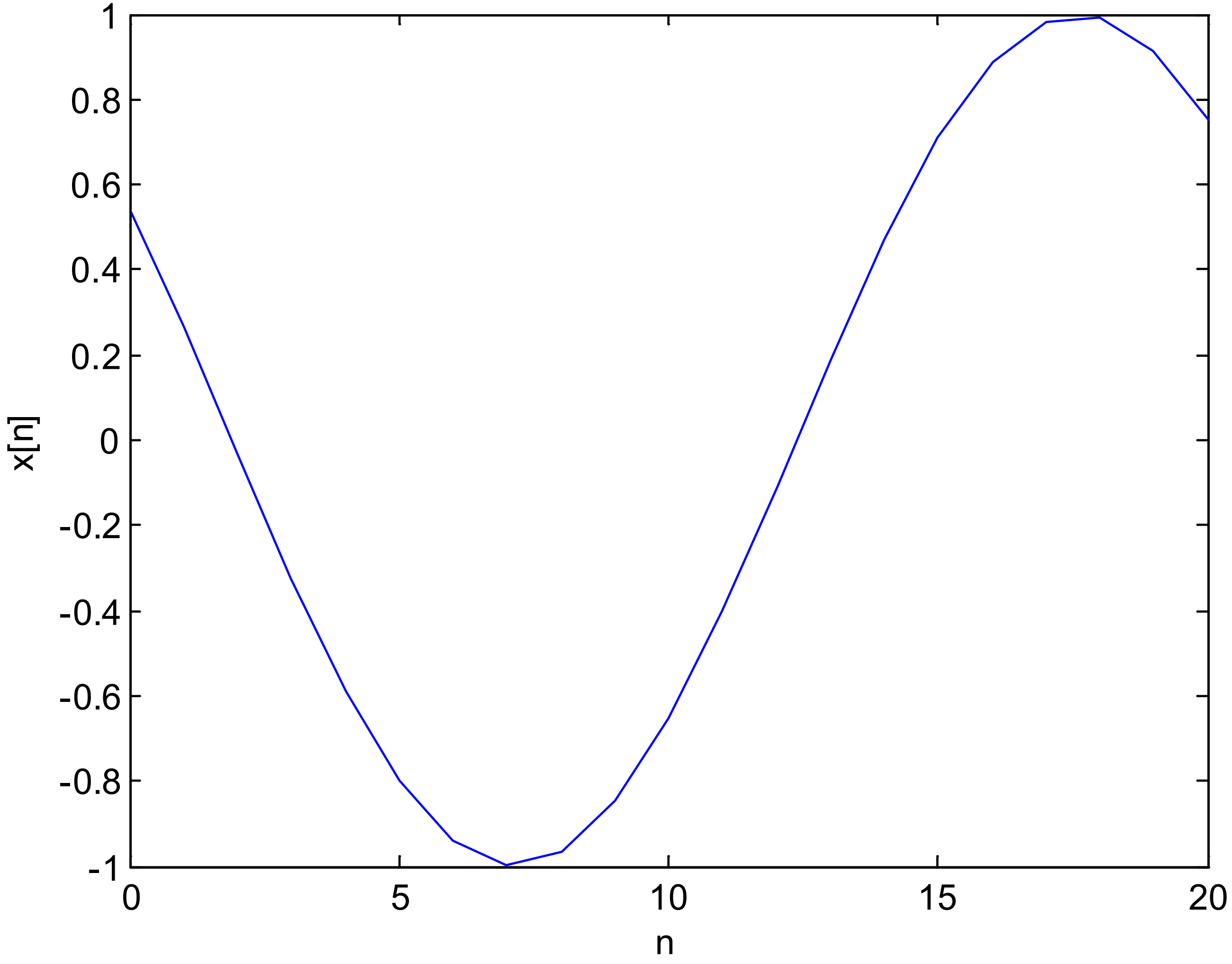


Fig.3.4: Plot of discrete-time sinusoid using plot

Apart from deterministic signal, random signal is another importance signal class. It cannot be described by mathematical expressions like deterministic signals but is characterized by its probability density function (PDF). MATLAB has commands to produce two common random signals, namely, uniform and Gaussian (normal) variables.

A uniform integer sequence $x[n]$ whose values are uniformly distributed between 0 and m, can be generated using:

$$x[n] = (a \cdot x[n-1]) \mod (m) \tag{3.8}$$

where a and m are very large positive integers, $x[n]$ is the reminder of dividing $a \cdot x[n-1]$ by m

Each admissible value of $0 \leq x[n] < m$ has the same probability of occurrence of approximately $1/m$

We also need an initial integer or seed, say, $x[-1] \in [0, m-1]$, for starting the generation of $x[n]$

(3.8) can be easily modified by properly scaling and shifting

e.g., a random number which is uniformly between −0.5 and 0.5, denoted by $y[n]$, is obtained from $x[n]$:

$$y[n] = \frac{x[n]}{m} - 0.5 \tag{3.9}$$

The MATLAB command `rand` is used to generate random numbers which are uniformly between 0 and 1

e.g., each realization of `stem(0:20,rand(1,21))` gives a distinct and random sequence, with values are bounded between 0 and 1

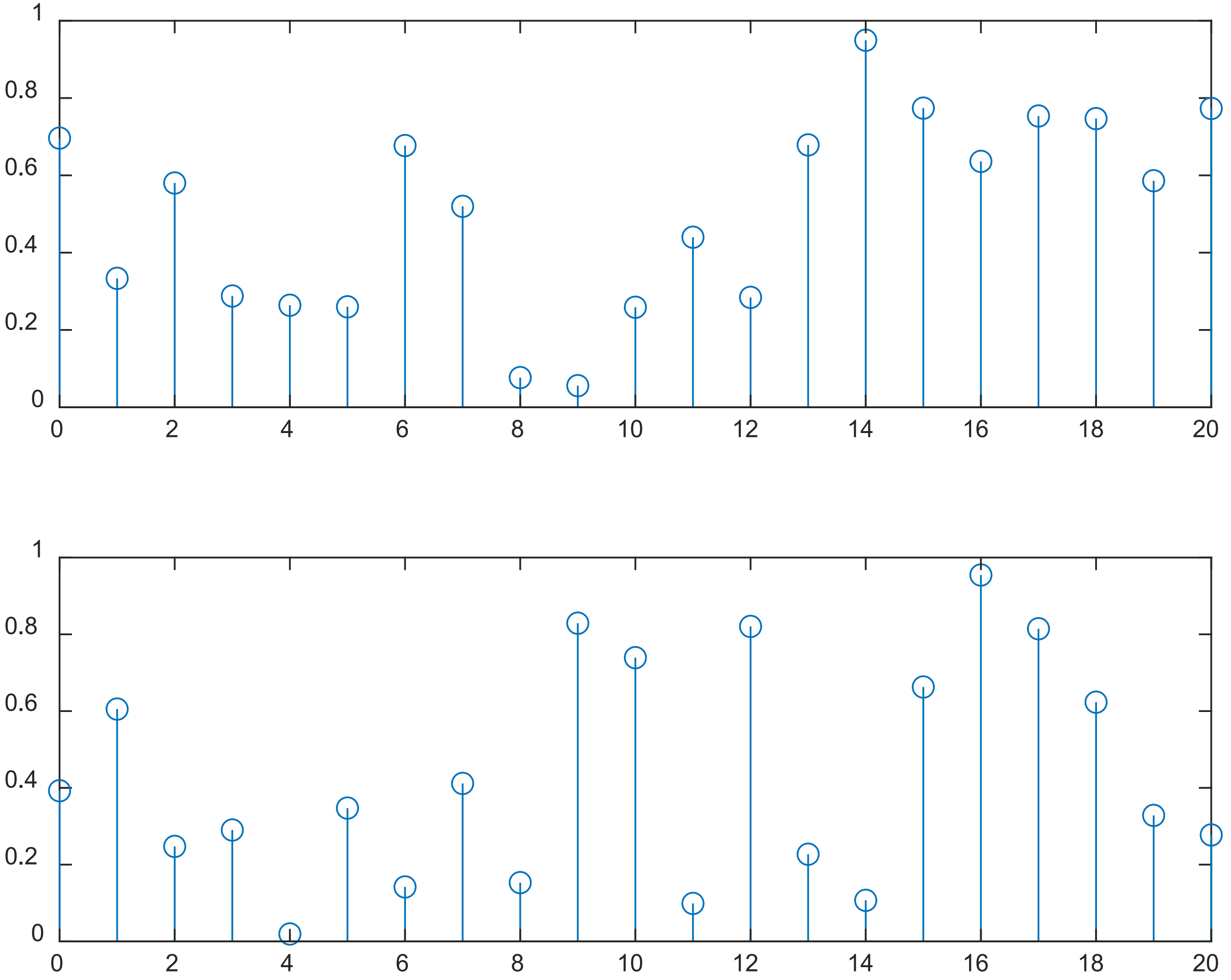


Fig.3.5: Uniform number realizations using `rand`

Example 3.2

Use MATLAB to generate a sequence of 10000 random numbers uniformly distributed between –0.5 and 0.5 based on the command `rand`. Verify its characteristics.

The PDF of `u`, denoted by $p(u)$, can be written as:

$$p(u) = \begin{cases} 1, & -0.5 \le u \le 0.5 \\ 0, & \text{otherwise} \end{cases}$$

such that $\int_{-\infty}^{\infty} p(u) du = 1$. The theoretical mean and power of u, are computed as

$$E\{u\} = \int_{-\infty}^{\infty} up(u)du = \int_{-0.5}^{0.5} udu = \frac{1}{2}u^2 \Big|_{-0.5}^{0.5} = 0$$

and

$$E\{u^2\} = \int_{-\infty}^{\infty} u^2 p(u) du = \int_{-0.5}^{0.5} u^2 p(u) du = \frac{1}{3}u^3 \bigg|_{-0.5}^{0.5} = \frac{1}{12}$$

The variance is then:

$$\text{var}(u) = E\{(u - E\{u\})^2\} = E\{u^2\} = \frac{1}{12}$$

According to (3.9), we use `u=rand(1,10000)-0.5` to generate the sequence. Empirically, the mean, power, and variance can be computed as:

$$\hat{\mu} = \frac{1}{10000}\sum_{i=1}^{10000} u_i \qquad \frac{1}{10000}\sum_{i=1}^{10000} u_i^2 \qquad \frac{1}{10000}\sum_{i=1}^{10000} (u_i - \hat{\mu})^2$$

where u_i represents the ith element of `u`.

The `mean` command can be used:

```
mean(u) % mean performs averaging over 10000 values
= -4.3942e-04

mean(u.*u) %empirical power
= 0.0829

mean((u-mean(u)).*(u-mean(u))) %empirical variance
= 0.0829
```

which can also be computed using `var(u,1)`. They all align with the theoretical calculations.

To visualize the distribution, we use:

```
histogram(u,'Normalization','pdf')
```

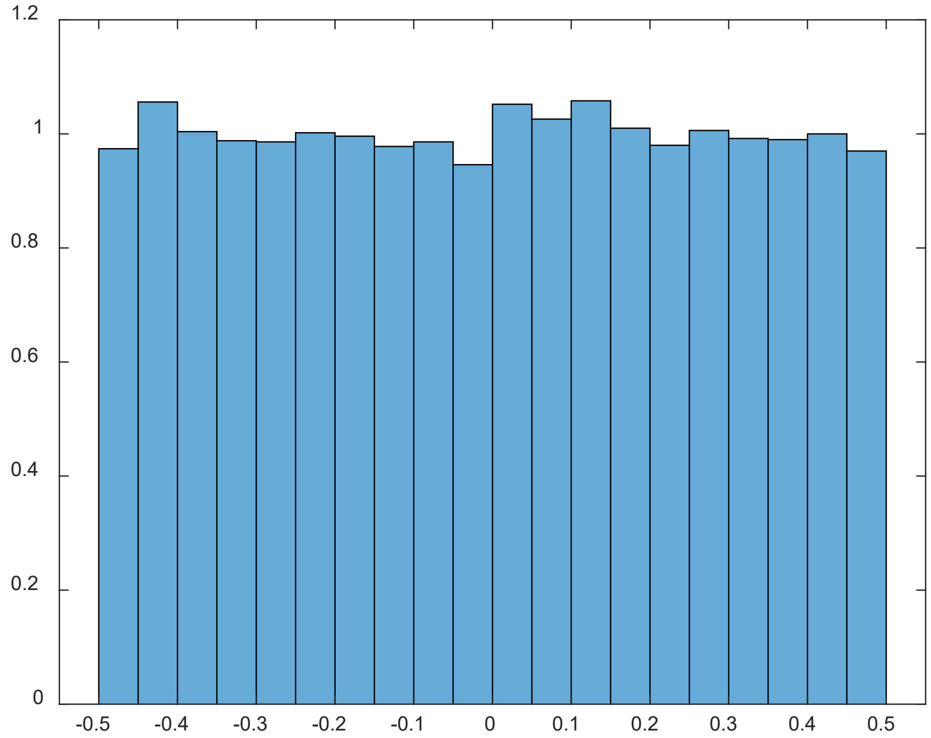


Fig.3.6: Empirical PDF for uniform sequence

We can see that all bins with width 0.05 is a rectangle with height close to 1, e.g.,

$$\int_{-0.5}^{-0.45} p(u)du \approx \cdots \approx \int_{0.45}^{0.5} p(u)du \approx 0.05$$

Gaussian numbers can be generated from the uniform variables

Given a pair of independent random numbers uniformly distributed between 0 and 1, (u_1, u_2) , a pair of independent Gaussian numbers (w_1, w_2), which have zero mean and power (or variance) of 1, can be generated from:

$$w_1 = \sqrt{-2\ln(u_1)} \cdot \cos(2\pi u_2) \tag{3.10}$$

and

$$w_2 = \sqrt{-2\ln(u_1)} \cdot \sin(2\pi u_2) \tag{3.11}$$

The MATLAB command is `randn`. Equations (3.10) and (3.11) are known as the Box-Mueller transformation

e.g., each realization of `stem(0:20,randn(1,21))` gives a distinct and random sequence, whose values are fluctuating around zero

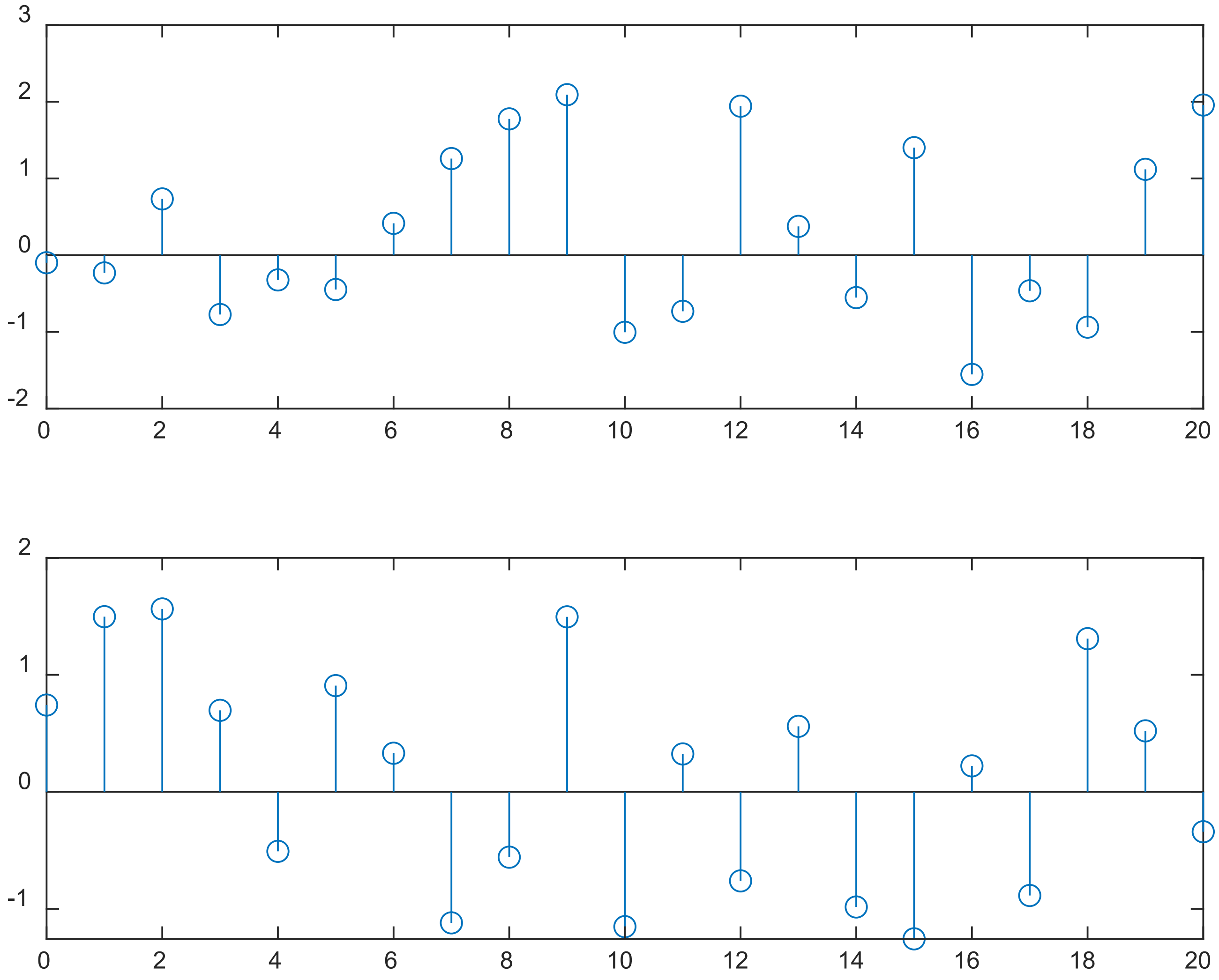


Fig.3.7: Gaussian number realizations using `randn`

Example 3.3

Use the MATLAB command `randn` to generate a zero-mean Gaussian sequence of length 10000 and unity power. Verify its characteristics.

In theory, each Gaussian variable w should have zero mean and power of 1, i.e.,

$$E\{w\} = 0$$

$$E\{w^2\} = 1$$

The variance can be easily shown to be:

$$\mathrm{var}(w) = E\{(w - E\{w\})^2\} = E\{w^2\} = 1$$

The standard deviation is defined as:

$$\sigma_w = \sqrt{\mathrm{var}(w)} = 1$$

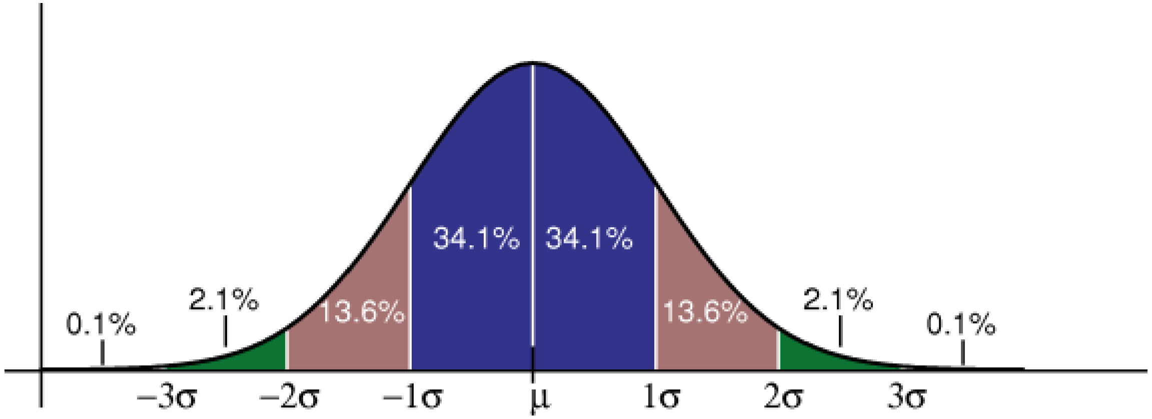
Note that 99.7% of the Gaussian data are within $\pm 3\sigma_w$

We use `w=randn(1,10000)` to generate the sequence and `histogram(w,'Normalization','pdf')` to show its distribution

The distribution aligns with Gaussian variables which is indicated by the bell shape

We see that most of the values are within $\pm 3\sigma_w$ with $\sigma_w = 1$

The empirical mean, power, and variance of `w` computed using `mean(w)`, `mean(w.*w)` and `var(w,1)` are 0.0015 and 0.9764 and 0.9764, respectively.



Gaussian Distribution (princeton.edu)

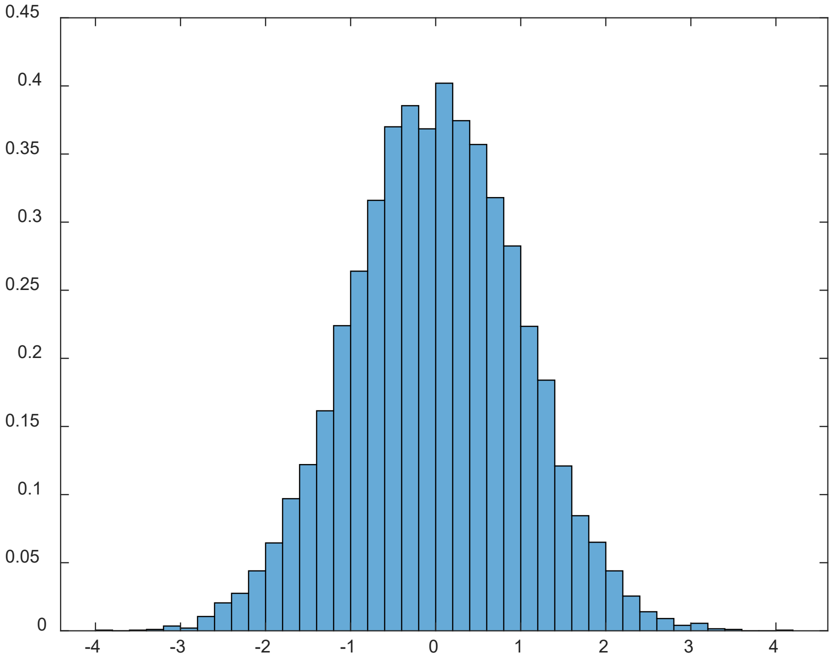


Fig.3.8: Empirical PDF for Gaussian sequence

Discrete-Time Systems

A discrete-time system is an operator $\mathcal{T}$ which maps an input sequence $x[n]$ into an output sequence $y[n]$:

$$y[n] = \mathcal{T}\{x[n]\} \tag{3.12}$$

- Memoryless: $y[n]$ at time n depends only on $x[n]$ at time n

Are they memoryless systems?

$$\mathbf{y}[\boldsymbol{n}]=(\mathbf{x}[\boldsymbol{n}])^2$$

$$\mathbf{y}[\boldsymbol{n}]=\mathbf{x}[\boldsymbol{n}]+\mathbf{x}[\boldsymbol{n}-2]$$

- Linear: obey principle of superposition, i.e., if

$$y_1[n] = \mathcal{T}\{x_1[n]\} \quad \text{and} \quad y_2[n] = \mathcal{T}\{x_2[n]\}$$

then

$$\mathcal{T}\{ax_1[n] + bx_2[n]\} = a\mathcal{T}\{x_1[n]\} + b\mathcal{T}\{x_2[n]\} = ay_1[n] + by_2[n] \tag{3.13}$$

Example 3.4

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not:

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \cdots + x[n-1] + x[n]$$

A standard approach to determine the linearity of a system is given as follows. Let

$$y_i[n] = \mathcal{T}\{x_i[n]\}, \quad i = 1, 2, 3$$

with

$$x_3[n] = ax_1[n] + bx_2[n]$$

If $y_3[n] = ay_1[n] + by_2[n]$, then the system is linear. Otherwise, the system is nonlinear.

Assigning $x_3[n] = ax_1[n] + bx_2[n]$, we have:

$$\begin{aligned} y_3[n] &= \sum_{k=-\infty}^{n} x_3[k] \\ &= \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]) \\ &= a \sum_{k=-\infty}^{n} x_1[k] + b \sum_{k=-\infty}^{n} x_2[k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Note that the outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = \sum_{k=-\infty}^{n} x_1[k]$ and $y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$

As a result, the system is linear

Example 3.5

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not:

$$y[n] = 3x^2[n] + 2x[n-3]$$

The system outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = 3x_1^2[n] + 2x_1[n-3]$ and $y_2[n] = 3x_2^2[n] + 2x_2[n-3]$. Assigning $x_3[n] = ax_1[n] + bx_2[n]$, its system output is then:

$$\begin{aligned}
y_3[n] &= 3x_3^2[n] + 2x_3[n-3] \\
&= 3\left(ax_1[n] + bx_2[n]\right)^2 + 2ax_1[n-3] + 2bx_2[n-3] \\
&= 3\left(a^2x_1^2[n] + b^2x_2^2[n] + 2abx_1[n]x_2[n]\right) + 2ax_1[n-3] + 2bx_2[n-3] \\
&\neq a\left(3x_1^2[n] + 2x_1[n-3]\right) + b\left(3x_2^2[n] + 2x_2[n-3]\right) \\
&= ay_1[n] + by_2[n]
\end{aligned}$$

As a result, the system is nonlinear

- Time-Invariant: a time-shift of input causes a corresponding shift in output, i.e., if

$$y[n] = \mathcal{T}\{x[n]\}$$

then

$$y[n-n_0] = \mathcal{T}\{x[n-n_0]\} \quad (3.14)$$

Example 3.6

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

A standard approach to determine the time-invariance of a system is given as follows.

Let

$$y_1[n] = \mathcal{T}\{x_1[n]\}$$

with $x_1[n] = x[n-n_0]$

If $y_1[n] = y[n-n_0]$, then the system is time-invariant. Otherwise, the system is time-variant.

From the given input-output relationship, $y[n-n_0]$ is:

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Let $x_1[n] = x[n-n_0]$, its system output is:

$$\begin{aligned} y_1[n] &= \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0] = \sum_{l=-\infty}^{n-n_0} x[l], \quad l = k - n_0 \\ &= y[n-n_0] \end{aligned}$$

As a result, the system is time-invariant.

Example 3.7

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not:

$$y[n] = 3x[3n]$$

From the given input-output relationship, $y[n-n_0]$ is of the form:

$$y[n-n_0] = 3x[3(n-n_0)] = 3x[3n-3n_0]$$

Let $x_1[n] = x[n-n_0]$, its system output is:

$$y_1[n] = 3x_1[3n] = 3x[3n-n_0] \neq y[n-n_0]$$

As a result, the system is time-variant.

- Causal: output $y[n]$ at time n depends on input $x[n]$ up to time n

  For linear time-invariant (LTI) systems, there is an alternative definition. A LTI system is causal if its impulse response $h[n]$ satisfies:

$$h[n] = 0, \quad n < 0 \tag{3.15}$$

  Analogous to the continuous-time case, $h[n]$ is discrete-time signal which is the output of a discrete-time LTI system when the input is the impulse $\delta[n]$

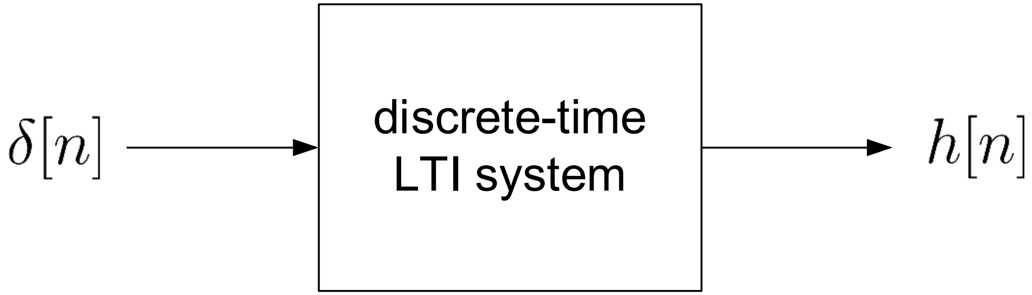


Fig. 3.9: Discrete-time impulse response

Are they causal systems?

y[*n*]=x[*n*]+x[n+1]

y[*n*]=x[*n*]+x[*n*-2]

- Stable: a bounded input $x[n]$ ($|x[n]| < \infty$) produces a bounded output $y[n]$ ($|y[n]| < \infty$)

  For LTI system, stability also corresponds to

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \qquad (3.16)$$

Are they stable systems?

y[*n*]=x[*n*]+x[n+1]

y[*n*]=1/x[*n*]

Note that these five definitions hold for continuous-time systems. That is, we just need to replace the discrete index n to the continuous t . In particular, for stability, $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ is changed to $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

Convolution

Convolution is used to describe the relationship between input, output and impulse response of a LTI system in time domain

Recall discrete-time impulse response $h[n] = \mathcal{T}\{\delta[n]\}$ of a LTI system and recall (3.4) that a discrete-time signal is a linear combination of impulses with different time-shifts:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \qquad (3.4)$$

Consider $x[n]$ as the system input with $y[n]$ being the output:

$$y[n] = \mathcal{T}\{x[n]\} = \mathcal{T}\left\{\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]\right\}$$
$$= \sum_{m=-\infty}^{\infty} x[m]\mathcal{T}\{\delta[n-m]\} \qquad (3.17)$$

due to the linearity property of (3.13)

Furthermore, using time-invariance property yields:

$$h[n-m] = \mathcal{T}\{\delta[n-m]\} \qquad (3.18)$$

Substituting (3.18) into (3.17), we obtain:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] \otimes h[n] \tag{3.19}$$

which is called the convolution of $x[n]$ and $h[n]$, and $\otimes$ denotes the convolution operator

According to (3.19), $h[n]$ completely characterizes the LTI system because for any input $x[n]$, the output $y[n]$ can be computed with the use of $h[n]$ via convolution

Compared with (2.24), discrete-time convolution is simpler because only multiplication and addition are involved

There are three properties in convolution:

- Commutative

$$\begin{aligned} x[n] \otimes h[n] &= h[n] \otimes x[n] \\ &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \end{aligned} \quad (3.20)$$

- Associative

$$x[n] \otimes (h_1[n] \otimes h_2[n]) = (x[n] \otimes h_1[n]) \otimes h_2[n] \quad (3.21)$$

Combining (3.20) and (3.21) yields:

$$\begin{aligned} y[n] &= x[n] \otimes h_1[n] \otimes h_2[n] \\ &= x[n] \otimes h_2[n] \otimes h_1[n] \\ &= x[n] \otimes (h_1[n] \otimes h_2[n]) \end{aligned} \quad (3.22)$$

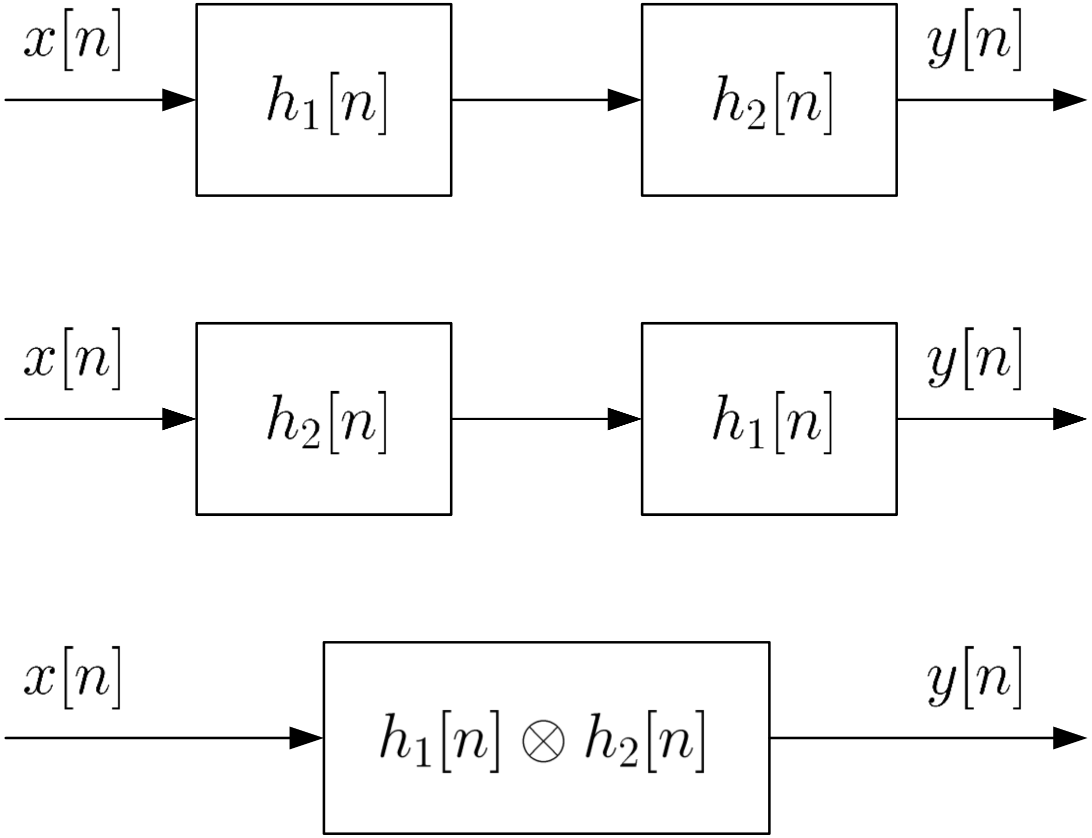


Fig.3.10: Cascade interconnection and convolution

- Distributive

$$y[n] = x[n] \otimes (h_1[n] + h_2[n]) = x[n] \otimes h_1[n] + x[n] \otimes h_2[n] \quad (3.23)$$

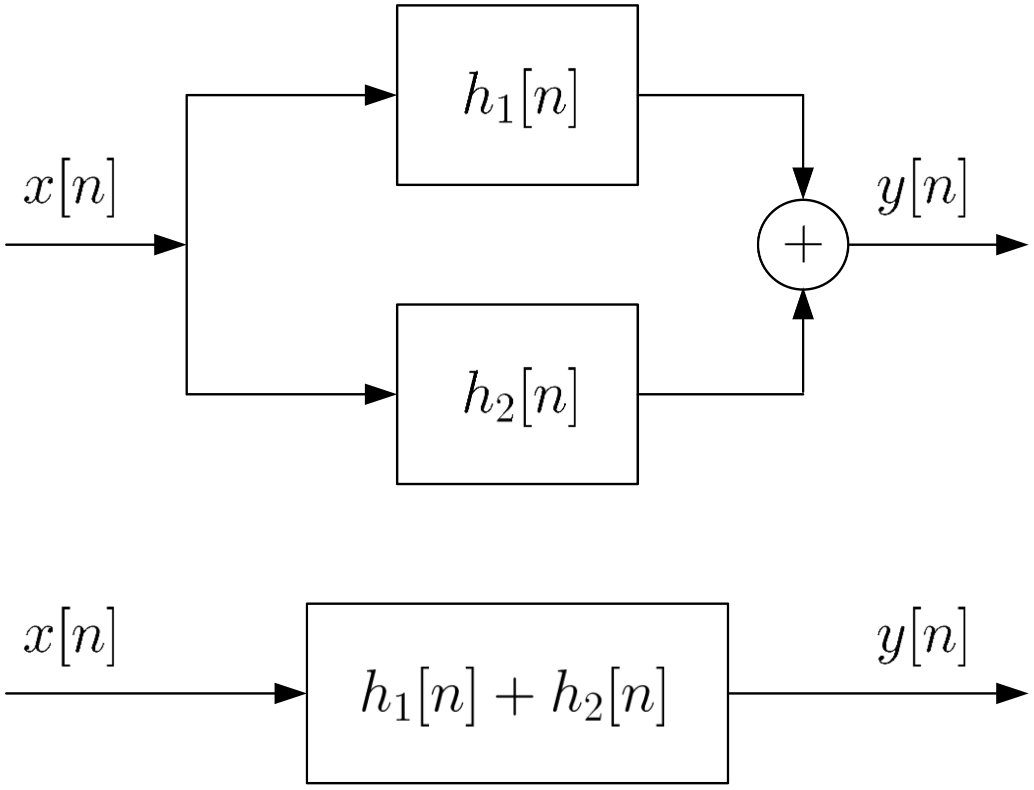
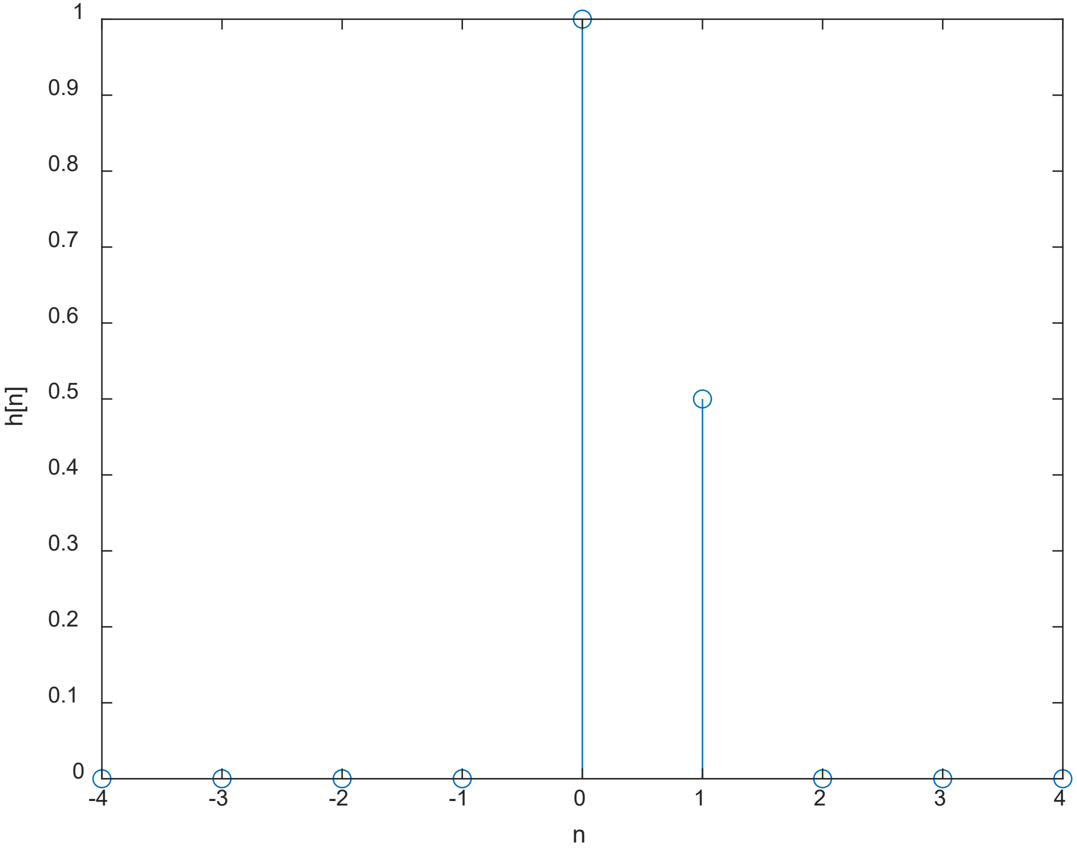


Fig.3.11: Parallel interconnection and convolution

Example 3.8

Compute the output $y[n]$ if the input is $x[n] = u[n]$ and the LTI system impulse response is $h[n] = \delta[n] + 0.5\delta[n-1]$. Determine the stability and causality of system.



```
>> n=-4:4;
   h=[ 0 0 0 0 1 0.5 0 0 0];
   stem(n,h);
   xlabel('n');
   ylabel('h[n]');
```

Using (3.19), we have:

$$
\begin{aligned}
y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} u[m] \left(\delta[n-m] + 0.5\delta[n-1-m] \right) \\
&= \sum_{m=0}^{\infty} \left(\delta[n-m] + 0.5\delta[n-1-m] \right) \\
&= \sum_{m=0}^{\infty} \delta[n-m] + 0.5 \sum_{m=0}^{\infty} \delta[n-1-m] = u[n] + 0.5u[n-1]
\end{aligned}
$$

Alternatively, we can first establish the general relationship between $y[n]$ and $x[n]$ with the specific $h[n]$ and (3.4):

$$\begin{aligned} y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{m=-\infty}^{\infty} x[m] \left(\delta[n-m] + 0.5\delta[n-1-m] \right) \\ &= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] + 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-1-m] \\ &= x[n] + 0.5x[n-1] \end{aligned}$$

Substituting $x[n] = u[n]$ yields the same $y[n]$.

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{1} |h[n]| = 1.5 < \infty$ and $h[n] = 0$ for $n < 0$ the system is stable and causal

Example 3.9

Compute the output $y[n]$ if the input is $x[n] = a^n u[n]$ and the LTI system impulse response is $h[n] = u[n] - u[n-10]$. Determine the stability and causality of system.

Using (3.19), we have:

$$\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} a^m u[m] \left(u[n-m] - u[n-10-m]\right) \\
&= \sum_{m=0}^{\infty} a^m \left(u[n-m] - u[n-10-m]\right) \\
&= \sum_{m=0}^{\infty} a^m u[n-m] - \sum_{m=0}^{\infty} a^m u[n-10-m]
\end{aligned}$$

Let $y_1[n]=\sum_{m=0}^{\infty} a^m u[n-m]$ and $y_2[n]=\sum_{m=0}^{\infty} a^m u[n-10-m]$ such that $y[n]=y_1[n]-y_2[n]$. By employing a change of variable, $y_1[n]$ is expressed as

$$\begin{aligned} y_1[n] &= \sum_{m=0}^{\infty} a^m u[n-m] = \sum_{k=n}^{-\infty} a^{n-k} u[k], \quad k=n-m \\ &= \sum_{k=-\infty}^{n} a^{n-k} u[k] \end{aligned}$$

Since $u[k]=0$ for $k<0$, $y_1[n]=0$ for $n<0$. For $n \geq 0$, $y_1[n]$ is:

$$y_1[n]=\sum_{k=0}^{n} a^{n-k} u[k] = 1+a+\cdots+a^n = \frac{1-a^{n+1}}{1-a}$$

That is,

$$y_1[n]=\frac{1-a^{n+1}}{1-a} u[n]$$

Similarly, $y_2[n]$ is:

$$\begin{aligned} y_2[n] &= \sum_{m=0}^{\infty} a^m u[n-10-m] \\ &= \sum_{k=-\infty}^{n-10} a^{n-10-k} u[k], \quad k = n-10-m \end{aligned}$$

Since $u[k]=0$ for $k<0$, $y_2[n]=0$ for $n<10$. For $n \geq 10$, $y_2[n]$ is:

$$y_2[n] = \sum_{k=0}^{n-10} a^{n-10-k} = 1 + a + \cdots + a^{n-10} = \frac{1-a^{n-9}}{1-a}$$

That is,

$$y_2[n] = \frac{1-a^{n-9}}{1-a} u[n-10]$$

Combining the results, we have:

$$y[n] = \frac{1-a^{n+1}}{1-a}u[n] - \frac{1-a^{n-9}}{1-a}u[n-10]$$

or

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \leq n < 10 \\ \dfrac{a^{n-9}\left(1-a^{10}\right)}{1-a}, & 10 \leq n \end{cases}$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{9} |h[n]| = 10 < \infty$, the system is stable. Moreover, the system is causal because $h[n] = 0$ for $n < 0$.

Example 3.10

Determine $y[n] = x[n] \otimes h[n]$ where $x[n]$ and $h[n]$ are

$$x[n] = \begin{cases} n^2+1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} n+1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Here, the lengths of both $x[n]$ and $h[n]$ are finite. More precisely, $x[0]=1$, $x[1]=2$, $x[2]=5$, $x[3]=10$, $h[0]=1$, $h[1]=2$, $h[2]=3$ and $h[3]=4$ while all other $x[n]$ and $h[n]$ have zero values.

We still use (3.19) but now it reduces to finite summation:

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3]
\end{aligned}
$$

By considering the non-zero values of $h[n]$, we obtain:

$$
y[n] = \begin{cases}
1, & n = 0 \\
4, & n = 1 \\
12, & n = 2 \\
30, & n = 3 \\
43, & n = 4 \\
50, & n = 5 \\
40, & n = 6 \\
0, & \text{otherwise}
\end{cases}
$$

Alternatively, for finite-length discrete-time signals, we can use the MATLAB command `conv` to compute the convolution of finite-length sequences:

```
n=0:3;
x=n.^2+1;
h=n+1;
y=conv(x,h)
```

The results are

```
y = 1      4      12      30      43      50      40
```

As the default starting time indices in both `h` and `x` are 1, we need to determine the appropriate time index for `y`

The correct index can be obtained by computing one value of $y[n]$ using (3.19). For simplicity, we may compute $y[0]$:

$$\begin{aligned} y[0] &= \sum_{m=-\infty}^{\infty} x[m]h[-m] \\ &= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + \cdots \\ &= x[0]h[0] \\ &= 1 \end{aligned}$$

In general, if the lengths of $x[n]$ and $h[n]$ are M and N, respectively, the length of $y[n] = x[n] \otimes h[n]$ is $(M + N - 1)$.

Linear Constant Coefficient Difference Equations

For a LTI system, its input $x[n]$ and output $y[n]$ are related via a Nth-order linear constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (3.24)$$

which is useful to check whether a system is both linear and time-invariant or not

Example 3.11

Determine if the following input-output relationships correspond to LTI systems:

(a) $y[n] = 0.1y[n-1] + x[n] + x[n-1]$

(b) $y[n] = x[n+1] + x[n]$

(c) $y[n] = 1/x[n]$

We see that (a) corresponds to a LTI system with $N = M = 1$, $a_0 = 1$, $a_1 = -0.1$ and $b_0 = b_1 = 1$

For (b), we reorganize the equation as:

$$y[n] = x[n+1] + x[n] \Rightarrow y[n-1] = x[n] + x[n-1]$$

which agrees with (3.24) when $N = M = 1$, $a_0 = 0$ and $a_1 = b_0 = b_1 = 1$. Hence (b) also corresponds to a LTI system

For (c), it does not correspond to a LTI system because $x[n]$ and $y[n]$ are not linear in the equation

Note that if a system cannot be fitted into (3.24), there are three possibilities: linear and time-variant; nonlinear and time-invariant; or nonlinear and time-variant

Example 3.12

Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n-1]$$

Expanding (3.20) as

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots \end{aligned}$$

we can easily deduce that only $h[0]$ and $h[1]$ are nonzero. That is, the impulse response is:

$$h[n] = \delta[n] - \delta[n-1]$$

The difference equation is also useful to generate the system output and input.

Assuming that $a_0 \neq 0$, $y[n]$ is computed as:

$$y[n] = \frac{1}{a_0}\left(-\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]\right) \qquad (3.25)$$

Assuming that $b_0 \neq 0$, $x[n]$ can be obtained from:

$$x[n] = \frac{1}{b_0}\left(\sum_{k=0}^{N} a_k y[n-k] - \sum_{k=1}^{M} b_k x[n-k]\right) \qquad (3.26)$$

Example 3.13

Given a LTI system with difference equation of $y[n] = 0.5y[n-1] + x[n] + x[n-1]$, compute the system output $y[n]$ for $0 \leq n \leq 50$ with an input of $x[n] = u[n]$. It is assumed that $y[-1] = 0$.

The MATLAB code is:

```
N=50;              %data length is N+1
y(1)=1;            %compute y[0], only x[n] is nonzero
for n=2:N+1
y(n)=0.5*y(n-1)+2; %compute y[1],y[2],…,y[50]
                   %x[n]=x[n-1]=1 for n>=1
end
n=[0:N];           %set time axis
stem(n,y);
```

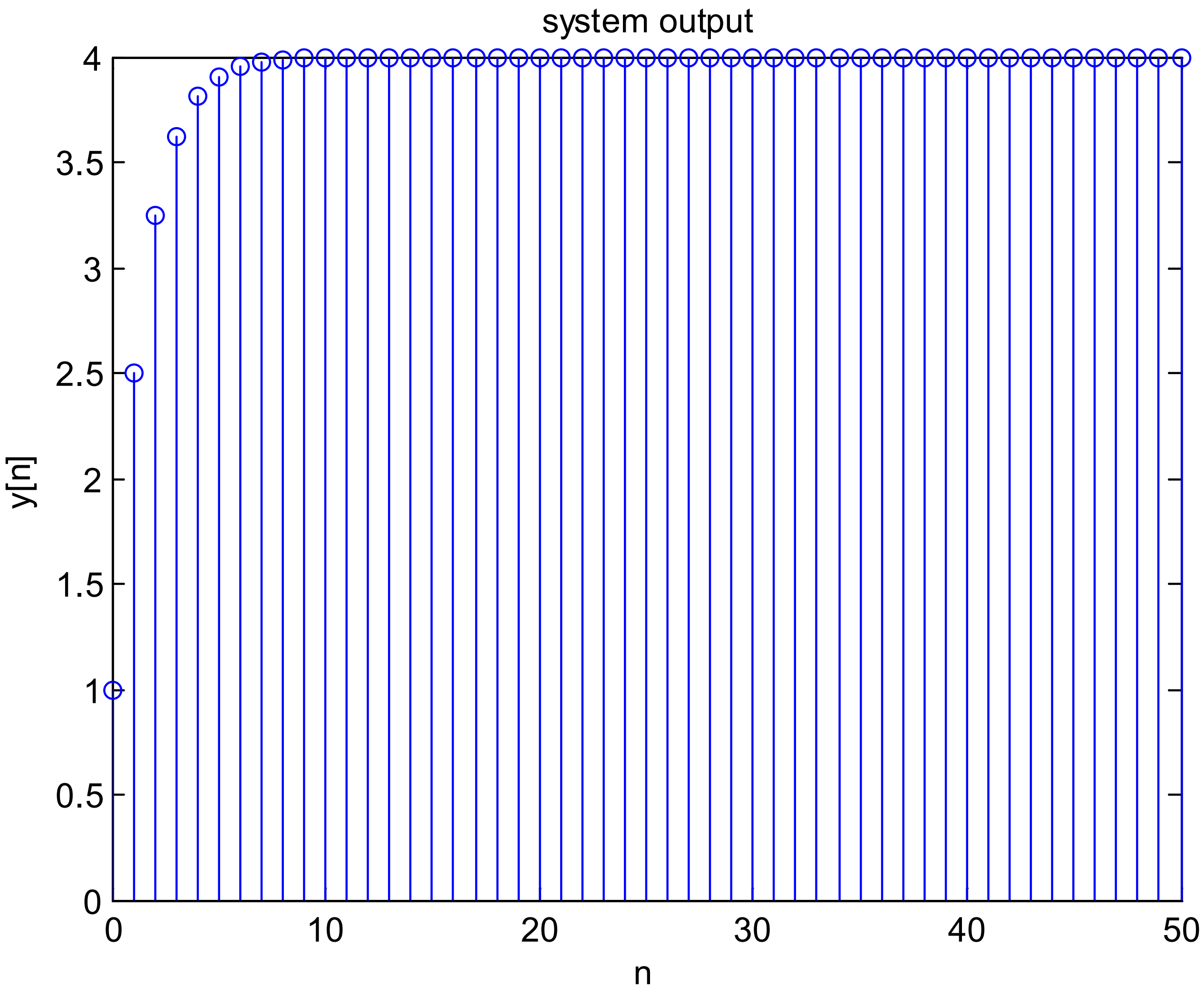


Fig.3.12: Output generation with difference equation

Alternatively, we can use the MATLAB command `filter` by rewriting the equation as:

$$y[n]-0.5y[n-1]=x[n]+x[n-1]$$

The corresponding MATLAB code is:

```
x=ones(1,51);               %define input
a=[1,-0.5];                 %define vector of a_k
b=[1,1];                    %define vector of b_k
y=filter(b,a,x);            %produce output
stem(0:length(y)-1,y)
```

The `x` is the input which has a value of 1 for $0 \le n \le 50$, while a and b are vectors which contain $\{a_k\}$ and $\{b_k\}$, respectively.

The MATLAB programs for this example are provided as `ex3_13.m` and `ex3_13_2.m`.