A Study of Two-Dimensional Sensor Placement using Time-Difference-of-Arrival Measurements

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Index Terms

time-difference-of-arrival, optimum array geometry, source localization

Abstract

Finding the position of a radiative source based on time-difference-of-arrival (TDOA) measurements from spatially separated receivers has important applications in sonar, radar, mobile communications and sensor networks. Each TDOA defines a hyperbolic locus on which the source must lie and the position estimate can then be determined with the knowledge of the sensor array geometry. While extensive research works have been performed on algorithm development for TDOA estimation and TDOA-based localization, limited attention has been paid in sensor array geometry design. In this paper, an optimum two-dimensional sensor placement strategy is derived with the use of optimum TDOA measurements, assuming that each sensor receives a white signal source in the presence of additive white noise. The minimum achievable Cramér-Rao lower bound is also produced.

I. Introduction

Passive source localization using measurements from an array of spatially separated sensors is an important problem in radar, sonar, mobile communications and wireless sensor networks. The time-difference-of-arrival (TDOA) method is a popular strategy for source localization and it usually proceeds in a two-step fashion as follows. TDOA measurements of the source signal received at the sensor array are first obtained. In the second step, the TDOA information is utilized to construct a set of hyperbolic equations that are highly nonlinear, from which the source position can be determined with the knowledge of the sensor array geometry.

Although extensive research has been performed in TDOA estimation [1]- [2] as well as TDOA-based localization [3]- [6], most of them do not consider the impact of the sensor array geometry on the localization accuracy. Nevertheless, Yang $et\ al.$ [7]- [9] have recently pioneered the theoretical study for sensor array placement strategies. In [7], the properties of the Cramér-Rao lower bound (CRLB) for TDOA-based positioning [3]- [4] and optimum sensor arrays are derived. It is proved that for two-dimensional (2D) source localization, uniform angular arrays (UAAs) and its superpositions can attain the minimum CRLB. Works in [8] and [9] are extensions to [7]. The former studies the performance loss for UAAs with reduced angular apertures while the latter develops the relationship between several sensor placement schemes. In spite of the elegant and rigorous mathematical treatment, it is invalid to assume uncorrelated TDOA measurements as they should be correlated [10]. In this paper, we will study the array geometry based on the correlated TDOA estimates, which are optimally computed using the sensor outputs where each of them receives an uncorrelated signal source in the presence of additive white Gaussian noise.

The rest of the paper is organized as follows. In Section II, we first develop the Fisher information matrices (FIMs) for the TDOA and 2D position estimates in the case of white signal source and noise. By minimizing the trace of the CRLB for positioning, it is shown that UAAs and its superpositions correspond to an optimum sensor placement and this is in agreement with [7]. The minimum achievable CRLB for positioning is also derived. Numerical examples are provided in Section III to validate our research findings. Finally, conclusions are drawn in Section IV.

II. Development of Optimum Sensor Placement

Suppose there are $L \geq 3$ sensors and the signal received at the lth sensor is modeled as

$$z_l(n) = s(n - D_l) + q_l(n), \qquad l = 1, 2, \cdots, L, \qquad n = 0, 1, \cdots, N-1 \tag{1}$$

where $s(n)$ is the white source signal, $q_l(n)$ and D_l are the additive white Gaussian noise and signal propagation delay, respectively, at the lth sensor, and N is the number of samples available at each sensor. Without loss of generality, we assign the first sensor as the reference and define the TDOA parameter vector as $\mathbf{d} = [d_{21}, d_{31}, \cdots d_{L1}]^T$ where T denotes the transpose operator and $d_{i1} = D_i - D_1$, $i = 2, 3, \cdots, L$. The FIM for $\mathbf{d}$, denoted by $\mathbf{F}(\mathbf{d})$, is [10]:

$$\mathbf{F}(\mathbf{d}) = \frac{N}{2\pi} \int_{-\pi}^{\pi} \omega^2 \frac{S^2(\omega)}{1 + \sum_{l=1}^{L} S(\omega)/Q_l(\omega)} \left[\mathrm{tr}(\mathbf{Q}^{-1}(\omega))\mathbf{Q}_p^{-1}(\omega) - \mathbf{Q}_p^{-1}(\omega)\mathbf{1}_{L-1}\mathbf{1}_{L-1}^T\mathbf{Q}_p^{-1}(\omega) \right] d\omega \tag{2}$$

where tr is the trace operator, $\mathbf{1}_i$ is the $i \times 1$ vector with all elements 1, $S(\omega)$ and $Q_l(\omega)$ represent the power spectra of $s(n)$ and $q_l(n)$, respectively, while $\mathbf{Q}(\omega) = \mathrm{diag}(Q_1(\omega), Q_2(\omega), \cdots, Q_L(\omega))$ and $\mathbf{Q}_p(\omega) = \mathrm{diag}(Q_2(\omega), Q_3(\omega), \cdots, Q_L(\omega))$ where $\mathrm{diag}(a_1, a_2, \cdots, a_n)$ is a diagonal matrix whose diagonal entries are $a_1, a_2, \cdots, a_n$. It is clear from (2) that the optimum TDOA estimates are correlated. Assuming that $s(n)$ and $\{q_l(n)\}$ are uncorrelated white Gaussian processes with variances σ_s^2 and σ_q^2, respectively, we have $S(\omega) = \sigma_s^2$, $Q_l(\omega) = \sigma_q^2$, $l = 1, 2, \cdots, L$, $\mathbf{Q}(\omega) = \sigma_q^2\mathbf{I}_L$ and $\mathbf{Q}_p(\omega) = \sigma_q^2\mathbf{I}_{L-1}$ where $\mathbf{I}_i$ represents the $i \times i$ identity matrix. Under the white signal and noise assumption, (2) can be simplified to [11]:

$$\mathbf{F}(\mathbf{d}) = \frac{\pi^2 N \Lambda^2 \left(L\mathbf{I}_{L-1} - \mathbf{1}_{L-1}\mathbf{1}_{L-1}^T \right)}{3\,(1 + L\Lambda)} \tag{3}$$

where $\Lambda = \sigma_s^2/\sigma_q^2$ is the signal-to-noise ratio (SNR). The covariance matrix of the optimum estimate for $\mathbf{d}$ when $N \to \infty$ is equal to the inverse of $\mathbf{F}(\mathbf{d})$, which has the form of [11]:

$$\mathbf{F}^{-1}(\mathbf{d}) = \frac{3(1 + L\Lambda)}{\pi^2 N \Lambda^2} \left[\mathbf{I}_{L-1} + \mathbf{1}_{L-1}\mathbf{1}_{L-1}^T \right] \tag{4}$$

Let the position of the source and sensors be $\mathbf{x} = [x, y]^T$ and $\mathbf{x}_l = [x_l, y_l]^T$, $l = 1, 2, \cdots, L$, respectively. The FIM for $\mathbf{x}$ using the optimum TDOA estimates, denoted by $\mathbf{F}(\mathbf{x})$, is [5]:

$$\mathbf{F}(\mathbf{x}) = \frac{\mathbf{G}\mathbf{F}(\mathbf{d})\mathbf{G}^T}{c^2} \tag{5}$$

where

$$\mathbf{G} = [\mathbf{g}_{21}, \mathbf{g}_{32}, \cdots \mathbf{g}_{L1}]$$

$$\mathbf{g}_{l1} = \mathbf{g}_l - \mathbf{g}_1$$

$$\mathbf{g}_l = \begin{bmatrix} g_{x,l} \\ g_{y,l} \end{bmatrix} = \begin{bmatrix} \frac{x-x_l}{\sqrt{(x-x_l)^2+(y-y_l)^2}} \\ \frac{y-y_l}{\sqrt{(x-x_l)^2+(y-y_l)^2}} \end{bmatrix} = \begin{bmatrix} \cos(\theta_l) \\ \sin(\theta_l) \end{bmatrix}$$

and c and θ_l denote the known signal propagation speed and the incline angle from the source to the lth sensor, respectively. With the use of (3), (5) becomes

$$\mathbf{F}(\mathbf{x}) = \frac{\pi^2 N \Lambda^2}{3c^2(1+L\Lambda)} \times \begin{bmatrix} (L-1)\sum_{l=1}^{L}\cos^2(\theta_l) - \sum_{i\neq j}^{L}\cos(\theta_i)\cos(\theta_j) & L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l) - \sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l) \\ L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l) - \sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l) & (L-1)\sum_{l=1}^{L}\sin^2(\theta_l) - \sum_{i\neq j}^{L}\sin(\theta_i)\sin(\theta_j) \end{bmatrix} \tag{6}$$

The optimum sensor placement strategy is obtained by minimizing the trace of the CRLB for positioning, that is, $\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x}))$. In Appendix A, we have proved that $\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x}))$ can be expressed as $f(\boldsymbol{\theta})$ where $\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_L]^T$, and $f(\boldsymbol{\theta})$ is of the form:

$$f(\boldsymbol{\theta}) = \frac{L(L-1) - 2\sum_{i>j}^{L}\cos(\theta_i - \theta_j)}{L(L-2)\sum_{i>j}^{L}\sin^2(\theta_l - \theta_j) + 2L\sum_{l=1}^{L}\sum_{i>j}^{L}\sin(\theta_l - \theta_j)\sin(\theta_i - \theta_l)} \tag{7}$$

Let the numerator and denominator of $f(\boldsymbol{\theta})$ be $u(\boldsymbol{\theta})$ and $v(\boldsymbol{\theta})$, respectively. As $f(\boldsymbol{\theta}) > 0$ because $\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x}))$ should be positive and $u(\boldsymbol{\theta})$ is obviously larger than 0, we have $v(\boldsymbol{\theta}) > 0$. Differentiating (7) with respect to θ_l, $l = 1, 2, \cdots, L$, and setting the resultant expressions to zero yield:

$$\left.\frac{\partial u(\boldsymbol{\theta})}{\partial \theta_l}\right|_{\theta_l=\hat{\theta}_l} = f(\boldsymbol{\theta})\left.\frac{\partial v(\boldsymbol{\theta})}{\partial \theta_l}\right|_{\theta_l=\hat{\theta}_l}, \quad l = 1, 2, \cdots, L \tag{8}$$

where the components of $[\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_L]^T$ are the optimum sensor directions with respect to the source. We have shown in Appendix B that (8) can be simplified to:

$$\left.\frac{\partial u(\boldsymbol{\theta})}{\partial \theta_l}\right|_{\theta_l=\hat{\theta}_l} = 0 \quad \text{and} \quad \left.\frac{\partial v(\boldsymbol{\theta})}{\partial \theta_l}\right|_{\theta_l=\hat{\theta}_l} = 0, \quad l = 1, 2, \cdots, L \tag{9}$$

which are only satisfied by the following conditions (See Appendix B):

$$\sum_{l=1}^{L}\cos(\hat{\theta}_l) = 0, \quad \sum_{l=1}^{L}\sin(\hat{\theta}_l) = 0, \quad \sum_{l=1}^{L}\cos(2\hat{\theta}_l) = 0 \quad \text{and} \quad \sum_{l=1}^{L}\sin(2\hat{\theta}_l) = 0 \tag{10}$$

An obvious solution for (10) is

$$\hat{\theta}_l = \frac{2\pi}{L}(l-1) + \phi, \qquad l = 1, 2, \cdots, L \tag{11}$$

where $\phi \in (0, 2\pi)$ which corresponds to an UAA. As an illustration, let $L = 3$ and the first incline angle is fixed at $\hat{\theta}_1 = 0$. After simple trigonometric manipulations, (10) is reduced to

$$1 + \cos(2\hat{\theta}_2) + \cos(2\hat{\theta}_3) = 0 \Rightarrow \sin^2(\hat{\theta}_2) + \sin^2(\hat{\theta}_3) = \frac{3}{4} \tag{12}$$

$$0 + \sin(2\hat{\theta}_2) + \sin(2\hat{\theta}_3) = 0 \Rightarrow \cos(\hat{\theta}_2)\sin(\hat{\theta}_2) + \cos(\hat{\theta}_3)\sin(\hat{\theta}_3) = 0 \tag{13}$$

$$1 + \cos(\hat{\theta}_2) + \cos(\hat{\theta}_3) = 0 \tag{14}$$

$$0 + \sin(\hat{\theta}_2) + \sin(\hat{\theta}_3) = 0 \Rightarrow \sin(\hat{\theta}_2) = -\sin(\hat{\theta}_3) \tag{15}$$

Solving (12) to (15) gives $\hat{\theta}_2 = 2\pi/3$ and $\hat{\theta}_3 = 4\pi/3$, which agree with (11). Nevertheless, a generalized solution is

$$\hat{\boldsymbol{\theta}}_{m,i} = \frac{2\pi}{L_m}(l_m - 1) + \phi_m, \qquad i = 1, 2, \cdots, L_m, \qquad m = 1, 2, \cdots, M \tag{16}$$

where the optimal direction vector, $\hat{\boldsymbol{\theta}}_{m,i} = [\hat{\theta}_{1,1}, \hat{\theta}_{1,2}, \cdots, \hat{\theta}_{1,L_1}, \hat{\theta}_{2,1}, \cdots \hat{\theta}_{2,L_2}, \cdots, \hat{\theta}_{M,L_M}]^T$ with $L_1 + L_2 + \cdots + L_M = L$, $\phi_m \in [0, 2\pi)$ and l_m is the index of L_m. It is noteworthy that these findings agree with [7], although the latter assumes uncorrelated TDOA information.

With the use of (10), $\mathbf{F}(\mathbf{x})$ with optimum array geometry is easily shown to be:

$$\mathbf{F}(\mathbf{x}) = \frac{\pi^2 N L^2 \Lambda^2}{6c^2(1 + L\text{SNR})}\mathbf{I}_2 \tag{17}$$

According to Appendix C, the minimum achievable trace of the CRLB for TDOA-based positioning is then:

$$\begin{aligned} \text{tr}(\mathbf{F}^{-1}(\mathbf{x})) &= \frac{12c^2(1 + L\Lambda)}{\pi^2 N L^2 \Lambda} \\ &= \frac{12c^2}{\pi^2 N L^2 \Lambda^2} + \frac{12c^2}{\pi^2 N L \Lambda} \end{aligned} \tag{18}$$

where we see that the CRLB decreases linearly with N. Moreover, the bound is inversely proportional to $L^2\Lambda^2$ for $\Lambda << 1$ while its value decreases linearly with L and Λ for $\Lambda >> 1$. It is worthy to point out that the result of (18) is essentially identical to [7] up to a scaling factor.

III. Numerical Examples

Simulation results are presented in this Section to validate our analytical findings. The source position is fixed at $\mathbf{x} = [0, 0]^T$. First the superiority of the UAA over other typical geometries, namely, nonuniform angular array (NAA), corners, L-shape and uniform linear array (ULA), in terms of mean square position error (MSPE) performance, is demonstrated in Figure 1. We consider a rectangular area of dimension 20m $\times$ 10m with four sensors whose coordinates for different placement strategies are tabulated in Table 1. As optimum TDOA measurements are assumed, all the MSPEs are computed using $\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x}))$ with $c = 360\text{ms}^{-1}$. It is observed in Figure 1 that the UAA strategy has 2dB to 7dB improvement over other standard placement schemes for different SNR conditions.

Second, we study the optimum angular separation by considering an angular array with $L = 3$ where the first sensor position is fixed at $\mathbf{x}_1 = [5\cos(0), 5\sin(0)]^T$. To avoid duplication, $\theta_3 > \theta_2$ is assigned. Figure 2 shows the MSPE versus different θ_2 and θ_3 at $\Lambda = 50$dB. We see that the MSPE is minimized when $\theta_2 = 2\pi/3$ and $\theta_3 = 4\pi/3$, which conform to our analytical calculations in (12) – (15).

IV. Conclusion

Via optimum time-difference-of-arrival (TDOA) estimation in an sensor array where each sensor receives a white source signal in the presence of white noise, we have found that uniform angular arrays and its superpositions correspond to an optimum sensor placement strategy in the two-dimensional scenarios. The minimum achievable Cramér-Rao lower bound is also produced. Our future works include extension of our development to arbitrary TDOA covariance matrices and/or three-dimensional positioning.

Appendix A

In this Appendix, we show that $\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x}))$ can be expressed as $f(\boldsymbol{\theta})$. Denoting the (i,j) entry of $3\ c^2\ (1\ +\ L\ \Lambda)\mathbf{F}^{-1}(\mathbf{x})/(\pi^2 N \Lambda^2)$ by f_{ij}, $\mathbf{F}^{-1}(\mathbf{x})$ is

$$\mathbf{F}^{-1}(\mathbf{x}) = \frac{3c^2(1+L\Lambda)}{\pi^2 N \Lambda^2} \times \frac{1}{f_{11}f_{22} - f_{12}f_{21}} \begin{bmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{bmatrix} \tag{A.1}$$

The trace of $\mathbf{F}^{-1}(\mathbf{x})$ is then

$$\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x})) = \frac{3c^2(1+L\Lambda)}{\pi^2 N \Lambda^2} \times \frac{f_{11}+f_{22}}{f_{11}f_{22} - f_{12}f_{21}} \tag{A.2}$$

With the use of (6), we have:

$$\begin{aligned} f_{11} + f_{22} &= (L-1)\sum_{l=1}^{L}\cos^2(\theta_l) - \sum_{i\neq j}^{L}\cos(\theta_i)\cos(\theta_j) + (L-1)\sum_{l=1}^{L}\sin^2(\theta_l) - \sum_{i\neq j}^{L}\sin(\theta_i)\sin(\theta_j) \\ &= L(L-1) - \sum_{i\neq j}^{L}\cos(\theta_i - \theta_j) = L(L-1) - 2\sum_{i>j}^{L}\cos(\theta_i-\theta_j) \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
f_{11}f_{22} &= \left[(L-1)\sum_{l=1}^{L}\cos^2(\theta_l) - \sum_{i\neq j}^{L}\cos(\theta_i)\cos(\theta_j)\right] \times \left[(L-1)\sum_{l=1}^{L}\sin^2(\theta_l) - \sum_{i\neq j}^{L}\sin(\theta_i)\sin(\theta_j)\right] \\
&= \left[L\sum_{l=1}^{L}\cos^2(\theta_l) - \left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2\right] \times \left[L\sum_{l=1}^{L}\sin^2(\theta_l) - \left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2\right] \\
&= L^2\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{l=1}^{L}\sin^2(\theta_l) + \left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2 \\
&\quad - L\sum_{l=1}^{L}\cos^2(\theta_l)\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2 - L\sum_{l=1}^{L}\sin^2(\theta_l)\left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2
\end{aligned} \tag{A.4}
$$

and

$$
\begin{aligned}
f_{12}f_{21} = f_{12}^2 &= \left[L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l) - \sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l)\right]^2 \\
&= L^2\left(\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\right)^2 + \left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2 \\
&\quad - 2L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l)
\end{aligned} \tag{A.5}
$$

From (A.4) and (A.5), we obtain:

$$
\begin{aligned}
f_{11}f_{22} - f_{12}^2 &= L^2\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{l=1}^{L}\sin^2(\theta_l) - L^2\left(\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\right)^2 \\
&\quad - L\sum_{l=1}^{L}\sin^2(\theta_l)\left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2 - L\sum_{l=1}^{L}\cos^2(\theta_l)\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2 \\
&\quad + 2L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l)
\end{aligned} \tag{A.6}
$$

The five terms in (A.6) are computed as

$$
L^2\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{l=1}^{L}\sin^2(\theta_l) = L^2\sum_{l=1}^{L}\cos^2(\theta_l)\sin^2(\theta_l) + L^2\sum_{i>j}^{L}\cos^2(\theta_i)\sin^2(\theta_j) + \sin^2(\theta_i)\cos^2(\theta_j) \tag{A.7}
$$

$$
L^2\left(\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\right)^2 = L^2\sum_{l=1}^{L}\cos^2(\theta_l)\sin^2(\theta_l) + 2L^2\sum_{i>j}^{L}\cos(\theta_i)\sin(\theta_i)\cos(\theta_j)\sin(\theta_j) \tag{A.8}
$$

$$
L\sum_{l=1}^{L}\sin^2(\theta_l)\left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2 = L\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{l=1}^{L}\sin^2(\theta_l) + 2L\sum_{l=1}^{L}\sin^2(\theta_l)\sum_{i>j}^{L}\cos(\theta_i)\cos(\theta_j) \tag{A.9}
$$

$$
L\sum_{l=1}^{L}\cos^2(\theta_l)\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2 = L\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{l=1}^{L}\sin^2(\theta_l) + 2L\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{i>j}^{L}\sin(\theta_i)\sin(\theta_j) \tag{A.10}
$$

and

$$
\begin{aligned}
&2L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l)\\
=&2L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\left(\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)+\sum_{i>j}^{L}\cos(\theta_i)\sin(\theta_j)+\sin(\theta_i)\cos(\theta_j)\right)\\
=&2L\left(\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\right)^2+2L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\sum_{i>j}^{L}\cos(\theta_i)\sin(\theta_j)+\sin(\theta_i)\cos(\theta_j)
\end{aligned}
\tag{A.11}
$$

Subtracting (A.8) from (A.7) yields

$$
\begin{aligned}
&L^2\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{l=1}^{L}\sin^2(\theta_l)-L^2\left(\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\right)^2\\
=&L^2\sum_{i>j}^{L}\cos^2(\theta_i)\sin^2(\theta_j)+L^2\sin^2(\theta_i)\cos^2(\theta_j)-2L^2\sum_{i>j}^{L}\cos(\theta_i)\sin(\theta_i)\cos(\theta_j)\sin(\theta_j)\\
=&L^2\sum_{i>j}^{L}\big(\cos(\theta_i)\sin(\theta_j)-L^2\sin(\theta_i)\cos(\theta_j)\big)^2=L^2\sum_{i>j}^{L}\sin^2(\theta_i-\theta_j)
\end{aligned}
\tag{A.12}
$$

Substituting (A.9)-(A.12) into (A.6) and with the use of (A.12), we get

$$
\begin{aligned}
&f_{11}f_{22}-f_{12}^2\\
&=L^2\sum_{i>j}^{L}\sin^2(\theta_i-\theta_j)-2L\left[\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{l=1}^{L}\sin^2(\theta_l)-\left(\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\right)^2\right]\\
&\quad+2L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l)\sum_{i>j}^{L}\cos(\theta_i)\sin(\theta_j)+\sin(\theta_i)\cos(\theta_j)\\
&\quad-2L\sum_{l=1}^{L}\cos^2(\theta_l)\sum_{i>j}^{L}\sin(\theta_i)\sin(\theta_j)-2L\sum_{l=1}^{L}\sin^2(\theta_l)\sum_{i>j}^{L}\cos(\theta_i)\cos(\theta_j)\\
&=(L^2-2L)\sum_{i>j}^{L}\sin^2(\theta_i-\theta_j)+2L\sum_{l=1}^{L}\sum_{i>j}^{L}\cos(\theta_l)\sin(\theta_l)\left[\cos(\theta_i)\sin(\theta_j)+\cos(\theta_j)\sin(\theta_i)\right]\\
&\quad-2L\sum_{l=1}^{L}\sum_{i>j}^{L}\sin^2(\theta_l)\cos(\theta_i)\cos(\theta_j)-2L\sum_{l=1}^{L}\sum_{i>j}^{L}\cos^2(\theta_l)\sin(\theta_i)\sin(\theta_j)\\
&=(L^2-2L)\sum_{i>j}^{L}\sin^2(\theta_i-\theta_j)+2L\sum_{l=1}^{L}\sum_{i>j}^{L}\sin(\theta_l-\theta_j)\sin(\theta_i-\theta_l)
\end{aligned}
\tag{A.13}
$$

Putting (A.3) and (A.13) into (A.2) yields (7).

Appendix B

In Appendix B, we first prove that (8) can be simplified to (9) and then derives (10). For an arbitrary index of $\boldsymbol{\theta}$, say, $s = 1, 2, \cdots, L$, the derivatives of $u(\boldsymbol{\theta})$ and $v(\boldsymbol{\theta})$ with respect to θ_s are

$$\begin{aligned}\frac{\partial\left(u(\boldsymbol{\theta})\right)}{\partial\theta_s} =& -2\frac{\partial\left(\sum_{i>j}^{L}\cos(\theta_i-\theta_j)\right)}{\partial\theta_s}\\ =&2\sum_{l=1}^{L}\sin(\theta_s-\theta_l) = 2\sin(\theta_s)\sum_{l=1}^{L}\cos(\theta_l) - 2\cos(\theta_s)\sum_{l=1}^{L}\sin(\theta_l)\end{aligned} \tag{B.1}$$

and

$$\frac{\partial\left(v(\boldsymbol{\theta})\right)}{\partial\theta_s} = L(L-2)\frac{\partial\left(\sum_{i>j}^{L}\sin^2(\theta_i-\theta_j)\right)}{\partial\theta_s} + 2L\frac{\partial\left(\sum_{l=1}^{L}\sum_{i>j}^{L}\sin(\theta_l-\theta_j)\sin(\theta_i-\theta_l)\right)}{\partial\theta_s} \tag{B.2}$$

with

$$\begin{aligned}\frac{\partial\left(\sum_{i>j}^{L}\sin^2(\theta_i-\theta_j)\right)}{\partial\theta_s} &= \sum_{s>j}^{L}2\sin(\theta_s-\theta_j)\cos(\theta_s-\theta_j) - \sum_{i>s}^{L}2\sin(\theta_i-\theta_s)\cos(\theta_i-\theta_s)\\ &= \sum_{l=1}^{L}2\sin(\theta_s-\theta_l)\cos(\theta_s-\theta_l)\\ &= \sum_{l=1}^{L}\sin(2(\theta_s-\theta_l)) = \sin(2\theta_s)\sum_{l=1}^{L}\cos(2\theta_l) - \cos(2\theta_s)\sum_{l=1}^{L}\sin(2\theta_l)\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}&\frac{\partial\left(\sum_{l=1}^{L}\sum_{i>j}^{L}\sin(\theta_l-\theta_j)\sin(\theta_i-\theta_l)\right)}{\partial\theta_s}\\ =&\sum_{\substack{l=1\\ s\neq l}}^{L}\left(\frac{\partial}{\partial\theta_s}\sum_{i>j}^{L}\sin(\theta_l-\theta_j)\sin(\theta_i-\theta_l)\right) + \frac{\partial}{\partial\theta_s}\sum_{i>j}^{L}\sin(\theta_s-\theta_j)\sin(\theta_i-\theta_s)\\ =&\sum_{\substack{l=1\\ s\neq l}}^{L}\sum_{l=1}^{L}\cos(\theta_s-\theta_l)\sin(\theta_l-\theta_i) - \sum_{i>j}^{L}\sin(2\theta_s-\theta_i-\theta_j)\\ =&\sum_{\substack{l=1\\ s\neq l}}^{L}\sum_{l=1}^{L}\cos(\theta_s-\theta_l)\sin(\theta_l-\theta_i) - \sin(2\theta_s)\sum_{i>j}^{L}\cos(\theta_i+\theta_j) - \cos(2\theta_s)\sum_{i>j}^{L}\sin(\theta_i+\theta_j)\end{aligned} \tag{B.4}$$

The three terms in (B.4) can be written as:

$$\begin{aligned}
&\sum_{\substack{l=1\\ s\neq l}}^{L}\sum_{l=1}^{L}\cos(\theta_s-\theta_l)\sin(\theta_l-\theta_i)\\
&=\sum_{\substack{l=1\\ s\neq l}}^{L}\cos(\theta_s-\theta_l)\sum_{i=1}^{L}\sin(\theta_l)\cos(\theta_i)-\sum_{\substack{l=1\\ s\neq l}}^{L}\cos(\theta_s-\theta_l)\sum_{i=1}^{L}\cos(\theta_l)\sin(\theta_i)\\
&=\sum_{\substack{l=1\\ s\neq l}}^{L}\cos(\theta_s-\theta_l)\sin(\theta_l)\sum_{i=1}^{L}\cos(\theta_i)-\sum_{\substack{l=1\\ s\neq l}}^{L}\cos(\theta_s-\theta_l)\cos(\theta_l)\sum_{i=1}^{L}\sin(\theta_i)
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
\sum_{i>j}^{L}\cos(\theta_i+\theta_j) &=\sum_{i=1}^{L}\sum_{j=1}^{L}\cos(\theta_i+\theta_j)-\sum_{l=1}^{L}\cos(2\theta_l)\\
&=\sum_{i=1}^{L}\cos(\theta_i)\sum_{j=1}^{L}\cos(\theta_j)+\sum_{i=1}^{L}\sin(\theta_i)\sum_{j=1}^{L}\sin(\theta_j)-\sum_{l=1}^{L}\cos(2\theta_l)\\
&=\left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2+\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2-\sum_{l=1}^{L}\cos(2\theta_l)
\end{aligned} \tag{B.6}$$

and

$$\begin{aligned}
\sum_{i>j}^{L}\sin(\theta_i+\theta_j) &=\sum_{i=1}^{L}\sum_{j=1}^{L}\sin(\theta_i+\theta_j)-\sum_{l=1}^{L}\sin(2\theta_l)\\
&=\sum_{i=1}^{L}\sin(\theta_i)\sum_{j=1}^{L}\cos(\theta_j)+\sum_{i=1}^{L}\cos(\theta_i)\sum_{j=1}^{L}\sin(\theta_j)-\sum_{l=1}^{L}\sin(2\theta_l)\\
&=2\sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l)-\sum_{l=1}^{L}\sin(2\theta_l)
\end{aligned} \tag{B.7}$$

Using (B.3)−(B.7), (B.2) becomes

$$\begin{aligned}
\frac{\partial\left(v(\boldsymbol{\theta})\right)}{\partial\theta_s} =&L(L-1)\left[\sin(2\theta_s)\sum_{l=1}^{L}\cos(2\theta_l)-\cos(2\theta_s)\sum_{l=1}^{L}\sin(2\theta_l)\right]+\\
&2L\left[\sum_{\substack{k=1\\ s\neq k}}^{L}\cos(\theta_s-\theta_k)\sin(\theta_k)\sum_{l=1}^{L}\cos(\theta_l)-\sum_{\substack{k=1\\ s\neq k}}^{L}\cos(\theta_s-\theta_k)\cos(\theta_k)\sum_{l=1}^{L}\sin(\theta_l)\right]\\
&-2L\sin(2\theta_s)\left[\left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2+\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2-\sum_{l=1}^{L}\cos(2\theta_l)\right]\\
&-2L\cos(2\theta_s)\left[2\sum_{l=1}^{L}\cos(\theta_l)\sum_{j=1}^{L}\sin(\theta_j)-\sum_{l=1}^{L}\sin(2\theta_l)\right]
\end{aligned} \tag{B.8}$$

From (7), when $\theta_s=\hat{\theta}_s$, $s=1,2,\cdots,L$, all the derivatives of $u(\boldsymbol{\theta})$ are identical and based on (B.1), we let

$$2\sin(\hat{\theta}_s)\sum_{l=1}^{L}\cos(\hat{\theta}_l)-2\cos(\hat{\theta}_s)\sum_{l=1}^{L}\sin(\hat{\theta}_l)=C \tag{B.9}$$

where C is a constant to be determined. Summing (B.9) for $s = 1, 2, \cdots, L$, we easily obtain $LC = 0$ which gives $C = 0$. That is, $\partial u(\boldsymbol{\theta})/\partial\theta_l|_{\theta_l=\hat{\theta}_l} = 0$. With the use of $f(\boldsymbol{\theta}) > 0$ and (8), we then get $\partial v(\boldsymbol{\theta})/\partial\theta_l|_{\theta_l=\hat{\theta}_l} = 0$. As a result simplification of (8) to (9) is shown.

To prove the uniqueness of the solutions given in (10), we employ propositional calculus. Based on (B.9), we construct the following statement:

$$\begin{aligned} &2\sin(\hat{\theta}_m)\sum_{l=1}^{L}\cos(\hat{\theta}_l) - 2\cos(\hat{\theta}_m)\sum_{l=1}^{L}\sin(\hat{\theta}_l) = 0 \\ &2\sin(\hat{\theta}_n)\sum_{l=1}^{L}\cos(\hat{\theta}_l) - 2\cos(\hat{\theta}_n)\sum_{l=1}^{L}\sin(\hat{\theta}_l) = 0, \quad m \neq n, \qquad m, n = 1, 2, \cdots, L \end{aligned} \tag{B.10}$$

We will prove that (B.10) contradicts with the opposite conditions of (10), namely, $\sum_{l=1}^{L}\cos(\hat{\theta}_l) = 0$ and $\sum_{l=1}^{L}\sin(\hat{\theta}_l) = 0$. This means $\sum_{l=1}^{L}\cos(\hat{\theta}_l) = 0$ and $\sum_{l=1}^{L}\sin(\hat{\theta}_l) = 0$ are the only solutions for (B.10). The existence of $\hat{\theta}_m$ and $\hat{\theta}_n$ is given by the following two statements, and the second statement can further divided into three sub-statements:

1) $\exists\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) \neq \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) \neq \cos(\theta_n)$
2) $\forall\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) = \sin(\hat{\theta}_n) \vee \cos(\hat{\theta}_m) = \cos(\hat{\theta}_n)$
   a) $\forall\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) \neq \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) = \cos(\hat{\theta}_n)$
   b) $\forall\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) = \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) \neq \cos(\hat{\theta}_n)$
   c) $\forall\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) = \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) = \cos(\hat{\theta}_n)$

where $\wedge$ and $\vee$ are conjunction and disjunction operators [12], respectively,

Now the above statements are examined one by one. For $\exists\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) \neq \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) \neq \cos(\hat{\theta}_n)$, we first assume the opposition of (10):

$$\sum_{l=1}^{L}\cos(\hat{\theta}_l) \neq 0 \vee \sum_{l=1}^{L}\sin(\hat{\theta}_l) \neq 0 \tag{B.11}$$

The assumption of (B.11) corresponds to the following three possibilities:

$$\begin{aligned} &\sum_{l=1}^{L}\cos(\hat{\theta}_l) \neq 0 \wedge \sum_{l=1}^{L}\sin(\hat{\theta}_l) = 0 \\ \Rightarrow &\sin(\hat{\theta}_1) = \sin(\hat{\theta}_2) = \cdots = \sin(\hat{\theta}_L) = 0 \lightning \sin(\hat{\theta}_m) \neq \sin(\hat{\theta}_n) \end{aligned} \tag{B.12}$$

$$\begin{aligned} &\sum_{l=1}^{L}\cos(\hat{\theta}_l) = 0 \wedge \sum_{l=1}^{L}\sin(\hat{\theta}_l) \neq 0 \\ \Rightarrow &\cos(\hat{\theta}_1) = \cos(\hat{\theta}_2) = \cdots = \cos(\hat{\theta}_L) = 0 \lightning \cos(\hat{\theta}_m) \neq \cos(\hat{\theta}_n) \end{aligned} \tag{B.13}$$

or

$$\sum_{l=1}^{L}\cos(\hat{\theta}_l) \neq 0 \wedge \sum_{l=1}^{L}\sin(\hat{\theta}_l) \neq 0$$
$$\Rightarrow \sin(\hat{\theta}_m)\cos(\hat{\theta}_n) = \sin(\hat{\theta}_n)\cos(\hat{\theta}_m) \Rightarrow \sin(\hat{\theta}_m - \hat{\theta}_n) = 0 \lightning L \geq 3 \tag{B.14}$$

where $\lightning$ is contradiction operator [12].

For $\forall\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) \neq \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) = \cos(\hat{\theta}_n)$, we have

$$\sin(\hat{\theta}_s) = \frac{1}{L}\sum_{l=1}^{L}\sin(\hat{\theta}_l), \text{ for } s = 1, 2, \cdots, L \lightning \sin(\hat{\theta}_m) \neq \sin(\hat{\theta}_n) \tag{B.15}$$

For $\forall\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) = \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) \neq \cos(\hat{\theta}_n)$, we have

$$\cos(\hat{\theta}_s) = \frac{1}{L}\sum_{l=1}^{L}\cos(\hat{\theta}_l), \text{ for } s = 1, 2, \cdots, L \lightning \cos(\hat{\theta}_m) \neq \cos(\hat{\theta}_n) \tag{B.16}$$

For $\forall\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) = \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) = \cos(\hat{\theta}_n)$, we have

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}_{2\times 2} \Rightarrow \mathbf{F}^{-1}(\mathbf{x}) \text{ is undefined} \lightning \mathbf{F}^{-1}(\mathbf{x}) \text{ is well defined} \tag{B.17}$$

Therefore, there is no solution for (B.10) except

$$\sum_{l=1}^{L}\cos(\hat{\theta}_l) = 0 \wedge \sum_{l=1}^{L}\sin(\hat{\theta}_l) = 0 \tag{B.18}$$

with $\exists\hat{\theta}_m, \hat{\theta}_n : \sin(\hat{\theta}_m) \neq \sin(\hat{\theta}_n) \wedge \cos(\hat{\theta}_m) \neq \cos(\hat{\theta}_n)$. Substituting (B.18) into (B.8) with $\theta_s = \hat{\theta}_s$, $s = 1, 2, \cdots, L$, yields:

$$(L+1)\sin(2\hat{\theta}_s)\sum_{l=1}^{L}\cos(2\hat{\theta}_l) - (L-3)\cos(2\hat{\theta}_s)\sum_{l=1}^{L}\sin(2\hat{\theta}_l) = 0 \tag{B.19}$$

By undergoing a similar procedure of (B.10)-(B.17), we obtain

$$\sum_{l=1}^{L}\cos(2\hat{\theta}_l) = 0 \wedge \sum_{l=1}^{L}\sin(2\hat{\theta}_l) = 0 \tag{B.20}$$

with $\exists\hat{\theta}_m, \hat{\theta}_n : \sin(2\hat{\theta}_m) \neq \sin(2\hat{\theta}_n) \wedge \cos(2\hat{\theta}_m) \neq \cos(2\hat{\theta}_n)$. Combining (B.18) and (B.20) yields (10).

APPENDIX C

In this Appendix, we produce the minimum value of the CRLB for TDOA-based positioning. Using (A.2) and $f_{12} = f_{21}$, we obtain

$$\frac{f_{11} + f_{22}}{f_{11}f_{22} - f_{12}f_{21}} \geq \frac{f_{11} + f_{22}}{f_{11}f_{22}} \tag{C.1}$$

Applying the arithmetic mean-geometric mean inequality of f_{11} and f_{22} yields

$$\frac{f_{11} + f_{22}}{2} \geq \sqrt{f_{11}f_{22}} \Rightarrow \frac{(f_{11} + f_{22})^2}{4} \geq f_{11}f_{22} \leq L^2 \tag{C.2}$$

From (C.1) and (C.2), we get $(f_{11}+f_{22})^2/4$, hence

$$\frac{f_{11}+f_{22}}{f_{11}f_{22}} \geq \frac{4}{f_{11}+f_{22}} \tag{C.3}$$

On the other hand, (A.3) can be written as

$$f_{11}+f_{22} = L^2 - 2\left(\sum_{l=1}^{L}\cos(\theta_l)\right)^2 - 2\left(\sum_{l=1}^{L}\sin(\theta_l)\right)^2 \leq L^2 \tag{C.4}$$

Using (C.3) and (C.4), we obtain

$$\frac{f_{11}+f_{22}}{f_{11}f_{22}-f_{12}f_{21}} \geq \frac{4}{L^2} \tag{C.5}$$

Including the scaling term of $\pi^2 N \mathrm{SNR}^2/(3c^2(1+L\Lambda))$ in (C.5) yields

$$\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x})) \geq \frac{12c^2(1+L\Lambda)}{\pi^2 N L^2 \Lambda^2} \tag{C.6}$$

which proves (18).

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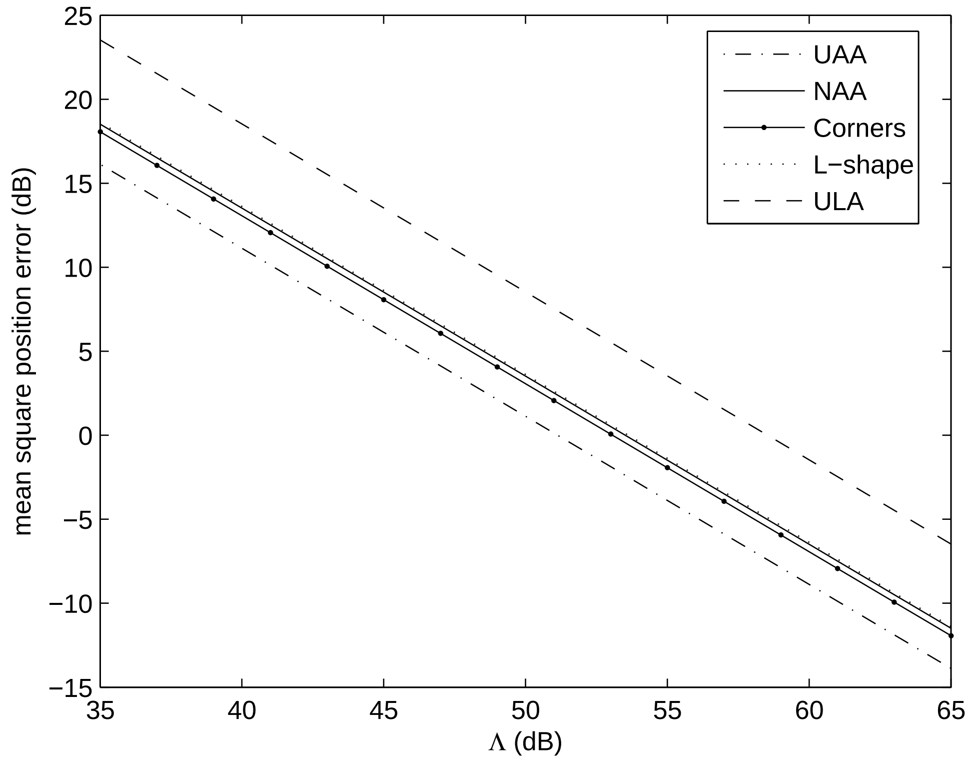

Fig. 1. Mean square position errors of different sensor geometries versus SNR when $L = 4$

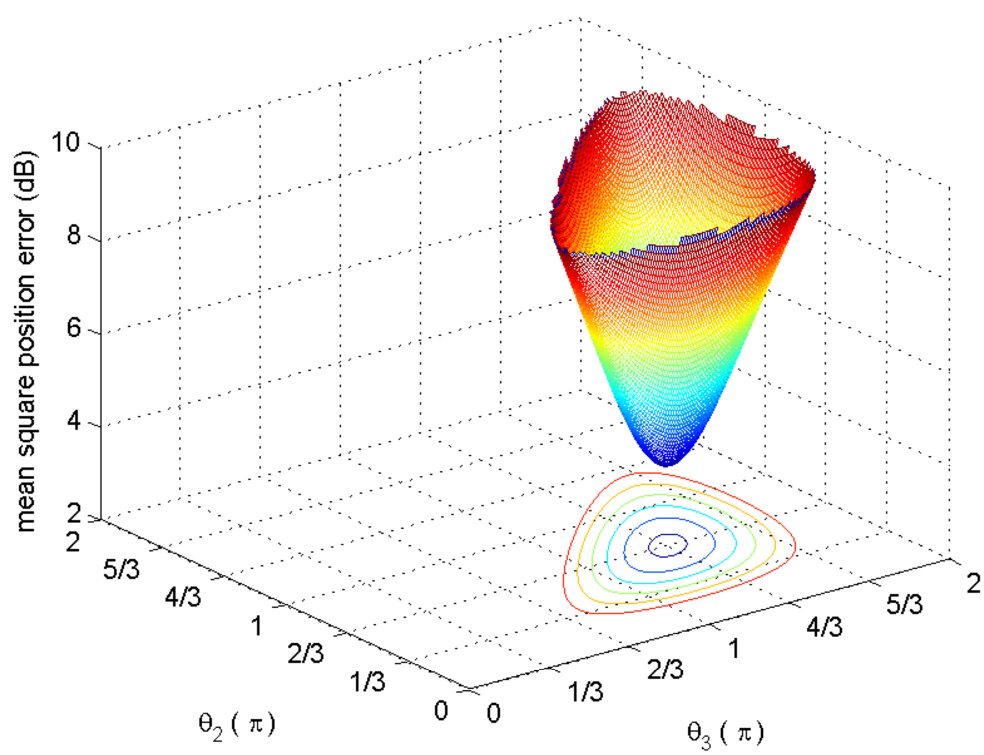

Fig. 2. Mean square position error versus angular separation when $L = 3$

Geometry	$\mathbf{x}_1$ (m)	$\mathbf{x}_2$ (m)	$\mathbf{x}_3$ (m)	$\mathbf{x}_4$ (m)
UAA	$[5\cos(0), 5\sin(0)]^T$	$[5\cos(\pi/2), 5\sin(\pi/2)]^T$	$[5\cos(\pi), 5\sin(\pi)]^T$	$[5\cos(3\pi/2), 5\sin(3\pi/2)]^T$
NAA	$[5\cos(\pi/3), 5\sin(\pi/3)]^T$	$[5\cos(2\pi/3), 5\sin(2\pi/3)]^T$	$[5\cos(\pi), 5\sin(\pi)]^T$	$[5\cos(4\pi/3), 5\sin(4\pi/3)]^T$
Corners	$[-10, 5]^T$	$[10, 5]^T$	$[10, -5]^T$	$[-10, -5]^T$
L-shape	$[-10, 5]^T$	$[0, -5]^T$	$[10, -5]^T$	$[-10, -5]^T$
ULA	$[-10, -5]^T$	$[10, -5]^T$	$[-2.5, -5]^T$	$[2.5, -5]^T$

Table 1. Senors positions for different placement strategies