IIR Filter Design

Chapter Intended Learning Outcomes:

(i) Ability to design analog Butterworth filters

(ii) Ability to design lowpass IIR filters according to predefined specifications based on analog filter theory and analog-to-digital filter transformation

(iii) Ability to construct frequency-selective IIR filters based on a lowpass IIR filter

Steps in Infinite Impulse Response Filter Design

The system transfer function of an IIR filter is:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad (11.1)$$

The task in IIR filter design is to find $\{a_k\}$ and $\{b_k\}$ such that $H(z)$ satisfies the given specifications.

Once $H(z)$ is computed, the filter can then be realized in hardware or software according to a direct, canonic, cascade or parallel form

We make use of the analog filter design to produce the required $H(z)$

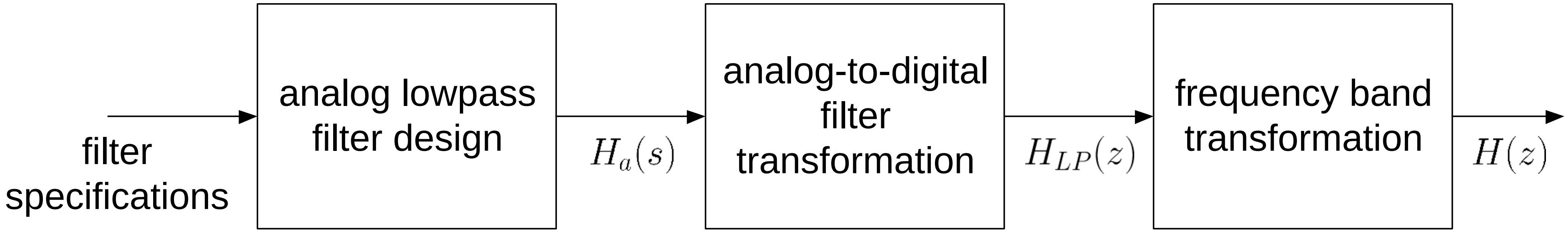

Fig.11.1: Steps in determining transfer function of IIR filter

Note that s is the Laplace transform parameter and substituting $s = j\Omega$ in $H_a(s)$ yields the Fourier transform of the filter, that is, $H_a(j\Omega)$

Main drawback is that there is no control over the phase response of $H(e^{j\omega})$, implying that the filter requirements can only be specified in terms of magnitude response

Butterworth Lowpass Filter Design

In analog lowpass filter design, we can only specify the magnitude of $H_a(j\Omega)$. Typically, we employ the magnitude square response, that is, $|H_a(j\Omega)|^2$:

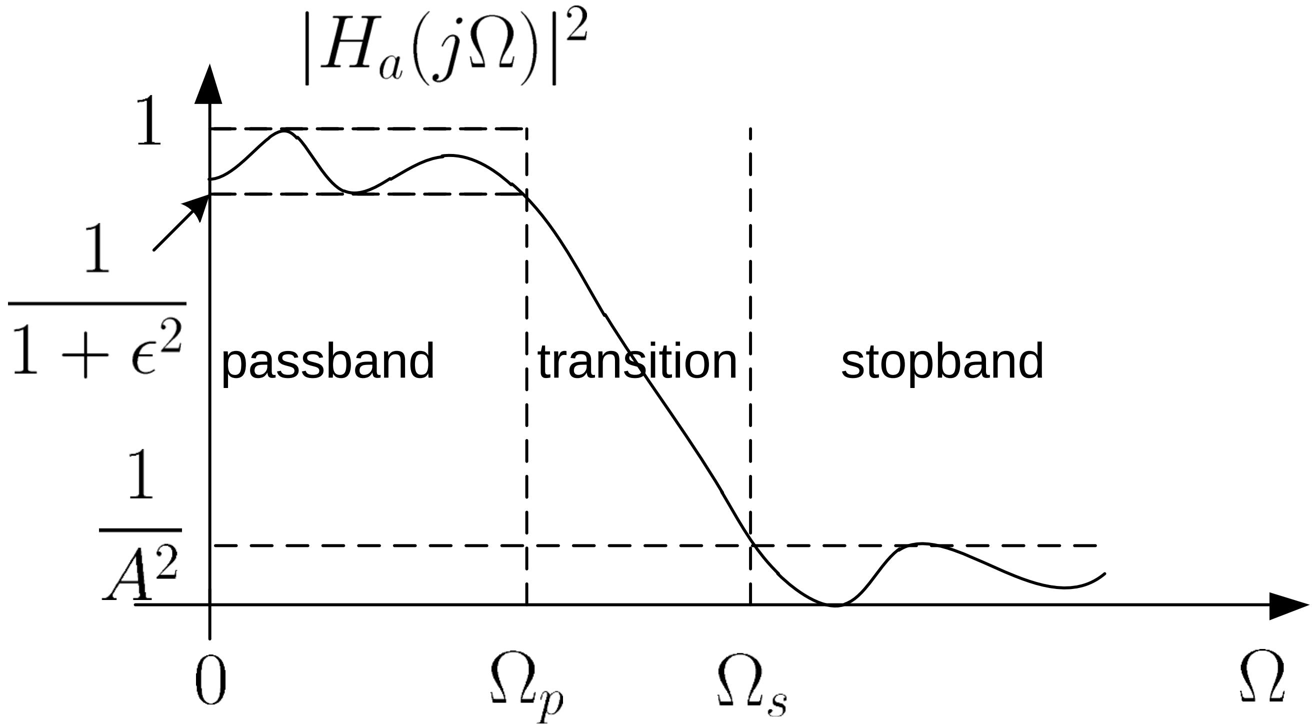

Fig.11.2: Specifications of analog lowpass filter

Passband corresponds to $\Omega \in [0, \Omega_p]$ where Ω_p is the passband frequency and ϵ is called the passband ripple

Stopband corresponds to $\Omega \in [\Omega_s, \infty)$ where Ω_s is the stopband frequency and A is called the stopband attenuation

Transition band corresponds to $\Omega \in [\Omega_p, \Omega_s]$

The specifications are represented as the two inequalities:

$$\frac{1}{1+\epsilon^2} \leq |H_a(j\Omega)|^2 \leq 1, \quad 0 \leq \Omega \leq \Omega_p \tag{11.2}$$

and

$$0 \leq |H_a(j\Omega)|^2 \leq \frac{1}{A^2}, \quad \Omega \geq \Omega_s \tag{11.3}$$

In particular, at $\Omega = \Omega_p$ and $\Omega = \Omega_s$, we have:

$$|H_a(j\Omega_p)|^2 = \frac{1}{1+\epsilon^2} \tag{11.4}$$

and

$$|H_a(j\Omega_s)|^2 = \frac{1}{A^2} \tag{11.5}$$

Apart from ϵ and A, it is also common to use their respective dB versions, denoted by R_p and A_s:

$$R_p = -10\log_{10}\left(\frac{1}{1+\epsilon^2}\right) \Rightarrow \epsilon = \sqrt{10^{R_p/10}-1} \tag{11.6}$$

and

$$A_s = -10\log_{10}\left(\frac{1}{A^2}\right) \Rightarrow A = 10^{A_s/20} \tag{11.7}$$

The magnitude square response of a Nth-order Butterworth lowpass filter is:

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \tag{11.8}$$

The filter is characterized by Ω_c and N, which represent the cutoff frequency and filter order

- $|H_a(j\Omega)|^2 = 1$ at $\Omega = 0$ and $|H_a(j\Omega)|^2 = 0.5$ at $\Omega = \Omega_c$ for all N
- $|H_a(j\Omega)|^2$ is a monotonically decreasing function of frequency which indicates that there is no ripple
- filter shape is closer to the ideal response as N increases, although the filter with order of $N \to \infty$ is not realizable.

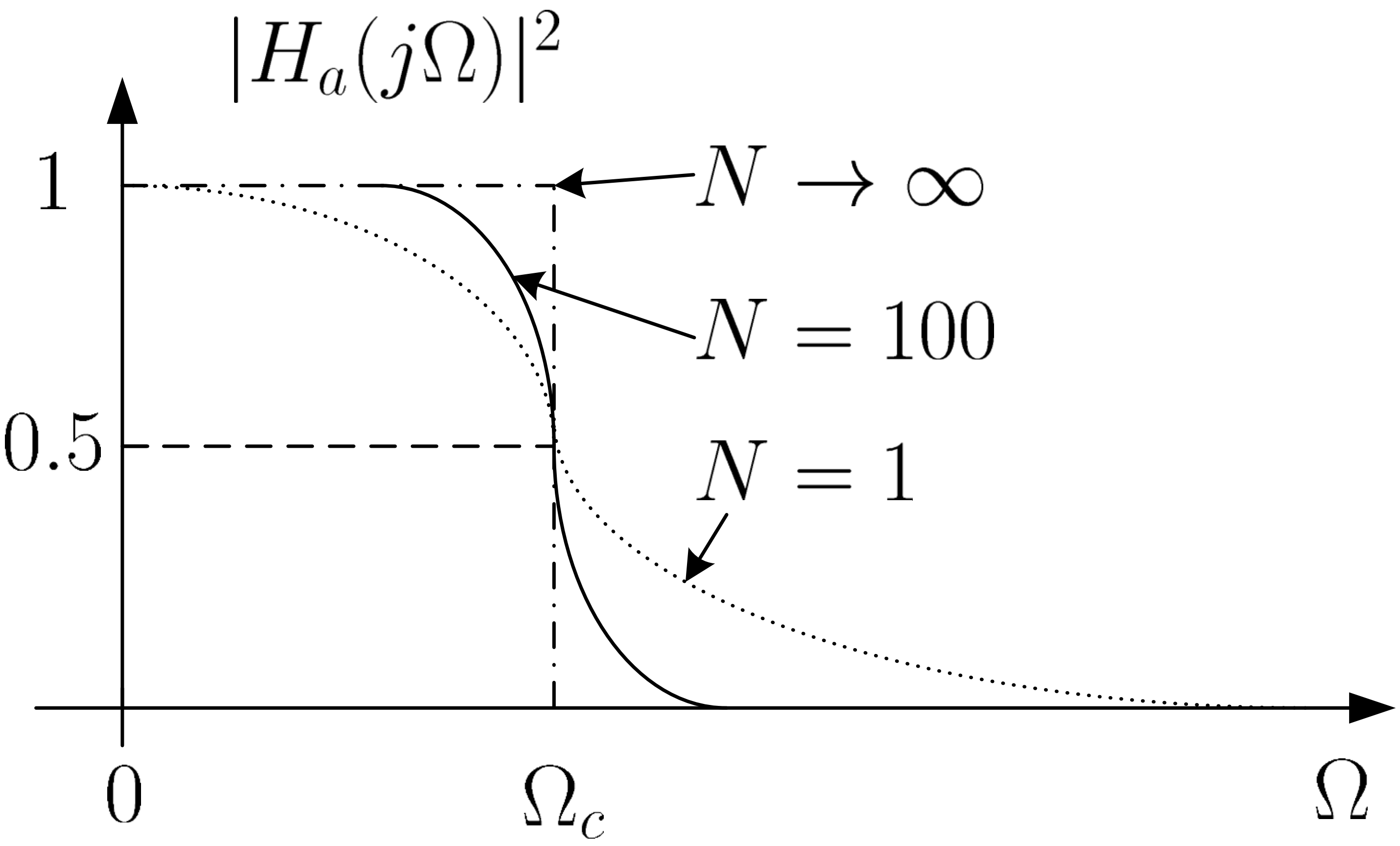

Fig.11.3: Magnitude square responses of Butterworth lowpass filter

To determine $H_a(s)$, we first make use of its relationship with $H_a(j\Omega)$:

$$H_a(s)\big|_{s=j\Omega} = H_a(j\Omega) \qquad (11.9)$$

From (11.8)-(11.9), we obtain:

$$H_a(s)H_a(-s) = \left.|H_a(j\Omega)|^2\right|_{\Omega=s/j} = \frac{(j\Omega_c)^{2N}}{s^{2N} + (j\Omega_c)^{2N}} \qquad (11.10)$$

The $2N$ poles of $H_a(s)H_a(-s)$, denoted by c_k, $k = 0, 1, \cdots, 2N-1$, are given as:

$$c_k = \begin{cases} \Omega_c e^{jk\pi/N}, & \text{odd } N \\ \Omega_c e^{jk\pi/N} \cdot e^{j\pi/(2N)}, & \text{even } N \end{cases} \qquad (11.11)$$

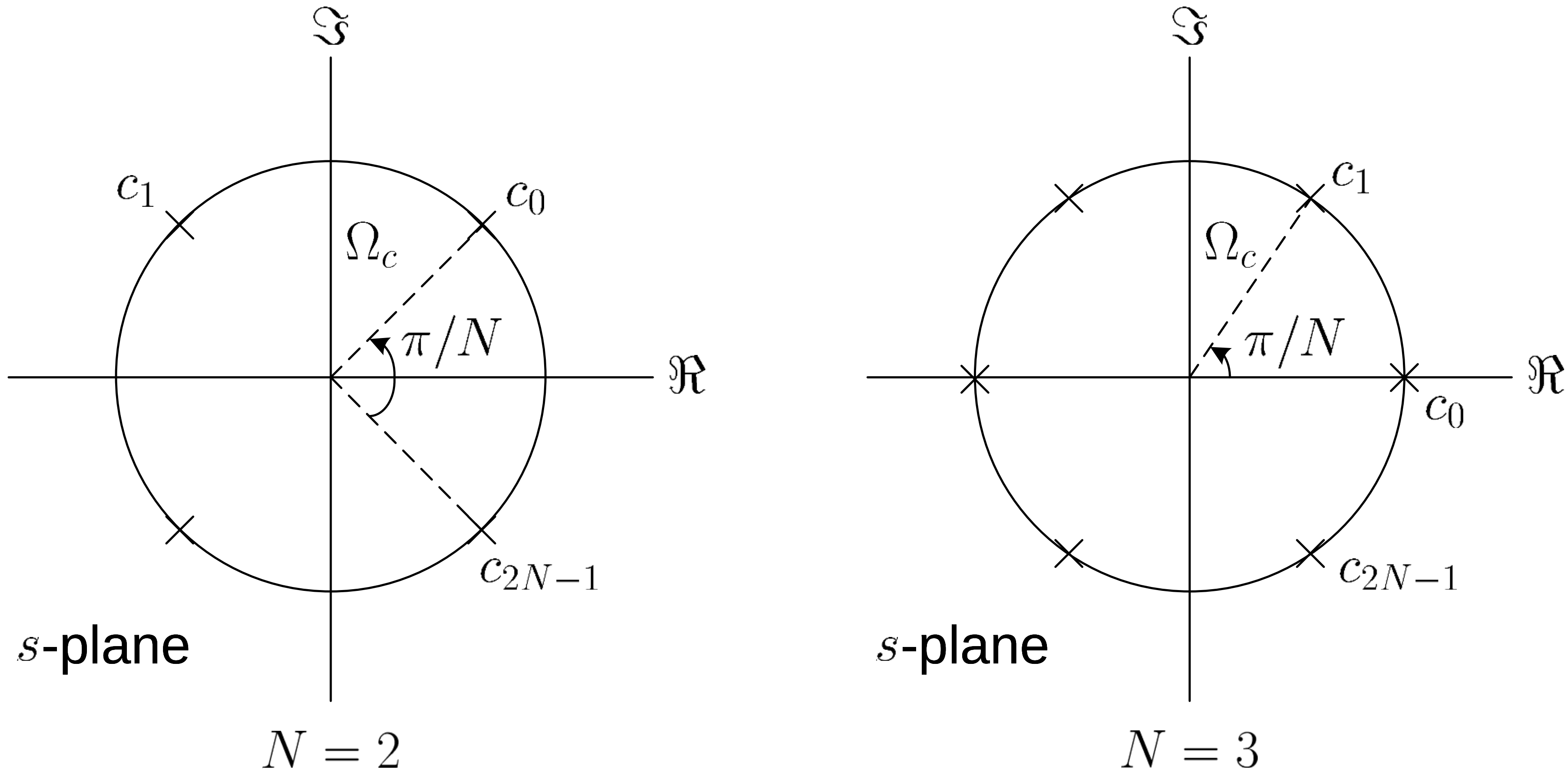

Fig.11.4: Poles of Butterworth lowpass filter

- $\{c_k\}$ are uniformly distributed on a circle of radius Ω_c with angular spacing of π/N in the s-plane

- poles are symmetrically located with respect to the imaginary axis

- there are two real-valued poles when N is odd

To extract $H_a(s)$ from (11.10), we utilize the knowledge that all poles of a stable and causal analog filter should be on the left half of the s-plane. As a result, $H_a(s)$ is:

$$H_a(s) = \frac{\Omega_c^N}{\prod_{\Re\{c_k\}<0} (s - c_k)} \tag{11.12}$$

Example 11.1

The magnitude square response of a Butterworth lowpass filter has the form of:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + 0.000001\Omega^6}$$

Determine the filter transfer function $H_a(s)$.

Expressing $|H_a(j\Omega)|^2$ as:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{10}\right)^{2\cdot 3}}$$

From (11.8), $\Omega_c = 10$ and $N = 3$

From (11.11):

$$c_k = 10e^{jk\pi/3}, \quad k = 2, 3, 4$$

Finally, we apply (11.12) to obtain:

$$\begin{aligned} H_a(s) &= \frac{\Omega_c^3}{(s - c_2)(s - c_3)(s - c_4)} \\ &= \frac{1000}{(s - 10e^{j2\pi/3})(s + 10)(s - 10e^{j4\pi/3})} \\ &= \frac{1000}{(s + 10)(s^2 + 10s + 100)} \end{aligned}$$

To find Ω_c and N given the passband and stopband requirements in terms of Ω_p, Ω_s, R_p and A_s, we exploit (11.4)-(11.5) together with (11.6)-(11.7) to obtain

$$-10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}}\right)=R_p \qquad (11.13)$$

and

$$-10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}}\right)=A_s \qquad (11.14)$$

Solving (11.13)-(11.14) and noting that N should be an integer, we get

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{R_p/10}-1\right)/\left(10^{A_s/10}-1\right)\right]}{2\log_{10}(\Omega_p/\Omega_s)} \right\rceil \tag{11.15}$$

where $\lceil u \rceil$ rounds up u to the nearest integer.

The Ω_c is then obtained from (11.13) or (11.14) so that the specification can be exactly met at Ω_p or Ω_s, respectively

From (11.13), Ω_c is computed as:

$$\Omega_c = \frac{\Omega_p}{\left(10^{R_p/10}-1\right)^{1/(2N)}} \tag{11.16}$$

From (11.14), Ω_c is computed as:

$$\Omega_c = \frac{\Omega_s}{\left(10^{A_s/10} - 1\right)^{1/(2N)}} \tag{11.17}$$

As a result, the admissible range of Ω_c is:

$$\Omega_c \in \left[\frac{\Omega_p}{\left(10^{R_p/10} - 1\right)^{1/(2N)}}, \frac{\Omega_s}{\left(10^{A_s/10} - 1\right)^{1/(2N)}}\right] \tag{11.18}$$

Example 11.2

Determine the transfer function of a Butterworth lowpass filter whose magnitude requirements are $\Omega_p = 4\pi \quad \mathrm{rads}^{-1}$, $\Omega_s = 6\pi \ \mathrm{rads}^{-1}$, $R_p = 8$ dB and $A_s = 16$ dB.

Employing (11.15) yields:

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{8/10}-1\right)/\left(10^{16/10}-1\right)\right]}{2\log_{10}(4\pi/(6\pi))} \right\rceil = \lceil 2.45 \rceil = 3$$

Putting $N=3$ in (11.18), the cutoff frequency is:

$$\Omega_c \in \left[\frac{4\pi}{\left(10^{8/10}-1\right)^{1/(2\cdot 3)}}, \frac{6\pi}{\left(10^{16/10}-1\right)^{1/(2\cdot 3)}}\right] = [9.5141, 10.2441]$$

For simplicity, we select $\Omega_c = 10$. Using Example 11.1, the filter transfer function $H_a(s)$ is:

$$H_a(s) = \frac{1000}{(s+10)(s^2+10s+100)} = \frac{1000}{s^3+20s^2+200s+1000}$$

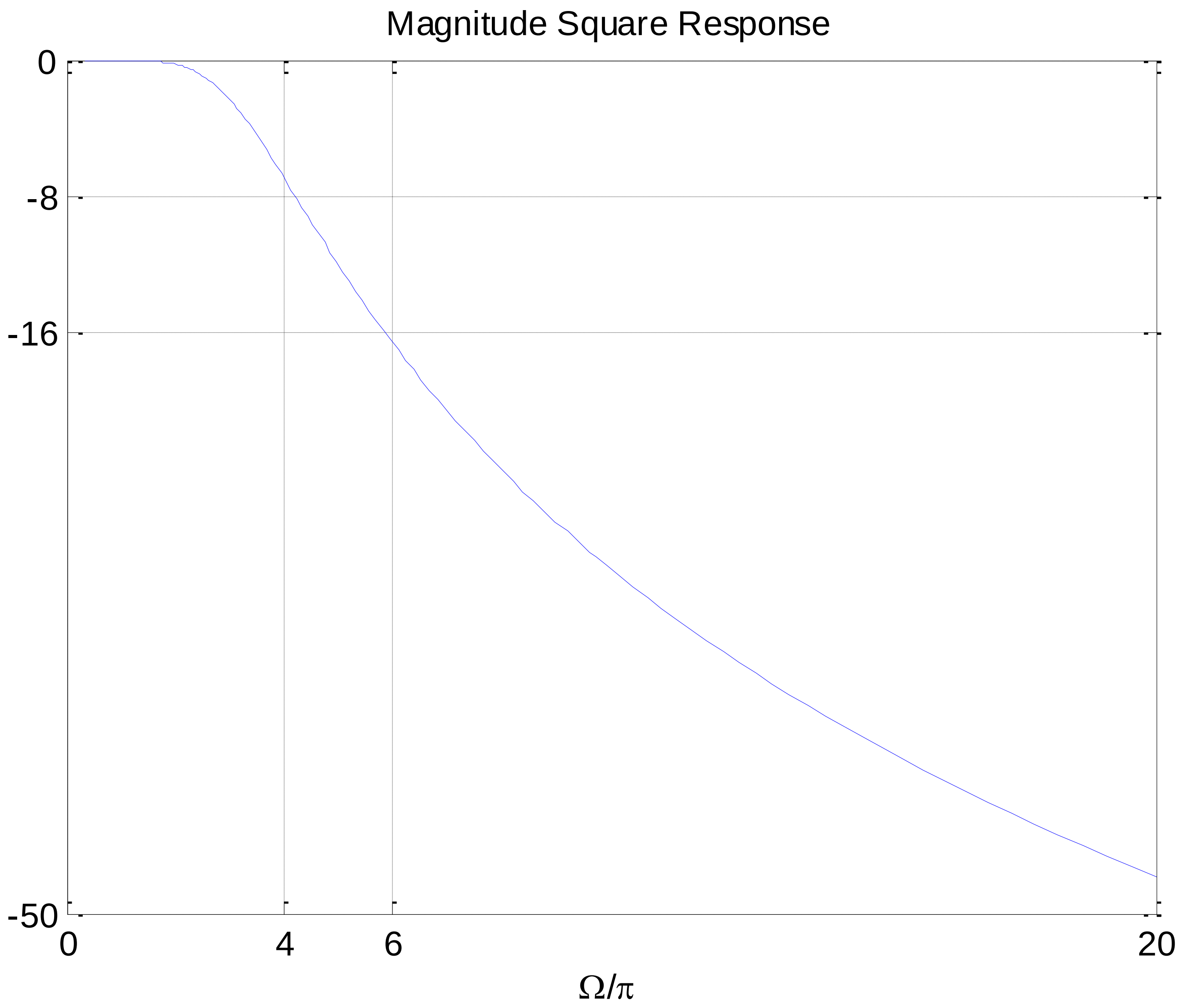

Fig.11.5: Magnitude square response of Butterworth lowpass filter

The MATLAB program is provided as `ex11_2.m` where the command `freqs`, which is analogous to `freqz`, is used to plot $|H_a(j\Omega)|^2$

Analog-to-Digital Filter Transformation

Typical methods include impulse invariance, bilinear transformation, backward difference approximation and matched-z transformation

Their common feature is that a stable analog filter will transform to a stable system with transfer function $H_{LP}(z)$.

Left half of s-plane maps into inside of unit circle in z-plane

Each has its pros and cons and thus optimal transformation does not exist

▪ Impulse Invariance

The idea is simply to sample impulse response of the analog filter $h_a(t)$ to obtain the digital lowpass filter impulse response $h_{LP}[n]$

The relationship between $h_{LP}[n]$ and $h_a(t)$ is

$$h_{LP}[n] = T \cdot h_a(nT), \quad n = \cdots -1, 0, 1, 2, \cdots \tag{11.19}$$

where T is the sampling interval

Why there is a scaling of *T* ?

With the use of (4.5) and (5.3)-(5.4), $H_{LP}(e^{j\omega})$ is:

$$H_{LP}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \quad (11.20)$$

where the analog and digital frequencies are related as:

$$\omega = \Omega T \quad (11.21)$$

The impulse response of the resultant IIR filter is similar to that of the analog filter

Aliasing due to the overlapping of $\{H_a\left(j\left(\omega - 2\pi k\right)/T\right)\}$ which are not bandlimited. However, $H_a(j\Omega)$ corresponds to a lowpass filter and thus aliasing effect is negligibly small particularly when T is chosen sufficiently small.

To derive the IIR filter transfer function $H_{LP}(z)$ from $H_a(s)$, we first obtain the partial fraction expansion:

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \qquad (11.22)$$

where $\{s_k\}$ are the poles on the left half of the s-plane

The inverse Laplace transform of (11.22) is given as:

$$h_a(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad (11.23)$$

Substituting (11.23) into (11.19), we have:

$$h_{LP}[n] = \sum_{k=1}^{N} T A_k e^{s_k n T} u[n] \quad (11.24)$$

The z transform of $h_{LP}[n]$ is:

$$H_{LP}(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - e^{s_k T} z^{-1}} \quad (11.25)$$

Comparing (11.22) and (11.25), it is seen that a pole of $s = s_k$ in the s-plane transforms to a pole at $z = e^{s_k T}$ in the z-plane:

$$z = e^{sT} \quad (11.26)$$

Expressing $s=\sigma+j\Omega$:

$$z=e^{\sigma T}\cdot e^{j\Omega T}=e^{\sigma T}\cdot e^{j(\Omega+2\pi k/T)T} \tag{11.27}$$

where k is any integer, indicating a many-to-one mapping

Each infinite horizontal strip of width $2\pi/T$ maps into the entire z-plane

$\sigma=0$ maps to $|z|=1$, that is, $j\Omega$ axis in the s-plane transforms to the unit circle in the z-plane

$\sigma<0$ maps to $|z|<1$, stable $H_a(s)$ produces stable $H_{LP}(z)$

$\sigma>0$ maps to $|z|>1$, right half of the s-plane maps into the outside of the unit circle in the z-plane

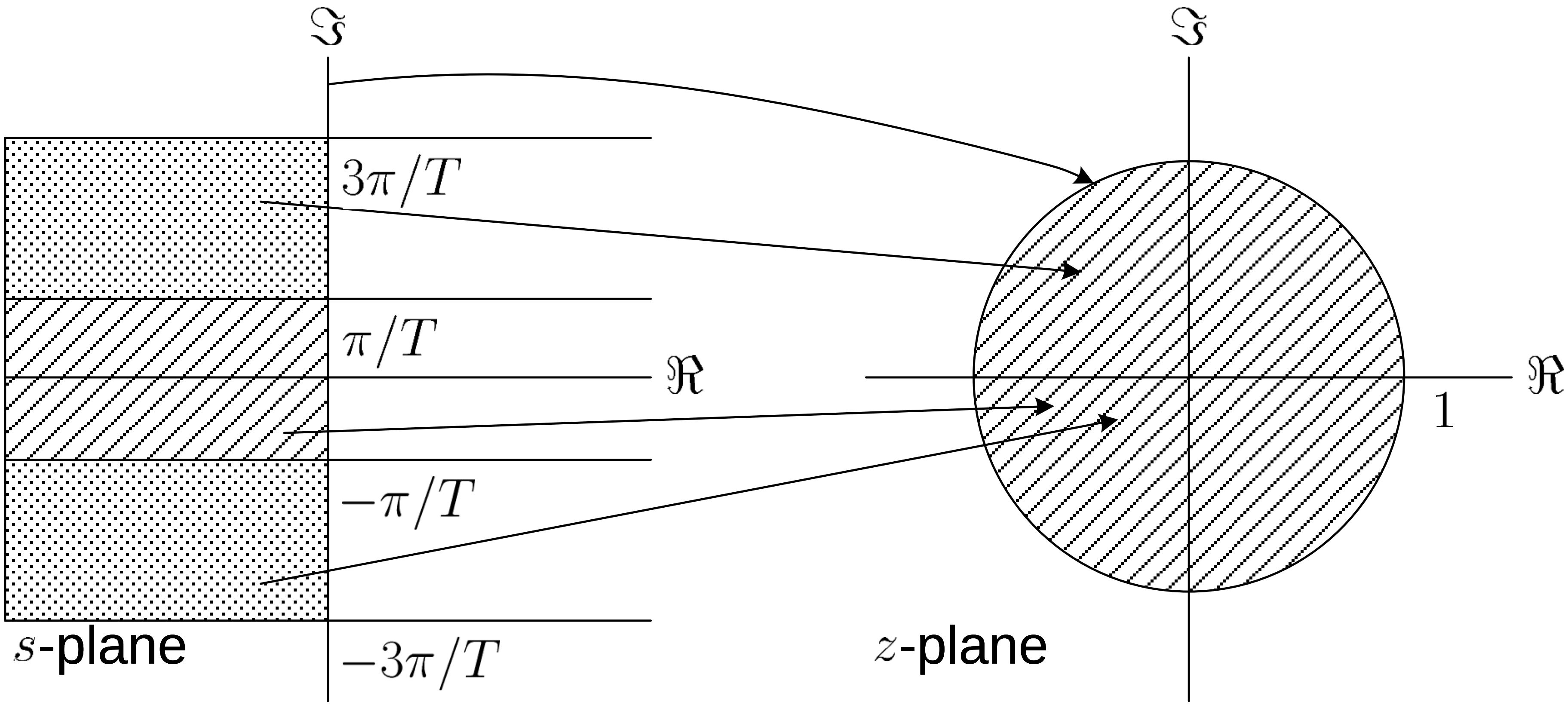

Fig.11.6: Mapping between s and z in impulse invariance method

Given the magnitude square response specifications of $H_{LP}(e^{j\omega})$ in terms of ω_p, ω_s, R_p and A_s, the design procedure for $H_{LP}(z)$ based on the impulse invariance method is summarized as the following steps:

(i) Select a value for the sampling interval T and then compute the passband and stopband frequencies for the analog lowpass filter according to $\Omega_p = \omega_p/T$ and $\Omega_s = \omega_s/T$

(ii) Design the analog Butterworth filter with transfer function $H_a(s)$ according to Ω_p, Ω_s, R_p and A_s

(iii) Perform partial fraction expansion on $H_a(s)$ as in (11.22)

(iv) Obtain $H_{LP}(z)$ using (11.25)

Example 11.3

The transfer function of an analog filter has the form of

$$H_a(s) = \frac{2s}{s^2 + 6s + 8}$$

Use impulse invariance method with sampling interval $T = 1$ to transform $H_a(s)$ to a digital filter transfer function $H(z)$.

Performing partial fraction expansion on $H_a(s)$:

$$H_a(s) = \frac{-2}{s+2} + \frac{4}{s+4}$$

Applying (11.25) with $T = 1$ yields

$$H(z) = \frac{-2}{1 - e^{-2}z^{-1}} + \frac{4}{1 - e^{-4}z^{-1}} = \frac{2 - 0.5047z^{-1}}{1 - 0.1537z^{-1} + 0.0025z^{-2}}$$

Example 11.4

Determine the transfer function $H_{LP}(z)$ of a digital lowpass filter whose magnitude requirements are $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $R_p = 8$ dB and $A_s = 16$ dB. Use the Butterworth lowpass filter and impulse invariance method in the design.

Selecting the sampling interval as $T = 0.1$, the analog frequency parameters are computed as:

$$\Omega_p = \frac{\omega_p}{T} = 4\pi$$

and

$$\Omega_s = \frac{\omega_s}{T} = 6\pi$$

Using Example 11.2, a Butterworth filter which meets the magnitude requirements are:

$$H_a(s) = \frac{1000}{(s+10)(s^2+10s+100)} = \frac{1000}{s^3+20s^2+200s+1000}$$

Performing partial fraction expansion on $H_a(s)$ with the use of the MATLAB command `residue`, we get

$$H_a(s) = \frac{10}{s+10} + \frac{-5-j2.8868}{s+5-j8.6603} + \frac{-5+j2.8868}{s+5+j8.6603}$$

Applying (11.25) with $T=0.1$ yields

$$
\begin{aligned}
H_{LP}(z) &= \frac{0.1 \cdot 10}{1 - e^{-10 \cdot 0.1} z^{-1}} + \frac{0.1 \cdot (-5 - j2.8868)}{1 - e^{(-5+j8.6603) \cdot 0.1} z^{-1}} + \frac{0.1 \cdot (-5 + j2.8868)}{1 - e^{(-5-j8.6603) \cdot 0.1} z^{-1}} \\
&= \frac{1}{1 - 0.3679 z^{-1}} + \frac{-0.5 - j0.2887}{1 - (0.3929 + j0.4620) z^{-1}} \\
&\quad + \frac{-0.5 + j0.2887}{1 - (0.3929 - j0.4620) z^{-1}} \\
&= \frac{1}{1 - 0.3679 z^{-1}} + \frac{-1 + 0.6597 z^{-1}}{1 - 0.7859 z^{-1} + 0.3679 z^{-2}} \\
&= \frac{0.2417 z^{-1} + 0.1262 z^{-2}}{1 - 1.1538 z^{-1} + 0.6570 z^{-2} - 0.1354 z^{-3}}
\end{aligned}
$$

The MATLAB program is provided as `ex11_4.m`.

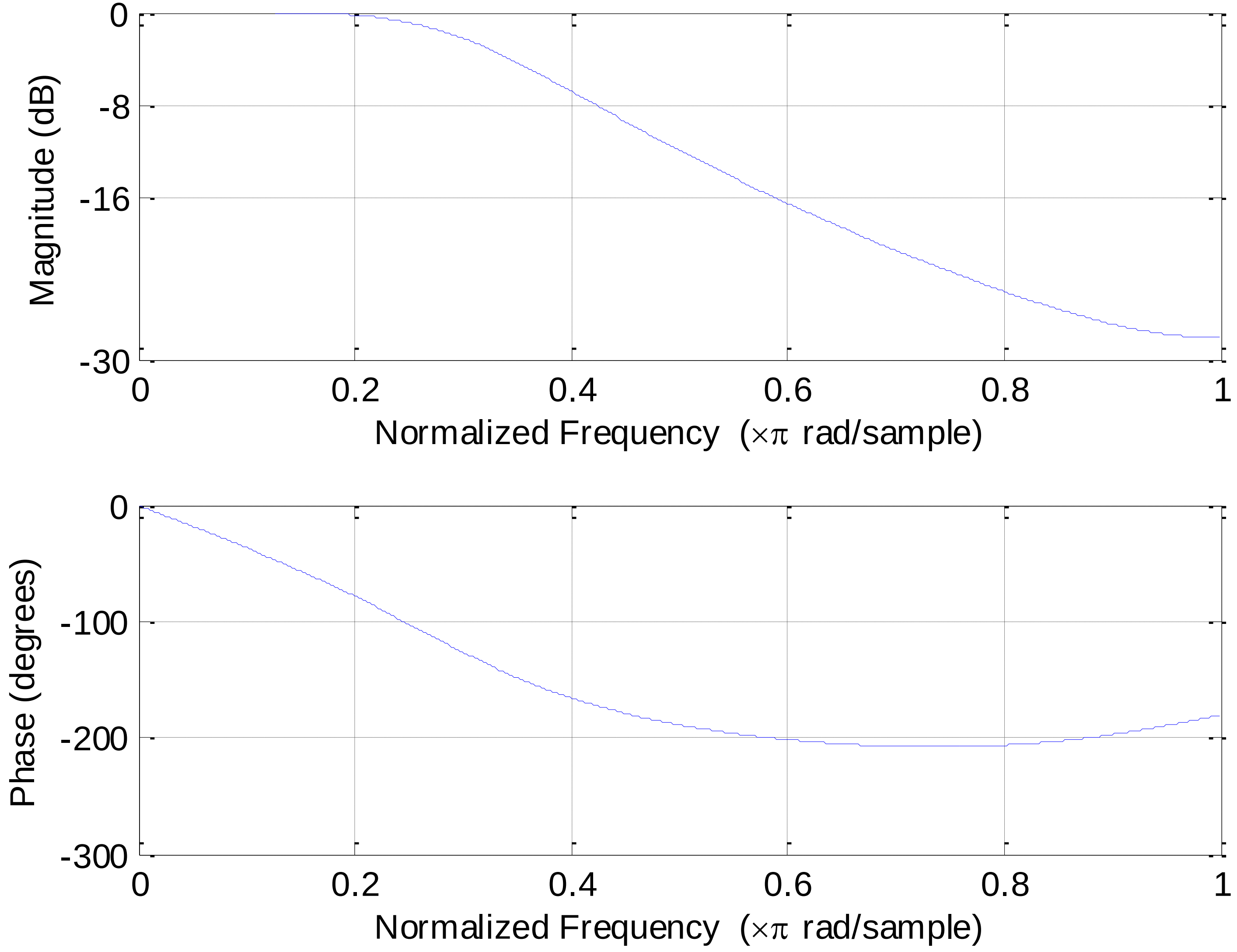

Fig.11.7: Magnitude and phase responses based on impulse invariance

- Bilinear Transformation

It is a conformal mapping that maps the $j\Omega$ axis of the s-plane into the unit circle of the z-plane only once, implying there is no aliasing problem as in the impulse invariance method

It is a one-to-one mapping

The relationship between s and z is:

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \Leftrightarrow z = \frac{1 + sT/2}{1 - sT/2} \qquad (11.28)$$

Employing $s = \sigma + j\Omega$, z can be expressed as:

$$z = \frac{(1 + \sigma T/2) + j\Omega T/2}{(1 - \sigma T/2) - j\Omega T/2} \qquad (11.29)$$

$\sigma = 0$ maps to $|z| = 1$, that is, $j\Omega$ axis in the s-plane transforms to the unit circle in the z-plane

$\sigma < 0$ maps to $|z| < 1$, stable $H_a(s)$ produces a stable $H_{LP}(z)$

$\sigma > 0$ maps to $|z| > 1$, right half of the s-plane maps into the outside of the unit circle in the z-plane

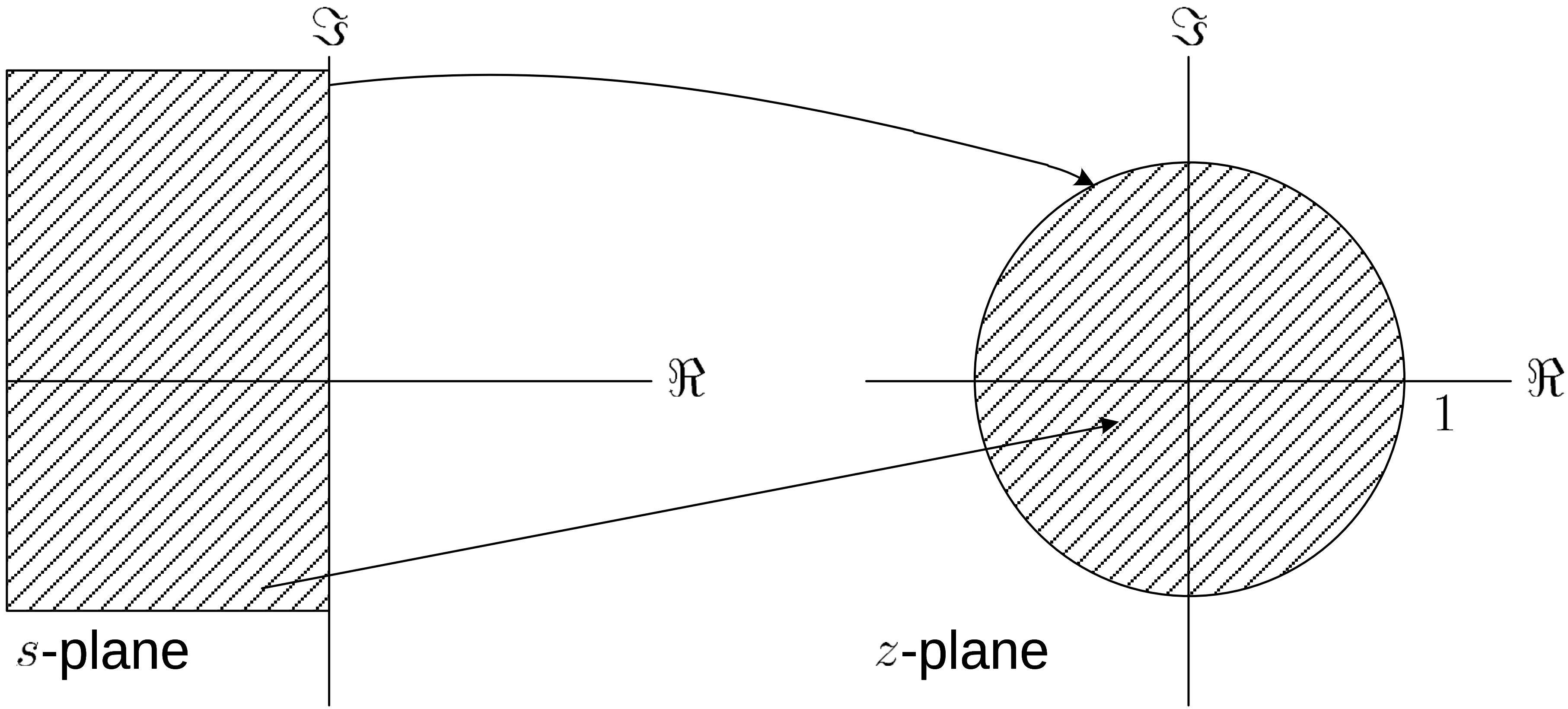

Fig.11.8: Mapping between s and z in bilinear transformation

Although aliasing is avoided, the drawback of the bilinear transformation is that there is no linear relationship between ω and Ω

Putting $z = e^{j\omega}$ and $s = j\Omega$ in (11.28), ω and Ω are related as:

$$
\begin{aligned}
j\Omega &= \frac{2}{T} \cdot \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2}{T} \cdot \frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}} = j\frac{2}{T}\tan\left(\frac{\omega}{2}\right) \\
&\Rightarrow \Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \Leftrightarrow \omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)
\end{aligned}
\qquad (11.30)
$$

Given the magnitude square response specifications of $H_{LP}(e^{j\omega})$ in terms of ω_p, ω_s, R_p and A_s, the design procedure for $H_{LP}(z)$ based on the bilinear transformation is summarized as the following steps:

(i) Select a value for T and then compute the passband and stopband frequencies for the analog lowpass filter according $\Omega_p = (2/T)\tan(\omega_p/2)$ and $\Omega_s = (2/T)\tan(\omega_s/2)$

(ii) Design the analog Butterworth filter with transfer function $H_a(s)$ according to Ω_p, Ω_s, R_p and A_s.

(iii) Obtain $H_{LP}(z)$ from $H_a(s)$ using the substitution of (11.28).

Example 11.5

The transfer function of an analog filter has the form of

$$H_a(s) = \frac{2s}{s^2 + 6s + 8}$$

Use the bilinear transformation with $T = 1$ to transform $H_a(s)$ to a digital filter with transfer function $H(z)$.

Applying (11.28) with $T=1$ yields

$$H(z)=\frac{2\cdot 2\frac{1-z^{-1}}{1+z^{-1}}}{\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^2+6\cdot 2\frac{1-z^{-1}}{1+z^{-1}}+8}=\frac{4-4z^{-2}}{15+14z^{-1}+9z^{-2}}$$

Example 11.6

Determine the transfer function $H_{LP}(z)$ of a digital lowpass filter whose magnitude requirements are $\omega_p=0.4\pi$, $\omega_s=0.6\pi$, $R_p=8$ dB and $A_s=16$ dB. Use the Butterworth lowpass filter and bilinear transformation in the design.

Selecting $T=0.1$, the analog frequency parameters are computed according to (11.30) as:

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = 14.5309$$

and

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 27.5276$$

Employing (11.15) yields:

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{8/10}-1\right)/\left(10^{16/10}-1\right)\right]}{2\log_{10}(14.5309/27.5276)} \right\rceil = \lceil 1.56 \rceil = 2$$

Putting $N=2$ in (11.18), the cutoff frequency is:

$$\Omega_c \in \left[\frac{14.5309}{\left(10^{8/10}-1\right)^{1/(2\cdot 2)}}, \frac{27.5276}{\left(10^{16/10}-1\right)^{1/(2\cdot 2)}}\right] = [9.5725, 11.0289]$$

For simplicity, $\Omega_c = 10$ is employed.

Following (11.11)-(11.12):

$$H_a(s) = \frac{100}{s^2 + 14.1421s + 100}$$

Finally, we use (11.28) with $T = 0.1$ to yield

$$\begin{aligned} H_{LP}(z) &= \frac{100}{\left(\frac{2}{0.1} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + 14.1421 \cdot \frac{2}{0.1} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} + 100} \\ &= \frac{1 + 2z^{-1} + z^{-2}}{7.8284 - 6z^{-1} + 2.1716z^{-2}} \end{aligned}$$

The MATLAB program is provided as `ex11_6.m`.

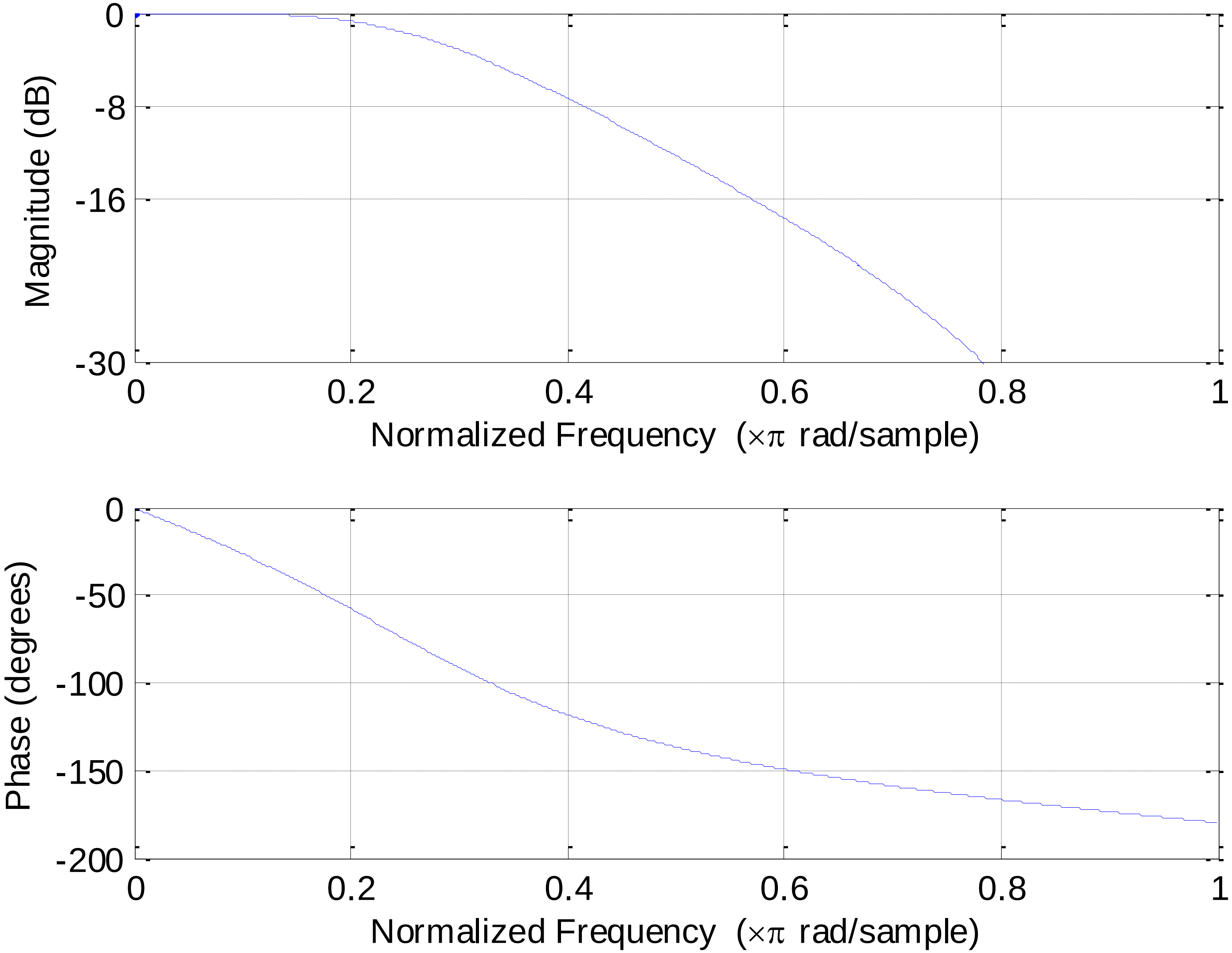

Fig.11.9: Magnitude and phase responses based on bilinear transformation

Frequency Band Transformation

The operations are similar to that of the bilinear transformation but now the mapping is performed only in the z-plane:

$$z_o^{-1} = T(z^{-1}) \tag{11.31}$$

where z_o and z correspond to the lowpass and resultant filters, respectively, and T denotes the transformation operator.

To ensure the transformed filter to be stable and causal, the unit circle and inside of the z_o-plane should map into those of the z-plane, respectively.

Filter Type	Transformation Operator	Design Parameter
Lowpass	$z_o^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\dfrac{\omega_{c_o} - \omega_c}{2}\right)}{\sin\left(\dfrac{\omega_{c_o} + \omega_c}{2}\right)}$
Highpass	$z_o^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\dfrac{\omega_{c_o} + \omega_c}{2}\right)}{\cos\left(\dfrac{\omega_{c_o} - \omega_c}{2}\right)}$
Bandpass	$z_o^{-1} = \dfrac{z^{-2} - \dfrac{2\alpha\beta}{\beta+1} z^{-1} + \dfrac{\beta-1}{\beta+1}}{\dfrac{\beta-1}{\beta+1} z^{-2} - \dfrac{2\alpha\beta}{\beta+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_{c_2} + \omega_{c_1}}{2}\right)}{\cos\left(\dfrac{\omega_{c_2} - \omega_{c_1}}{2}\right)}$ $\beta = \cot\left(\dfrac{\omega_{c_2} - \omega_{c_1}}{2}\right)\tan\left(\dfrac{\omega_{c_o}}{2}\right)$

Bandstop	$z_o^{-1} = \dfrac{z^{-2} - \dfrac{2\alpha}{1+\beta} z^{-1} + \dfrac{1-\beta}{1+\beta}}{\dfrac{1-\beta}{1+\beta} z^{-2} - \dfrac{2\alpha}{1+\beta} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_{c_2}+\omega_{c_1}}{2}\right)}{\cos\left(\dfrac{\omega_{c_2}-\omega_{c_1}}{2}\right)}$ $\beta = \cot\left(\dfrac{\omega_{c_2}-\omega_{c_1}}{2}\right)\tan\left(\dfrac{\omega_{c_o}}{2}\right)$

Table 11.1: Frequency band transformation operators

Example 11.7

Determine the transfer function $H(z)$ of a digital highpass filter whose magnitude requirements are $\omega_p = 0.6\pi$, $\omega_s = 0.4\pi$, $R_p = 8$ dB and $A_s = 16$ dB. Use the Butterworth lowpass filter and bilinear transformation in the design.

Using Example 11.6, the corresponding lowpass filter transfer function $H_{LP}(z_o)$ is:

$$H_{LP}(z_o) = \frac{1 + 2z_o^{-1} + z_o^{-2}}{7.8284 - 6z_o^{-1} + 2.1716z_o^{-2}}$$

Assigning the cutoff frequencies as the midpoints between the passband and stopband frequencies, we have

$$\omega_{c_o} = \omega_c = \frac{0.4\pi + 0.6\pi}{2} = 0.5\pi$$

With the use of Table 11.1, the corresponding value of α is:

$$\alpha = -\frac{\cos\left(\frac{\omega_{c_o} + \omega_c}{2}\right)}{\cos\left(\frac{\omega_{c_o} - \omega_c}{2}\right)} = -\frac{\cos(0.5\pi)}{\cos(0)} = 0$$

which gives the transformation operator:

$$z_o^{-1} = -\frac{z^{-1} + 0}{1 + 0 \cdot z^{-1}} = -z^{-1}$$

As a result, the digital highpass filter transfer function is:

$$H(z) = H_{LP}(z_o)\big|_{z_o^{-1} = -z^{-1}} = \frac{1 - 2z^{-1} + z^{-2}}{7.8284 + 6z^{-1} + 2.1716z^{-2}}$$

The MATLAB program is provided as `ex11_7.m`.

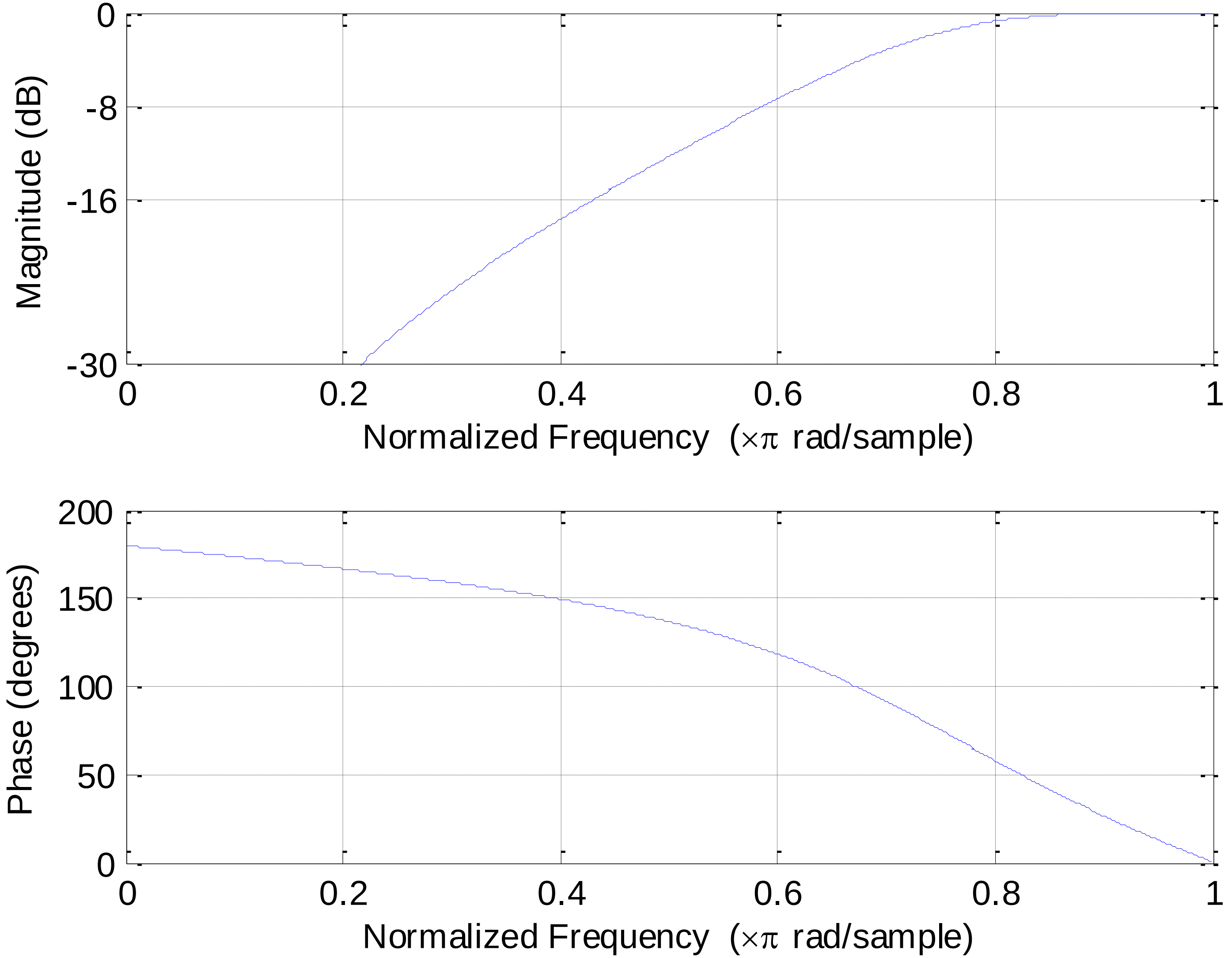

Fig.11.10: Magnitude and phase responses based on frequency band transformation