**IIR Filter Design**

Chapter Intended Learning Outcomes:

(i) Ability to design analog Butterworth filters

(ii) Ability to design lowpass IIR filters according to predefined specifications based on analog filter theory and analog-to-digital filter transformation

(iii) Ability to construct frequency-selective IIR filters based on a lowpass IIR filter
Steps in Infinite Impulse Response Filter Design

The system transfer function of an IIR filter is:

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \]  

(11.1)

The task in IIR filter design is to find \( \{a_k\} \) and \( \{b_k\} \) such that \( H(z) \) satisfies the given specifications.

Once \( H(z) \) is computed, the filter can then be realized in hardware or software according to a direct, canonic, cascade or parallel form.
We make use of the analog filter design to produce the required $H(z)$

![Diagram showing the process of filter design](image)

**Fig.11.1: Steps in determining transfer function of IIR filter**

Note that $s$ is the Laplace transform parameter and substituting $s = j\Omega$ in $H_a(s)$ yields the Fourier transform of the filter, that is, $H_a(j\Omega)$

Main drawback is that there is no control over the phase response of $H(e^{j\omega})$, implying that the filter requirements can only be specified in terms of magnitude response
Butterworth Lowpass Filter Design

In analog lowpass filter design, we can only specify the magnitude of $H_a(j\Omega)$. Typically, we employ the magnitude square response, that is, $|H_a(j\Omega)|^2$:

![Graph showing the magnitude square response of a Butterworth filter with passband, transition, and stopband regions labeled.](Fig.11.2: Specifications of analog lowpass filter)
Passband corresponds to $\Omega \in [0, \Omega_p]$ where $\Omega_p$ is the passband frequency and $\epsilon$ is called the passband ripple.

Stopband corresponds to $\Omega \in [\Omega_s, \infty)$ where $\Omega_s$ is the stopband frequency and $A$ is called the stopband attenuation.

Transition band corresponds to $\Omega \in [\Omega_p, \Omega_s]$

The specifications are represented as the two inequalities:

$$\frac{1}{1 + \epsilon^2} \leq |H_a(j\Omega)|^2 \leq 1, \quad 0 \leq \Omega \leq \Omega_p \quad (11.2)$$

and

$$0 \leq |H_a(j\Omega)|^2 \leq \frac{1}{A^2}, \quad \Omega \geq \Omega_s \quad (11.3)$$
In particular, at $\Omega = \Omega_p$ and $\Omega = \Omega_s$, we have:

$$|H_a(j\Omega_p)|^2 = \frac{1}{1 + \epsilon^2} \quad (11.4)$$

and

$$|H_a(j\Omega_s)|^2 = \frac{1}{A^2} \quad (11.5)$$

Apart from $\epsilon$ and $A$, it is also common to use their respective dB versions, denoted by $R_p$ and $A_s$:

$$R_p = -10 \log_{10} \left( \frac{1}{1 + \epsilon^2} \right) \Rightarrow \epsilon = \sqrt{10^{R_p/10} - 1} \quad (11.6)$$

and

$$A_s = -10 \log_{10} \left( \frac{1}{A^2} \right) \Rightarrow A = 10^{A_s/20} \quad (11.7)$$
The magnitude square response of a $N$th-order Butterworth lowpass filter is:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$  \hspace{1cm} (11.8)

The filter is characterized by $\Omega_c$ and $N$, which represent the cutoff frequency and filter order.

- $|H_a(j\Omega)|^2 = 1$ at $\Omega = 0$ and $|H_a(j\Omega)|^2 = 0.5$ at $\Omega = \Omega_c$ for all $N$
- $|H_a(j\Omega)|^2$ is a monotonically decreasing function of frequency which indicates that there is no ripple
- filter shape is closer to the ideal response as $N$ increases, although the filter with order of $N \to \infty$ is not realizable.
Fig. 11.3: Magnitude square responses of Butterworth lowpass filter
To determine $H_a(s)$, we first make use of its relationship with
$H_a(j\Omega)$:

$$H_a(s)|_{s=j\Omega} = H_a(j\Omega) \quad (11.9)$$

From (11.8)-(11.9), we obtain:

$$H_a(s)H_a(-s) = |H_a(j\Omega)|^2\bigg|_{\Omega=s/j} = \frac{(j\Omega_c)^{2N}}{s^{2N} + (j\Omega_c)^{2N}} \quad (11.10)$$

The $2N$ poles of $H_a(s)H_a(-s)$, denoted by $c_k$, $k = 0, 1, \ldots, 2N - 1$, are given as:

$$c_k = \begin{cases} 
\Omega_c e^{jk\pi/N}, & \text{odd } N \\
\Omega_c e^{jk\pi/N} \cdot e^{j\pi/(2N)}, & \text{even } N
\end{cases} \quad (11.11)$$
Fig. 11.4: Poles of Butterworth lowpass filter
\[ \{c_k\} \text{ are uniformly distributed on a circle of radius } \Omega_c \text{ with angular spacing of } \frac{\pi}{N} \text{ in the } s\text{-plane} \]

\[ \text{poles are symmetrically located with respect to the imaginary axis} \]

\[ \text{there are two real-valued poles when } N \text{ is odd} \]

To extract \( H_a(s) \) from (11.10), we utilize the knowledge that all poles of a stable and causal analog filter should be on the left half of the \( s\)-plane. As a result, \( H_a(s) \) is:

\[ H_a(s) = \frac{\Omega_c^N}{\prod_{\Re\{c_k\}<0} (s - c_k)} \quad (11.12) \]
Example 11.1
The magnitude square response of a Butterworth lowpass filter has the form of:

\[ |H_a(j\Omega)|^2 = \frac{1}{1 + 0.000001\Omega^6} \]

Determine the filter transfer function \( H_a(s) \).

Expressing \( |H_a(j\Omega)|^2 \) as:

\[ |H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{10}\right)^{2.3}} \]

From (11.8), \( \Omega_c = 10 \) and \( N = 3 \)
From (11.11):

\[ c_k = 10e^{jk\pi/3}, \quad k = 2, 3, 4 \]

Finally, we apply (11.12) to obtain:

\[
H_a(s) = \frac{\Omega_c^3}{(s - c_2)(s - c_3)(s - c_4)} \cdot \frac{1000}{(s - 10e^{j2\pi/3})(s + 10)(s - 10e^{j4\pi/3})} \cdot \frac{1000}{(s + 10)(s^2 + 10s + 100)}
\]
To find $\Omega_c$ and $N$ given the passband and stopband requirements in terms of $\Omega_p$, $\Omega_s$, $R_p$ and $A_s$, we exploit (11.4)-(11.5) together with (11.6)-(11.7) to obtain

\[ -10 \log_{10} \left( \frac{1}{1 + \left( \frac{\Omega_p}{\Omega_c} \right)^{2N}} \right) = R_p \quad (11.13) \]

and

\[ -10 \log_{10} \left( \frac{1}{1 + \left( \frac{\Omega_s}{\Omega_c} \right)^{2N}} \right) = A_s \quad (11.14) \]
Solving (11.13)-(11.14) and noting that $N$ should be an integer, we get

$$N = \left\lceil \frac{\log_{10} \left( \frac{10^{R_p/10} - 1}{10^{A_s/10} - 1} \right)}{2 \log_{10}(\Omega_p/\Omega_s)} \right\rceil$$  \hspace{1cm} (11.15)$$

where $\lceil u \rceil$ rounds up $u$ to the nearest integer.

The $\Omega_c$ is then obtained from (11.13) or (11.14) so that the specification can be exactly met at $\Omega_p$ or $\Omega_s$, respectively.

From (11.13), $\Omega_c$ is computed as:

$$\Omega_c = \frac{\Omega_p}{\left(10^{R_p/10} - 1\right)^{1/(2N)}}$$  \hspace{1cm} (11.16)$$
From (11.14), $\Omega_c$ is computed as:

$$\Omega_c = \frac{\Omega_s}{(10^{A_s/10} - 1)^{1/(2N)}} \quad (11.17)$$

As a result, the admissible range of $\Omega_c$ is:

$$\Omega_c \in \left[ \frac{\Omega_p}{(10^{R_p/10} - 1)^{1/(2N)}}, \frac{\Omega_s}{(10^{A_s/10} - 1)^{1/(2N)}} \right] \quad (11.18)$$

**Example 11.2**
Determine the transfer function of a Butterworth lowpass filter whose magnitude requirements are $\Omega_p = 4\pi$ rad$^{-1}$, $\Omega_s = 6\pi$ rad$^{-1}$, $R_p = 8$ dB and $A_s = 16$ dB.
Employing (11.15) yields:

\[
N = \left[ \log_{10} \left( \frac{(10^{8/10} - 1) / (10^{16/10} - 1)}{2 \log_{10}(4\pi/(6\pi))} \right) \right] = [2.45] = 3
\]

Putting \( N = 3 \) in (11.18), the cutoff frequency is:

\[
\Omega_c \in \left[ \frac{4\pi}{(10^{8/10} - 1)^{1/(2.3)}}, \frac{6\pi}{(10^{16/10} - 1)^{1/(2.3)}} \right] = [9.5141, 10.2441]
\]

For simplicity, we select \( \Omega_c = 10 \). Using Example 11.1, the filter transfer function \( H_a(s) \) is:

\[
H_a(s) = \frac{1000}{(s + 10)(s^2 + 10s + 100)} = \frac{1000}{s^3 + 20s^2 + 200s + 1000}
\]
Fig. 11.5: Magnitude square response of Butterworth lowpass filter
The MATLAB program is provided as `ex11_2.m` where the command `freqs`, which is analogous to `freqz`, is used to plot $|H_a(j\Omega)|^2$

**Analog-to-Digital Filter Transformation**

Typical methods include impulse invariance, bilinear transformation, backward difference approximation and matched-$z$ transformation

Their common feature is that a stable analog filter will transform to a stable system with transfer function $H_{LP}(z)$.

Left half of $s$-plane maps into inside of unit circle in $z$-plane

Each has its pros and cons and thus optimal transformation does not exist
**Impulse Invariance**

The idea is simply to sample impulse response of the analog filter $h_a(t)$ to obtain the digital lowpass filter impulse response $h_{LP}[n]$

The relationship between $h_{LP}[n]$ and $h_a(t)$ is

$$h_{LP}[n] = T \cdot h_a(nT), \quad n = \cdots -1, 0, 1, 2, \cdots$$

(11.19)

where $T$ is the sampling interval

**Why there is a scaling of $T$?**
With the use of (4.5) and (5.3)-(5.4), \( H_{LP}(e^{j\omega}) \) is:

\[
H_{LP}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a \left( j \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right) \tag{11.20}
\]

where the analog and digital frequencies are related as:

\[
\omega = \Omega T \tag{11.21}
\]

The impulse response of the resultant IIR filter is similar to that of the analog filter.

Aliasing due to the overlapping of \( \{H_a(j(\omega - 2\pi k)/T)\} \) which are not bandlimited. However, \( H_a(j\Omega) \) corresponds to a lowpass filter and thus aliasing effect is negligibly small particularly when \( T \) is chosen sufficiently small.
To derive the IIR filter transfer function $H_{LP}(z)$ from $H_a(s)$, we first obtain the partial fraction expansion:

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \quad (11.22)$$

where $\{s_k\}$ are the poles on the left half of the $s$-plane.

The inverse Laplace transform of (11.22) is given as:

$$h_a(t) = \begin{cases} 
\sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\
0, & t < 0 
\end{cases} \quad (11.23)$$
Substituting (11.23) into (11.19), we have:

\[ h_{LP}[n] = \sum_{k=1}^{N} TA_k e^{s_k n T} u[n] \]  \hspace{1cm} (11.24)

The \( z \)-transform of \( h_{LP}[n] \) is:

\[ H_{LP}(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{s_k T} z^{-1}} \]  \hspace{1cm} (11.25)

Comparing (11.22) and (11.25), it is seen that a pole of \( s = s_k \) in the \( s \)-plane transforms to a pole at \( z = e^{s_k T} \) in the \( z \)-plane:

\[ z = e^{sT} \]  \hspace{1cm} (11.26)
Expressing $s = \sigma + j\Omega$:

$$z = e^{\sigma T} \cdot e^{j\Omega T} = e^{\sigma T} \cdot e^{j(\Omega + 2\pi k/T)T} \quad (11.27)$$

where $k$ is any integer, indicating a many-to-one mapping.

Each infinite horizontal strip of width $2\pi/T$ maps into the entire $z$-plane.

$\sigma = 0$ maps to $|z| = 1$, that is, $j\Omega$ axis in the $s$-plane transforms to the unit circle in the $z$-plane.

$\sigma < 0$ maps to $|z| < 1$, stable $H_a(s)$ produces stable $H_{LP}(z)$.

$\sigma > 0$ maps to $|z| > 1$, right half of the $s$-plane maps into the outside of the unit circle in the $z$-plane.
Fig.11.6: Mapping between $s$ and $z$ in impulse invariance method
Given the magnitude square response specifications of $H_{LP}(e^{j\omega})$ in terms of $\omega_p$, $\omega_s$, $R_p$ and $A_s$, the design procedure for $H_{LP}(z)$ based on the impulse invariance method is summarized as the following steps:

(i) Select a value for the sampling interval $T$ and then compute the passband and stopband frequencies for the analog lowpass filter according to $\Omega_p = \omega_p/T$ and $\Omega_s = \omega_s/T$

(ii) Design the analog Butterworth filter with transfer function $H_a(s)$ according to $\Omega_p$, $\Omega_s$, $R_p$ and $A_s$

(iii) Perform partial fraction expansion on $H_a(s)$ as in (11.22)

(iv) Obtain $H_{LP}(z)$ using (11.25)
Example 11.3
The transfer function of an analog filter has the form of

\[ H_a(s) = \frac{2s}{s^2 + 6s + 8} \]

Use impulse invariance method with sampling interval \( T = 1 \) to transform \( H_a(s) \) to a digital filter transfer function \( H(z) \).

Performing partial fraction expansion on \( H_a(s) \):

\[ H_a(s) = \frac{-2}{s + 2} + \frac{4}{s + 4} \]

Applying (11.25) with \( T = 1 \) yields

\[ H(z) = \frac{-2}{1 - e^{-2}z^{-1}} + \frac{4}{1 - e^{-4}z^{-1}} = \frac{2 - 0.5047z^{-1}}{1 - 0.1537z^{-1} + 0.0025z^{-2}} \]
Example 11.4
Determine the transfer function $H_{LP}(z)$ of a digital lowpass filter whose magnitude requirements are $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $R_p = 8$ dB and $A_s = 16$ dB. Use the Butterworth lowpass filter and impulse invariance method in the design.

Selecting the sampling interval as $T = 0.1$, the analog frequency parameters are computed as:

$$\Omega_p = \frac{\omega_p}{T} = 4\pi$$

and

$$\Omega_s = \frac{\omega_s}{T} = 6\pi$$
Using Example 11.2, a Butterworth filter which meets the magnitude requirements are:

\[ H_a(s) = \frac{1000}{(s + 10)(s^2 + 10s + 100)} = \frac{1000}{s^3 + 20s^2 + 200s + 1000} \]

Performing partial fraction expansion on \( H_a(s) \) with the use of the MATLAB command `residue`, we get

\[ H_a(s) = \frac{10}{s + 10} + \frac{-5 - j2.8868}{s + 5 - j8.6603} + \frac{-5 + j2.8868}{s + 5 + j8.6603} \]

Applying (11.25) with \( T = 0.1 \) yields
\[
H_{LP}(z) = \frac{0.1 \cdot 10}{1 - e^{-10 \cdot 0.1} z^{-1}} + \frac{0.1 \cdot (-5 - j2.8868)}{1 - e^{(-5+j8.6603) \cdot 0.1} z^{-1}} + \frac{0.1 \cdot (-5 + j2.8868)}{1 - e^{(-5-j8.6603) \cdot 0.1} z^{-1}}
\]
\[
= \frac{1}{1 - 0.3679 z^{-1}} + \frac{-0.5 - j0.2887}{1 - (0.3929 + j0.4620) z^{-1}} + \frac{-0.5 + j0.2887}{1 - (0.3929 - j0.4620) z^{-1}}
\]
\[
= \frac{1}{1 - 0.3679 z^{-1}} + \frac{-1 + 0.6597 z^{-1}}{1 - 0.7859 z^{-1} + 0.3679 z^{-2}} + \frac{0.2417 z^{-1} + 0.1262 z^{-2}}{1 - 1.1538 z^{-1} + 0.6570 z^{-2} - 0.1354 z^{-3}}
\]

The MATLAB program is provided as `ex11_4.m`. 
Fig. 11.7: Magnitude and phase responses based on impulse invariance
• Bilinear Transformation

It is a conformal mapping that maps the $j\Omega$ axis of the $s$-plane into the unit circle of the $z$-plane only once, implying there is no aliasing problem as in the impulse invariance method.

It is a one-to-one mapping.

The relationship between $s$ and $z$ is:

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \iff z = \frac{1 + sT/2}{1 - sT/2}$$

(11.28)
Employing \( s = \sigma + j\Omega \), \( z \) can be expressed as:

\[
\begin{aligned}
z &= \frac{(1 + \sigma T/2) + j\Omega T/2}{(1 - \sigma T/2) - j\Omega T/2}
\end{aligned}
\]  

(11.29)

\( \sigma = 0 \) maps to \( |z| = 1 \), that is, \( j\Omega \) axis in the \( s \)-plane transforms to the unit circle in the \( z \)-plane.

\( \sigma < 0 \) maps to \( |z| < 1 \), stable \( H_a(s) \) produces a stable \( H_{LP}(z) \).

\( \sigma > 0 \) maps to \( |z| > 1 \), right half of the \( s \)-plane maps into the outside of the unit circle in the \( z \)-plane.
Fig. 11.8: Mapping between $s$ and $z$ in bilinear transformation
Although aliasing is avoided, the drawback of the bilinear transformation is that there is no linear relationship between $\omega$ and $\Omega$.

Putting $z = e^{j\omega}$ and $s = j\Omega$ in (11.28), $\omega$ and $\Omega$ are related as:

$$j\Omega = \frac{2}{T} \cdot \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{2}{T} \cdot \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}} = j\frac{2}{T} \tan \left(\frac{\omega}{2}\right)$$

$$\Rightarrow \Omega = \frac{2}{T} \tan \left(\frac{\omega}{2}\right) \Leftrightarrow \omega = 2\tan^{-1} \left(\frac{\Omega T}{2}\right)$$ (11.30)

Given the magnitude square response specifications of $H_{LP}(e^{j\omega})$ in terms of $\omega_p$, $\omega_s$, $R_p$ and $A_s$, the design procedure for $H_{LP}(z)$ based on the bilinear transformation is summarized as the following steps:
(i) Select a value for $T$ and then compute the passband and stopband frequencies for the analog lowpass filter according $\Omega_p = (2/T) \tan(\omega_p/2)$ and $\Omega_s = (2/T) \tan(\omega_s/2)$

(ii) Design the analog Butterworth filter with transfer function $H_a(s)$ according to $\Omega_p$, $\Omega_s$, $R_p$ and $A_s$.

(iii) Obtain $H_{LP}(z)$ from $H_a(s)$ using the substitution of (11.28).

**Example 11.5**
The transfer function of an analog filter has the form of

$$H_a(s) = \frac{2s}{s^2 + 6s + 8}$$

Use the bilinear transformation with $T = 1$ to transform $H_a(s)$ to a digital filter with transfer function $H(z)$. 
Applying (11.28) with $T = 1$ yields

$$H(z) = \frac{2 \cdot 2 \frac{1 - z^{-1}}{1 + z^{-1}}}{\left(2 \frac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + 6 \cdot 2 \frac{1 - z^{-1}}{1 + z^{-1}} + 8} = \frac{4 - 4z^{-2}}{15 + 14z^{-1} + 9z^{-2}}$$

Example 11.6
Determine the transfer function $H_{LP}(z)$ of a digital lowpass filter whose magnitude requirements are $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $R_p = 8$ dB and $A_s = 16$ dB. Use the Butterworth lowpass filter and bilinear transformation in the design.

Selecting $T = 0.1$, the analog frequency parameters are computed according to (11.30) as:
\[
\Omega_p = \frac{2}{T} \tan \left( \frac{\omega_p}{2} \right) = 14.5309
\]

and

\[
\Omega_s = \frac{2}{T} \tan \left( \frac{\omega_s}{2} \right) = 27.5276
\]

Employing (11.15) yields:

\[
N = \left\lfloor \frac{\log_{10} \left[ \frac{(10^{8/10} - 1)}{(10^{16/10} - 1)} \right]}{2 \log_{10}(14.5309/27.5276)} \right\rfloor = [1.56] = 2
\]

Putting \( N = 2 \) in (11.18), the cutoff frequency is:

\[
\Omega_c \in \left[ \frac{14.5309}{(10^{8/10} - 1)^{1/(2.2)}}, \frac{27.5276}{(10^{16/10} - 1)^{1/(2.2)}} \right] = [9.5725, 11.0289]
\]
For simplicity, $\Omega_c = 10$ is employed.

Following (11.11)-(11.12):

$$H_a(s) = \frac{100}{s^2 + 14.1421s + 100}$$

Finally, we use (11.28) with $T = 0.1$ to yield

$$H_{LP}(z) = \frac{100}{\left( \frac{2}{0.1} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \right)^2 + 14.1421 \cdot \frac{2}{0.1} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} + 100 \cdot \frac{1 + 2z^{-1} + z^{-2}}{7.8284 - 6z^{-1} + 2.1716z^{-2}}}$$

The MATLAB program is provided as ex11_6.m.
Fig.11.9: Magnitude and phase responses based on bilinear transformation
Frequency Band Transformation

The operations are similar to that of the bilinear transformation but now the mapping is performed only in the $z$-plane:

$$z_o^{-1} = T(z^{-1}) \quad (11.31)$$

where $z_o$ and $z$ correspond to the lowpass and resultant filters, respectively, and $T$ denotes the transformation operator.

To ensure the transformed filter to be stable and causal, the unit circle and inside of the $z_o$-plane should map into those of the $z$-plane, respectively.
<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Transformation Operator</th>
<th>Design Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowpass</td>
<td>$z_o^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$</td>
<td>$\alpha = \sin \left( \frac{\omega_{c_0} - \omega_c}{2} \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\beta = \cot \left( \frac{\omega_{c_2} - \omega_{c_1}}{2} \right) \tan \left( \frac{\omega_{c_0}}{2} \right)$</td>
</tr>
<tr>
<td>Highpass</td>
<td>$z_o^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$</td>
<td>$\alpha = \cos \left( \frac{\omega_{c_0} + \omega_c}{2} \right)$</td>
</tr>
<tr>
<td>Bandpass</td>
<td>$z_o^{-1} = \frac{z^{-2} - \frac{2\alpha\beta}{\beta + 1} z^{-1} + \frac{\beta - 1}{\beta + 1}}{\beta - 1 z^{-2} - \frac{2\alpha\beta}{\beta + 1} z^{-1} + 1}$</td>
<td>$\beta = \cot \left( \frac{\omega_{c_2} - \omega_{c_1}}{2} \right)$</td>
</tr>
</tbody>
</table>
Example 11.7
Determine the transfer function $H(z)$ of a digital highpass filter whose magnitude requirements are $\omega_p = 0.6\pi$, $\omega_s = 0.4\pi$, $R_p = 8$ dB and $A_s = 16$ dB. Use the Butterworth lowpass filter and bilinear transformation in the design.

Using Example 11.6, the corresponding lowpass filter transfer function $H_{LP}(z_o)$ is:
Assigning the cutoff frequencies as the midpoints between the passband and stopband frequencies, we have

\[
\omega_{c_o} = \omega_c = \frac{0.4\pi + 0.6\pi}{2} = 0.5\pi
\]

With the use of Table 11.1, the corresponding value of \( \alpha \) is:

\[
\alpha = -\frac{\cos \left( \frac{\omega_{c_o} + \omega_c}{2} \right)}{\cos \left( \frac{\omega_{c_o} - \omega_c}{2} \right)} = -\frac{\cos(0.5\pi)}{\cos(0)} = 0
\]
which gives the transformation operator:

\[
\tilde{z}_o^{-1} = -\frac{z^{-1} + 0}{1 + 0 \cdot z^{-1}} = -z^{-1}
\]

As a result, the digital highpass filter transfer function is:

\[
H(z) = H_{LP}(z_o)\big|_{z_o^{-1} = -z^{-1}} = \frac{1 - 2z^{-1} + z^{-2}}{7.8284 + 6z^{-1} + 2.1716z^{-2}}
\]

The MATLAB program is provided as `ex11_7.m`. 
Fig. 11.10: Magnitude and phase responses based on frequency band transformation