Discrete-Time Fourier Transform (DTFT)

Chapter Intended Learning Outcomes:

(i) Understanding the characteristics and properties of DTFT

(ii) Ability to perform discrete-time signal conversion between the time and frequency domains using DTFT and inverse DTFT

Definition

DTFT is a frequency analysis tool for aperiodic discrete-time signals

The DTFT of $x[n]$, $X(e^{j\omega})$, has been derived in (5.4):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{6.1}$$

The derivation is based on taking the Fourier transform of $x_s(t)$ of (5.2)

As in Fourier transform, $X(e^{j\omega})$ is also called spectrum and is a continuous function of the frequency parameter ω

Is DTFT complex? Is it periodic?

To convert $X(e^{j\omega})$ to $x[n]$, we use inverse DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \tag{6.2}$$

which can be proved by putting (6.1) into LHS of (6.2):

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right]e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\int_{-\pi}^{\pi} e^{-j\omega m}e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\frac{2\sin((n-m)\pi)}{n-m} \\
&= x[n] \qquad (6.3)
\end{aligned}$$

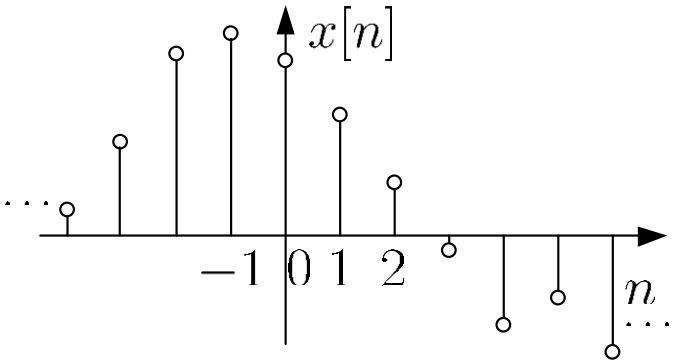
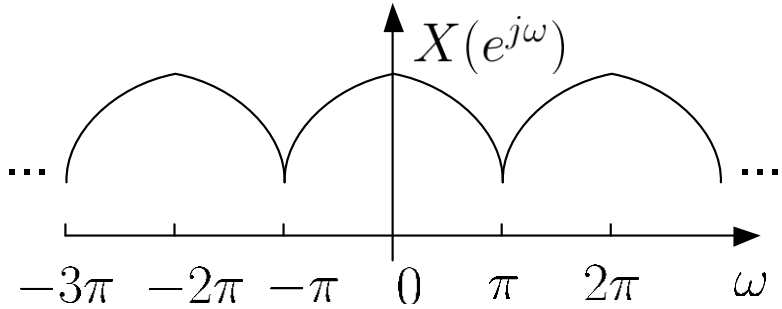
time domain	frequency domain
	
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \Longrightarrow$	$\Longleftarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
discrete and aperiodic	continuous and periodic

Fig.6.1: Illustration of DTFT

$X(e^{j\omega})$ is continuous and periodic with a period of 2π

$X(e^{j\omega})$ is generally complex, we can illustrate $X(e^{j\omega})$ using the magnitude and phase spectra, i.e., $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$:

$$|X(e^{j\omega})| = \sqrt{(\Re\{X(e^{j\omega})\})^2 + (\Im\{X(e^{j\omega})\})^2} \tag{6.4}$$

and

$$\angle(X(e^{j\omega})) = \tan^{-1}\left(\frac{\Im\{X(e^{j\omega})\}}{\Re\{X(e^{j\omega})\}}\right) \tag{6.5}$$

where both are continuous in frequency and periodic.

Convergence of DTFT

The DTFT of a sequence $x[n]$ converges if

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]| \cdot \left|e^{-j\omega n}\right| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad (6.6)$$

Recall (5.10) and assume the z transform of $x[n]$ converges for region of convergence (ROC) of $R_+ < |z| < R_-$:

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} \left|x[n]z^{-n}\right| < \infty, \quad R_+ < |z| < R_- \quad (6.7)$$

When ROC includes the unit circle:

$$\left.\sum_{n=-\infty}^{\infty} \left|x[n]z^{-n}\right|\right|_{|z|=1} = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad (6.8)$$

which leads to the convergence condition for $X(e^{j\omega})$. This also proves the P2 property of the z transform.

Let $h[n]$ be the impulse response of a linear time-invariant (LTI) system, the following three statements are equivalent:

S1. ROC for the z transform of $h[n]$ includes unit circle

S2. The system is stable so that $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

S3. The DTFT of $h[n]$, i.e., $H(e^{j\omega})$, converges

Note that $H(e^{j\omega})$ is also known as system frequency response

<u>Example 6.1</u>
Determine the DTFT of $x[n] = u[n]$.

Using (6.1), the DTFT of $x[n]$ is computed as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n}$$

Since

$$\sum_{n=0}^{\infty} \left|e^{-j\omega n}\right| = \sum_{n=0}^{\infty} 1 = \infty$$

$X(e^{j\omega})$ does not exist.

Alternatively, employing the stability condition:

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} 1 = \infty$$

which also indicates that the DTFT does not converge

Furthermore, the z transform of $x[n]$ is:

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1$$

Because $|z| > 1$ does not include the unit circle, there is no DTFT for $x[n]$.

<u>Example 6.2</u>

Find the DTFT of $x[n] = u[n] - u[n-N]$. Plot the magnitude and phase spectra for $N = 10$.

Using (6.1), we have

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{e^{-j\omega \cdot 0}\left(1-e^{-j\omega N}\right)}{1-e^{-j\omega}} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

Alternatively, we can first use z transform because

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}$$

The z transform of $x[n]$ is evaluated as

$$X(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}}, \quad |z| > 0$$

As the ROC includes the unit circle, its DTFT exists and the same result is obtained by the substitution of $z = e^{j\omega}$.

There are two advantages of z transform over DTFT:

- z transform is a generalization of DTFT and it encompasses a broader class of signals since DTFT does not converge for all sequences
- notation convenience of writing z instead of $e^{j\omega}$.

To plot the magnitude and phase spectra, we express $X(e^{j\omega})$:

$$X(e^{j\omega}) = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \cdot \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{j\omega/2}} = e^{-j\omega(N-1)/2} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

In doing so, $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$ can be written in closed-forms as:

$$|X(e^{j\omega})| = \left|\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right|$$

and

$$\angle(X(e^{j\omega})) = -\frac{\omega(N-1)}{2} + \angle\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)$$

Note that we generally employ (6.4) and (6.5) for magnitude and phase computation

In using MATLAB to plot $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$, we utilize the command `sinc` so that there is no need to separately handle the "0/0" cases due to the sine functions

Recall the definition of sinc function:

$$\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

As a result, we have:

$$\begin{aligned}\frac{\sin(\omega N/2)}{\sin(\omega/2)} &= \frac{\sin(\omega \cdot N\pi/(2\pi))}{\omega \cdot N\pi/(2\pi)} \cdot \frac{\omega \cdot N\pi}{2\pi} \cdot \frac{\omega\pi/(2\pi)}{\sin(\omega\pi/(2\pi))} \cdot \frac{2\pi}{\omega\pi} \\ &= N \cdot \frac{\mathrm{sinc}(\omega N/(2\pi))}{\mathrm{sinc}(\omega/(2\pi))}\end{aligned}$$

The key MATLAB code for $N = 10$ is

```
N=10;                    %N=10
w=0:0.01*pi:2*pi;        %successive frequency point
                         %separation is 0.01pi
dtft=N.*sinc(w.*N./2./pi)./(sinc(w./2./pi)).*exp(-
j.*w.*(N-1)./2);         %define DTFT function
subplot(2,1,1)
Mag=abs(dtft);           %compute magnitude
plot(w./pi,Mag);         %plot magnitude
subplot(2,1,2)
Pha=angle(dtft);         %compute phase
plot(w./pi,Pha);         %plot phase
```

Analogous to Example 4.4, there are 201 uniformly-spaced points to approximate the continuous functions $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$.

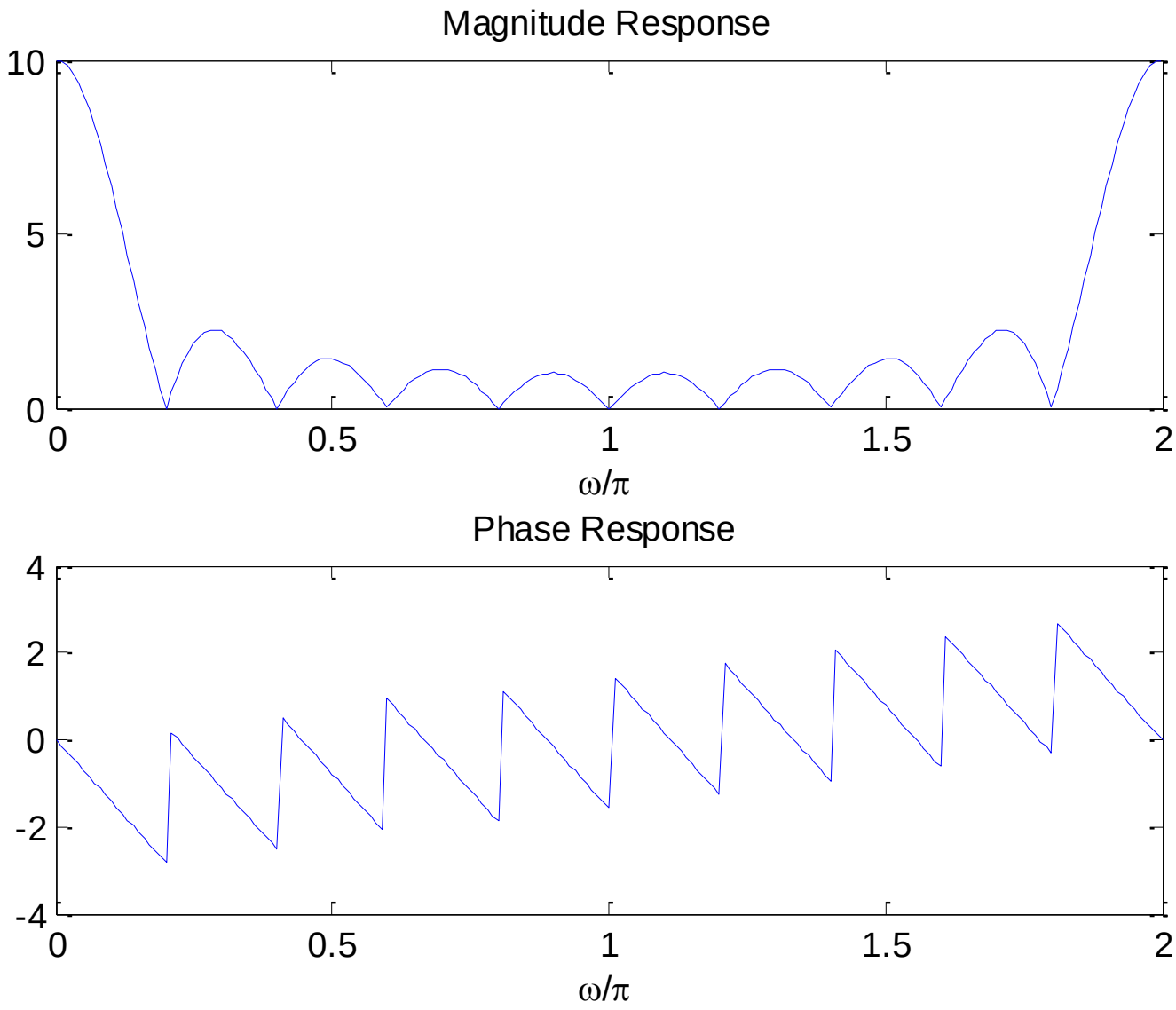


Fig.6.2: DTFT plots using `abs` and `angle`

Alternatively, we can use the command `freqz`:

$$X(z) = \frac{1 - z^{-N}}{1 - z^{-1}}$$

which is ratio of two polynomials in z^{-1}

The corresponding MATLAB code is:

```
N=10;                        %N=10
a=[1,-1];                    %vector for denominator
b=[1,zeros(1,N-1),-1];       %vector for numerator
freqz(b,a)                   %plot magnitude & phase (dB)
```

Note that it is also possible to use $X(z) = \sum_{n=0}^{N-1} z^{-n}$ and in this case we have `b=ones(N,1)` and `a=1`.

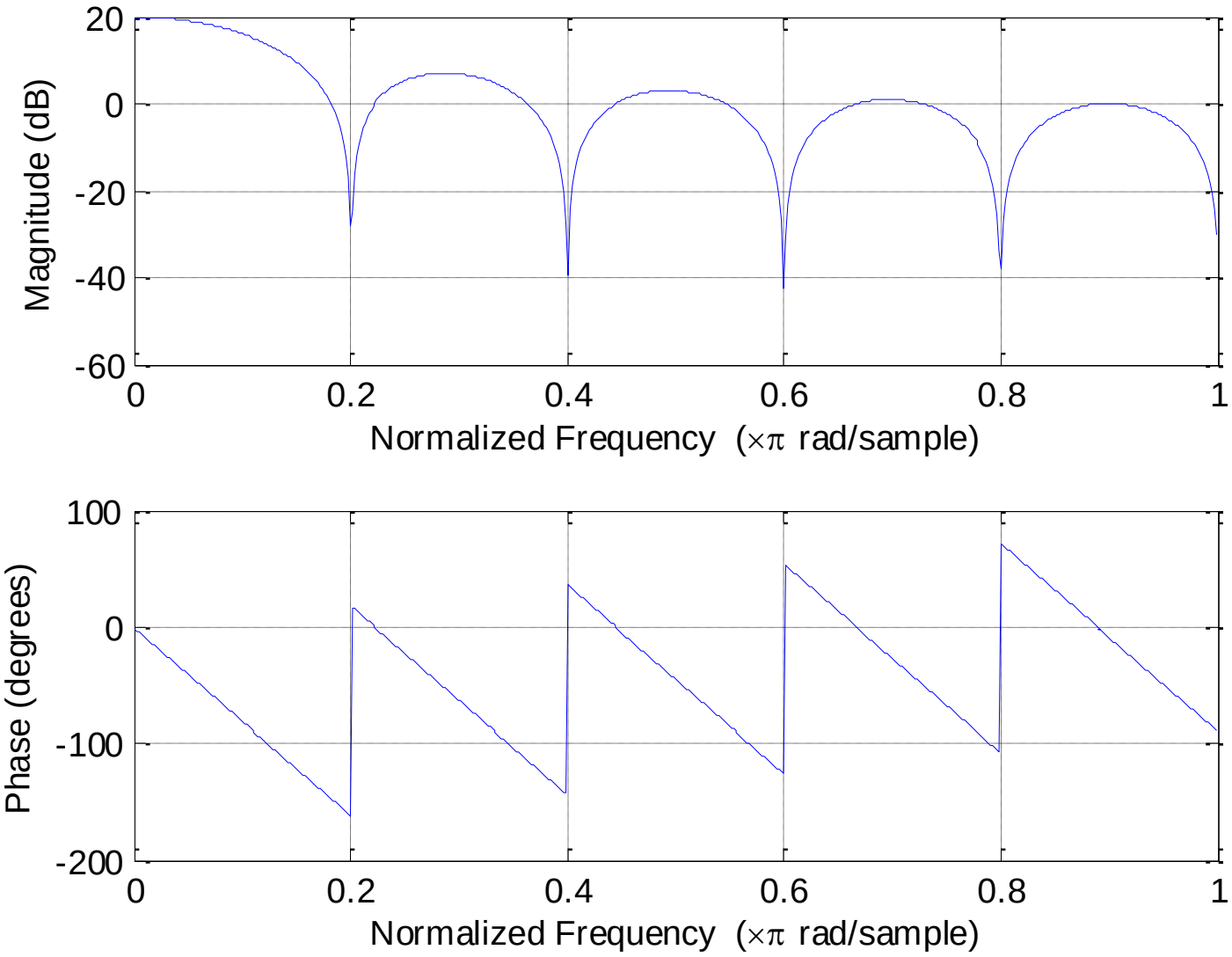


Fig.6.3: DTFT plots using `freqz`

The results in Figs. 6.2 and 6.3 are identical, although their presentations are different:

- $|X(e^{j\omega})| = 10$ at $\omega = 0$ in Fig. 6.2 while that of Fig. 6.3 is 20 dB. It is easy to verify that 10 corresponds to $20\log_{10} 10 = 20$ dB

- units of phase spectra in Figs. 6.2 and 6.3 are radian and degree, respectively. To make the phase values in both plots identical, we also need to take care of the 2π phase ambiguity.

The MATLAB programs for this example are provided as `ex6_2.m` and `ex6_2_2.m`.

Example 6.3

Find the inverse DTFT of $X(e^{j\omega})$ which is a rectangular pulse within $\omega \in [-\pi, \pi]$:

$$X(e^{j\omega}) = \begin{cases} 1, & -w_0 < \omega < w_0 \\ 0, & \text{otherwise} \end{cases}$$

where $0 < w_0 < \pi$.

Using (6.2), we get:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-w_0}^{w_0} e^{j\omega n} d\omega = \frac{\sin(w_0 n)}{\pi n} = \frac{w_0}{\pi} \text{sinc}\left(\frac{w_0 n}{\pi}\right)$$

That is, $x[n]$ is an infinite-duration sequence whose values are drawn from a scaled sinc function.

Example 6.4

Determine the inverse DTFT of $X(e^{j\omega})$ which has the form of:

$$X(e^{j\omega}) = \frac{e^{-j2\omega}}{1 + 0.7e^{-j\omega}}$$

With the use of $z = e^{j\omega}$, the corresponding z transform is

$$X(z) = \frac{z^{-2}}{1 + 0.7z^{-1}}, \quad |z| > 0.7$$

Note that ROC should include the unit circle as DTFT exists

Employing the time shifting property, we get

$$x[n] = (-0.7)^{n-2}u[n-2]$$

Properties of DTFT

Since DTFT is closely related to z transform, its properties follow those of z transform. Note that ROC is not involved because it should include unit circle in order for DTFT exists

1. Linearity

If $(x_1[n], X_1(e^{j\omega}))$ and $(x_2[n], X_2(e^{j\omega}))$ are two DTFT pairs, then:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega}) \tag{6.9}$$

2. Time Shifting

A shift of n_0 in $x[n]$ causes a multiplication of $e^{-j\omega n_0}$ in $X(e^{j\omega})$:

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega}) \tag{6.10}$$

3. Multiplication by an Exponential Sequence

Multiplying $x[n]$ by $e^{j\omega_0 n}$ in time domain corresponds to a shift of ω_0 in the frequency domain:

$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)}) \tag{6.11}$$

which agrees with (5.33) by putting $z = e^{j\omega}$ and $z_0 = e^{j\omega_0}$

4. Differentiation

Differentiating $X(e^{j\omega})$ with respect to ω corresponds to multiplying $x[n]$ by n:

$$nx[n] \leftrightarrow -j\frac{dX(e^{j\omega})}{d\omega} \tag{6.12}$$

Note the RHS can be obtained from (5.34) by putting $z = e^{j\omega}$:

$$-e^{j\omega}\frac{dX(e^{j\omega})}{de^{j\omega}} = -e^{j\omega}\frac{dX(e^{j\omega})}{d\omega}\cdot\left(\frac{de^{j\omega}}{d\omega}\right)^{-1} = -j\frac{dX(e^{j\omega})}{d\omega} \quad (6.13)$$

5. Conjugation

The DTFT pair for $x^*[n]$ is given as:

$$x^*[n] \leftrightarrow X^*(e^{-j\omega}) \quad (6.14)$$

6. Time Reversal

The DTFT pair for $x[-n]$ is given as:

$$x[-n] \leftrightarrow X(e^{-j\omega}) \quad (6.15)$$

7. Convolution

If $(x_1[n], X_1(e^{j\omega}))$ and $(x_2[n], X_2(e^{j\omega}))$ are two DTFT pairs, then:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1(e^{j\omega})X_2(e^{j\omega}) \tag{6.16}$$

In particular, for a LTI system with input $x[n]$, output $y[n]$ and impulse response $h[n]$, we have:

$$y[n] = x[n] \otimes h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{6.17}$$

which is analogous to (2.24) for continuous-time LTI systems

8. Multiplication

Multiplication in the time domain corresponds to convolution in the frequency domain:

$$x_1[n] \cdot x_2[n] \leftrightarrow X_1(e^{j\omega}) \tilde{\otimes} X_2(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\tau}) X_2(e^{j(\omega-\tau)}) d\tau \qquad (6.18)$$

where $\tilde{\otimes}$ denotes convolution within one period

9. Parseval's Relation

The Parseval's relation addresses the energy of a sequence:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega \qquad (6.19)$$

With the use of (6.2), the proof is:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]|^2 &= \sum_{n=-\infty}^{\infty} x[n]x^*[n] \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \right)^* \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right) d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega \qquad (6.20)
\end{aligned}$$