Discrete Fourier Series & Discrete Fourier Transform

Chapter Intended Learning Outcomes

(i) Understanding the relationships between the z transform, discrete-time Fourier transform (DTFT), discrete Fourier series (DFS), discrete Fourier transform (DFT) and fast Fourier transform (FFT)

(ii) Understanding the characteristics and properties of DFS and DFT

(iii) Ability to perform discrete-time signal conversion between the time and frequency domains using DFS and DFT and their inverse transforms

Discrete Fourier Series

DTFT may not be practical for analyzing $x[n]$ because $X(e^{j\omega})$ is a function of the continuous frequency variable ω and we cannot use a digital computer to calculate a continuum of functional values

DFS is a frequency analysis tool for periodic infinite-duration discrete-time signals which is practical because it is discrete in frequency

The DFS is derived from the Fourier series as follows.

Let $\tilde{x}[n]$ be a periodic sequence with fundamental period N where N is a positive integer. Analogous to (2.2), we have:

$$\tilde{x}[n] = \tilde{x}[n + rN] \tag{7.1}$$

for any integer value of r.

Let $x(t)$ be the continuous-time counterpart of $\tilde{x}[n]$. According to Fourier series expansion, $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T_p}} \tag{7.2}$$

which has frequency components at $\Omega = 0, \pm\Omega_0, \pm 2\Omega_0, \cdots$. Substituting $x(t) = \tilde{x}[n]$, $T_p = N$ and $t = n$:

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kn}{N}} \tag{7.3}$$

Note that (7.3) is valid for discrete-time signals as only the sample points of $x(t)$ are considered.

It is seen that $\tilde{x}[n]$ has frequency components at $\omega = 0, \pm 2\pi/N, \pm(2\pi/N)(2), \cdots$, and the respective complex exponentials are $e^{j(2\pi/N(0))}, e^{\pm j(2\pi/N(1))}, e^{\pm j(2\pi/N(2))}, \cdots$.

Nevertheless, there are only N distinct frequencies in $\tilde{x}[n]$ due to the periodicity of $e^{j2\pi k/N}$.

Without loss of generality, we select the following N distinct complex exponentials, $e^{j(2\pi/N(0))}, e^{j(2\pi/N(1))}, \cdots, e^{j(2\pi/N(N-1))}$, and thus the infinite summation in (7.3) is reduced to:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{N}} \quad (7.4)$$

Defining $\tilde{X}[k] = Na_k$, $k = 0, 1, \cdots, N-1$, as the DFS coefficients, the inverse DFS formula is given as:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{\frac{j2\pi kn}{N}} \quad (7.5)$$

The formula for converting $\tilde{x}[n]$ to $\tilde{X}[k]$ is derived as follows.

Multiplying both sides of (7.5) by $e^{-j(2\pi/N)rn}$ and summing from $n=0$ to $n=N-1$:

$$\begin{aligned}\sum_{n=0}^{N-1}\tilde{x}[n]e^{\frac{-j2\pi rn}{N}} &= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{\frac{j2\pi kn}{N}}\right)e^{\frac{-j2\pi rn}{N}} \\ &= \sum_{n=0}^{N-1}\frac{1}{N}\left(\sum_{k=0}^{N-1}\tilde{X}[k]e^{\frac{j2\pi(k-r)n}{N}}\right) \\ &= \sum_{k=0}^{N-1}\tilde{X}[k]\left[\frac{1}{N}\sum_{n=0}^{N-1}e^{\frac{j2\pi(k-r)n}{N}}\right] \qquad (7.6)\end{aligned}$$

Using the orthogonality identity of complex exponentials:

$$\frac{1}{N}\sum_{n=0}^{N-1}e^{\frac{j2\pi(k-r)n}{N}} = \begin{cases}1, & k-r=mN, \quad m \quad \text{is} \quad \text{an} \quad \text{integer}\\ 0, & \text{otherwise}\end{cases} \qquad (7.7)$$

(7.6) is reduced to

$$\sum_{n=0}^{N-1} \tilde{x}[n] e^{-\frac{j2\pi rn}{N}} = \tilde{X}[r] \tag{7.8}$$

which is also periodic with period N.

Let

$$W_N = e^{-\frac{j2\pi}{N}} \tag{7.9}$$

The DFS analysis and synthesis pair can be written as:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \tag{7.10}$$

and

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \tag{7.11}$$

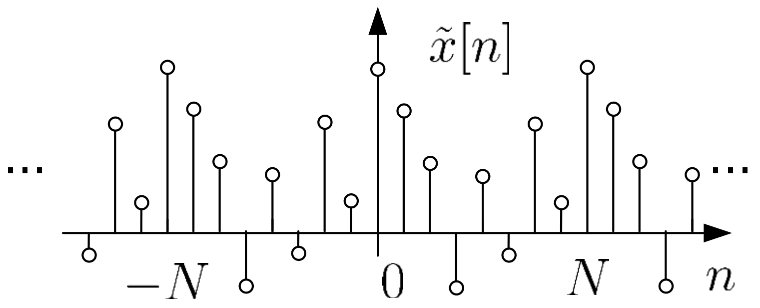
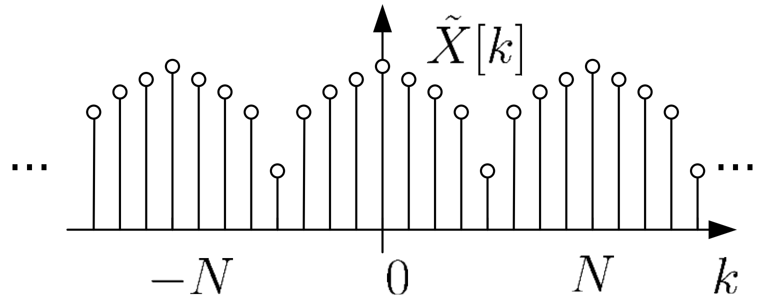
time domain	frequency domain
	
$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \Longrightarrow$	
	$\Longleftarrow \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$
discrete and periodic	discrete and periodic

Fig.7.1: Illustration of DFS

Example 7.1

Find the DFS coefficients of the periodic sequence $\tilde{x}[n]$ with a period of $N=5$. Plot the magnitudes and phases of $\tilde{X}[k]$. Within one period, $\tilde{x}[n]$ has the form of:

$$\tilde{x}[n] = \begin{cases} 1, & n = 0,1,2 \\ 0, & n = 3,4 \end{cases}$$

Using (7.10), we have

$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\
&= W_5^0 + W_5^k + W_5^{2k} \\
&= 1 + e^{-\frac{j2\pi k}{5}} + e^{-\frac{j4\pi k}{5}} \\
&= e^{-\frac{j2\pi k}{5}} \left(e^{\frac{j2\pi k}{5}} + 1 + e^{-\frac{j2\pi k}{5}} \right) \\
&= e^{-\frac{j2\pi k}{5}} \left[1 + 2\cos\left(\frac{2\pi k}{5}\right) \right]
\end{aligned}$$

Similar to Example 6.2, we get:

$$|\tilde{X}[k]| = \left|1 + 2\cos\left(\frac{2\pi k}{5}\right)\right|$$

and

$$\angle(\tilde{X}[k]) = -\frac{2\pi k}{5} + \angle\left(1 + 2\cos\left(\frac{2\pi k}{5}\right)\right)$$

The key MATLAB code for plotting DFS coefficients is

```
N=5;
x=[1 1 1 0 0];
k=-N:2*N;                              %plot for 3 periods
Xm=abs(1+2.*cos(2*pi.*k/N));%magnitude computation
Xa=angle(exp(-2*j*pi.*k/5).*(1+2.*cos(2*pi.*k/N)));
                                       %phase computation
```

The MATLAB program is provided as `ex7_1.m`.

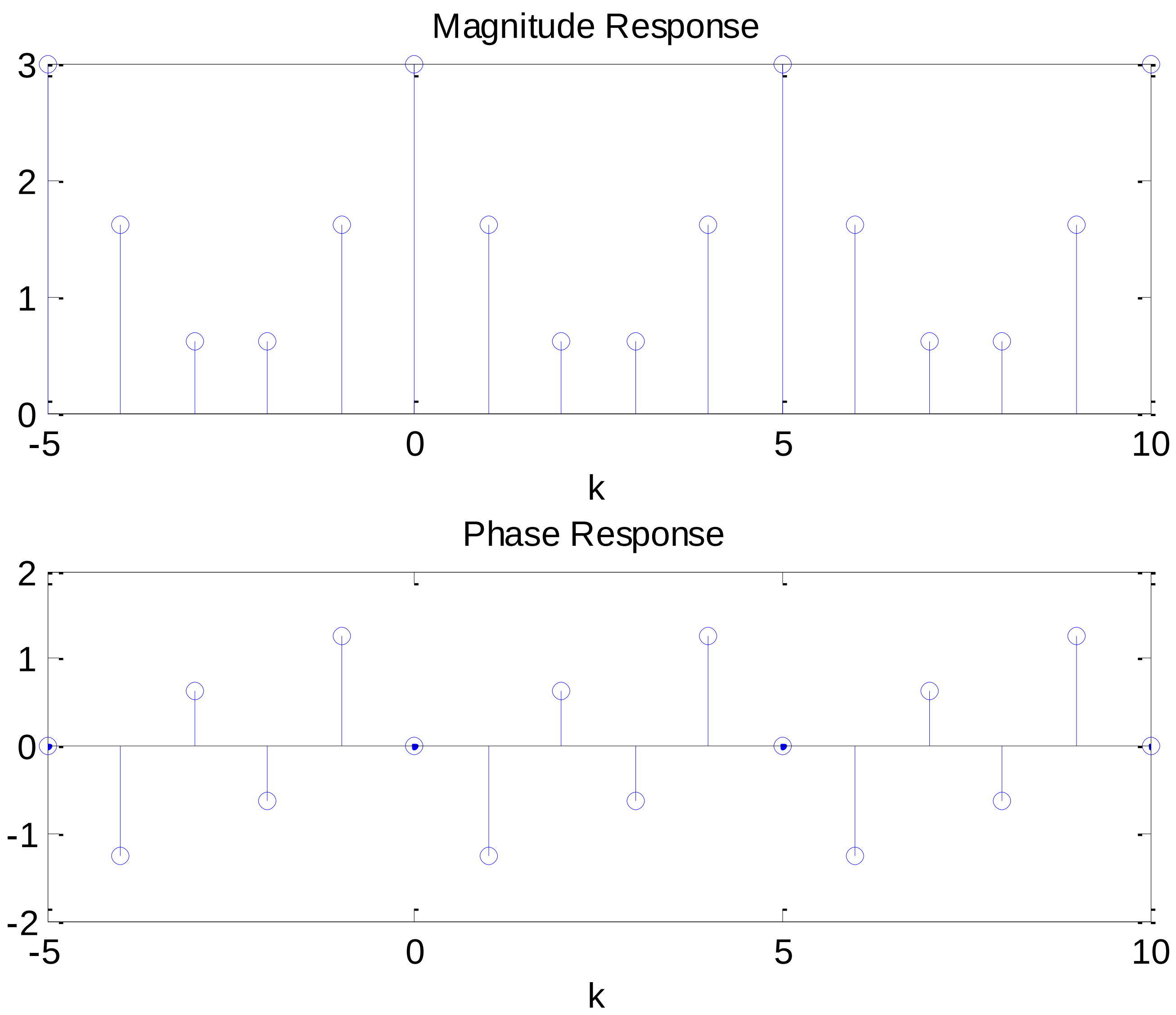


Fig.7.2: DFS plots

Relationship with DTFT

Let $x[n]$ be a finite-duration sequence which is extracted from a periodic sequence $\tilde{x}[n]$ of period N:

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (7.12)$$

Recall (6.1), the DTFT of $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad (7.13)$$

With the use of (7.12), (7.13) becomes

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\omega n} \quad (7.14)$$

Comparing the DFS and DTFT in (7.8) and (7.14), we have:

$$\tilde{X}[k] = X(e^{j\omega})\big|_{\omega=\frac{2\pi k}{N}} \qquad (7.15)$$

That is, $\tilde{X}[k]$ is equal to $X(e^{j\omega})$ sampled at N distinct frequencies between $\omega \in [0, 2\pi]$ with a uniform frequency spacing of $2\pi/N$.

Samples of $X(e^{j\omega})$ or DTFT of a finite-duration sequence $x[n]$ can be computed using the DFS of an infinite-duration periodic sequence $\tilde{x}[n]$, which is a periodic extension of $x[n]$.

Relationship with z Transform

$X(e^{j\omega})$ is also related to z transform of $x[n]$ according to (5.8):

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} \tag{7.16}$$

Combining (7.15) and (7.16), $\tilde{X}[k]$ is related to $X(z)$ as:

$$\tilde{X}[k] = X(z)\big|_{z=e^{\frac{j2\pi k}{N}}} = X(e^{\frac{j2\pi k}{N}}) \tag{7.17}$$

That is, $\tilde{X}[k]$ is equal to $X(z)$ evaluated at N equally-spaced points on the unit circle, namely, $1, e^{j2\pi/N}, \cdots, e^{j2(N-1)\pi/N}$.

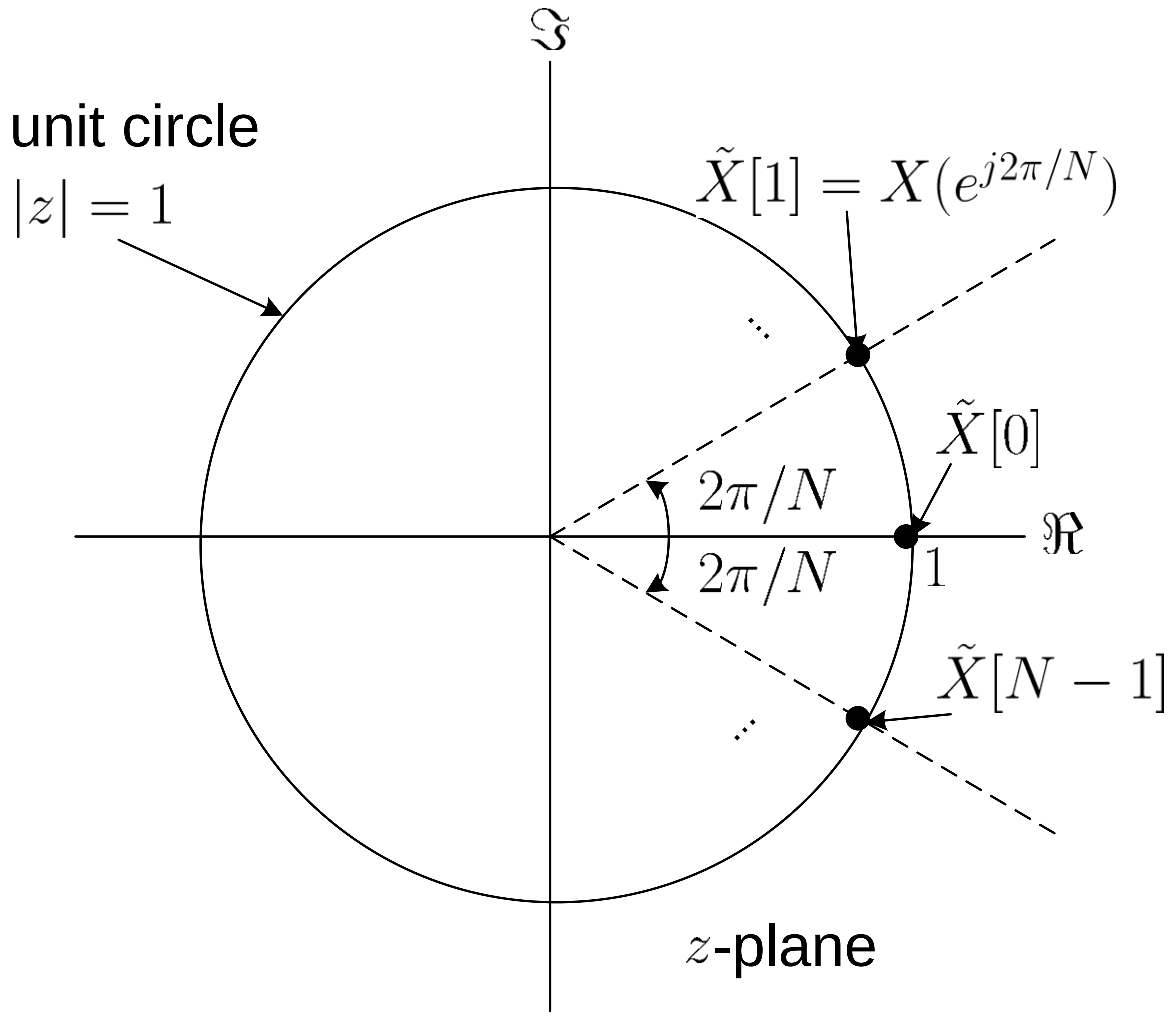


Fig.7.3: Relationship between $\tilde{X}[k]$, $X(e^{j\omega})$ and $X(z)$

Example 7.2

Determine the DTFT of a finite-duration sequence $x[n]$:

$$x[n] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Then compare the results with those in Example 7.1.

Using (6.1), the DTFT of $x[n]$ is computed as:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= 1 + e^{-j\omega} + e^{-j2\omega} \\ &= e^{-j\omega}\left(e^{j\omega} + 1 + e^{-j\omega}\right) \\ &= e^{-j\omega}\left[1 + 2\cos(\omega)\right] \end{aligned}$$

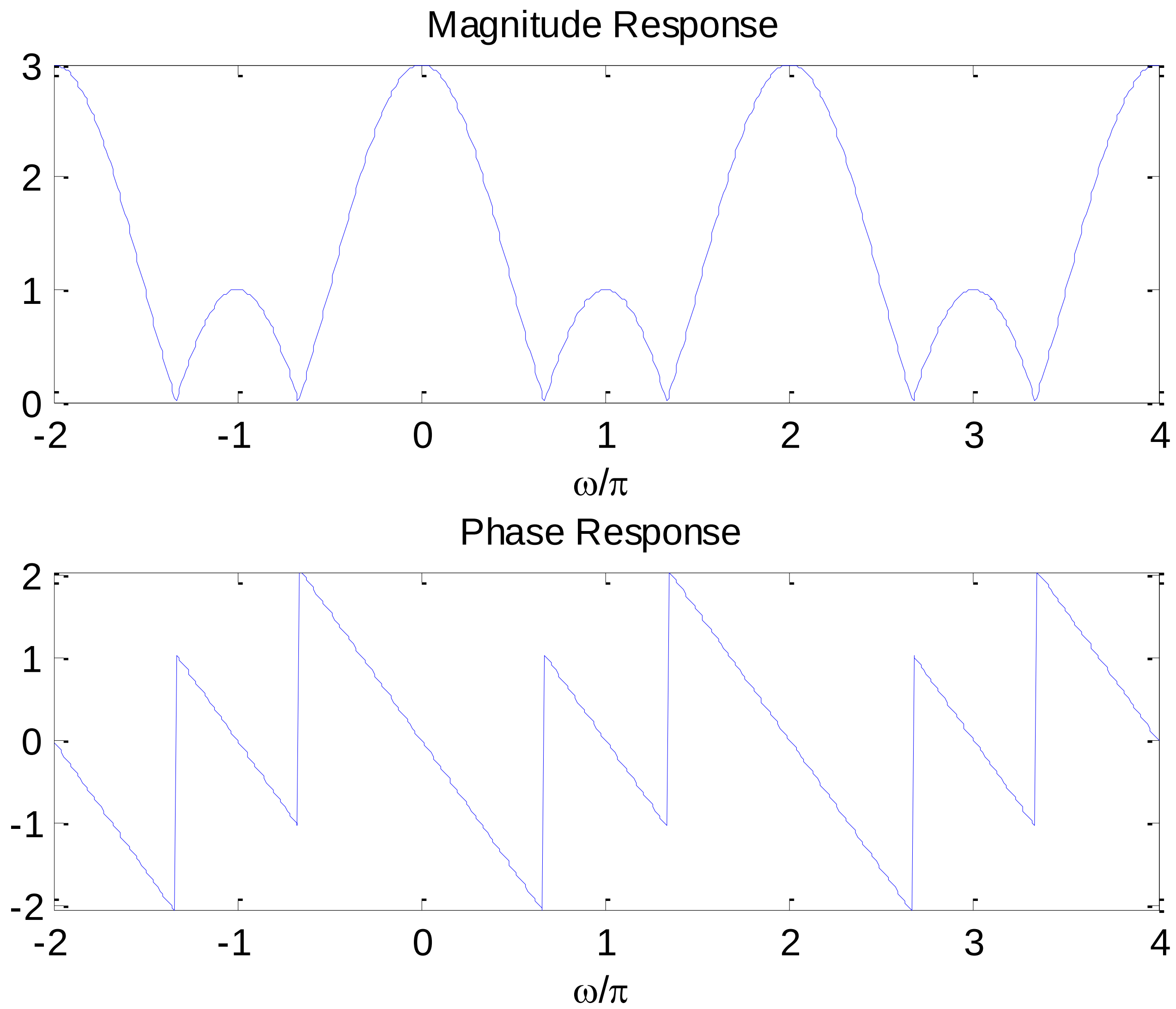


Fig.7.4: DTFT plots

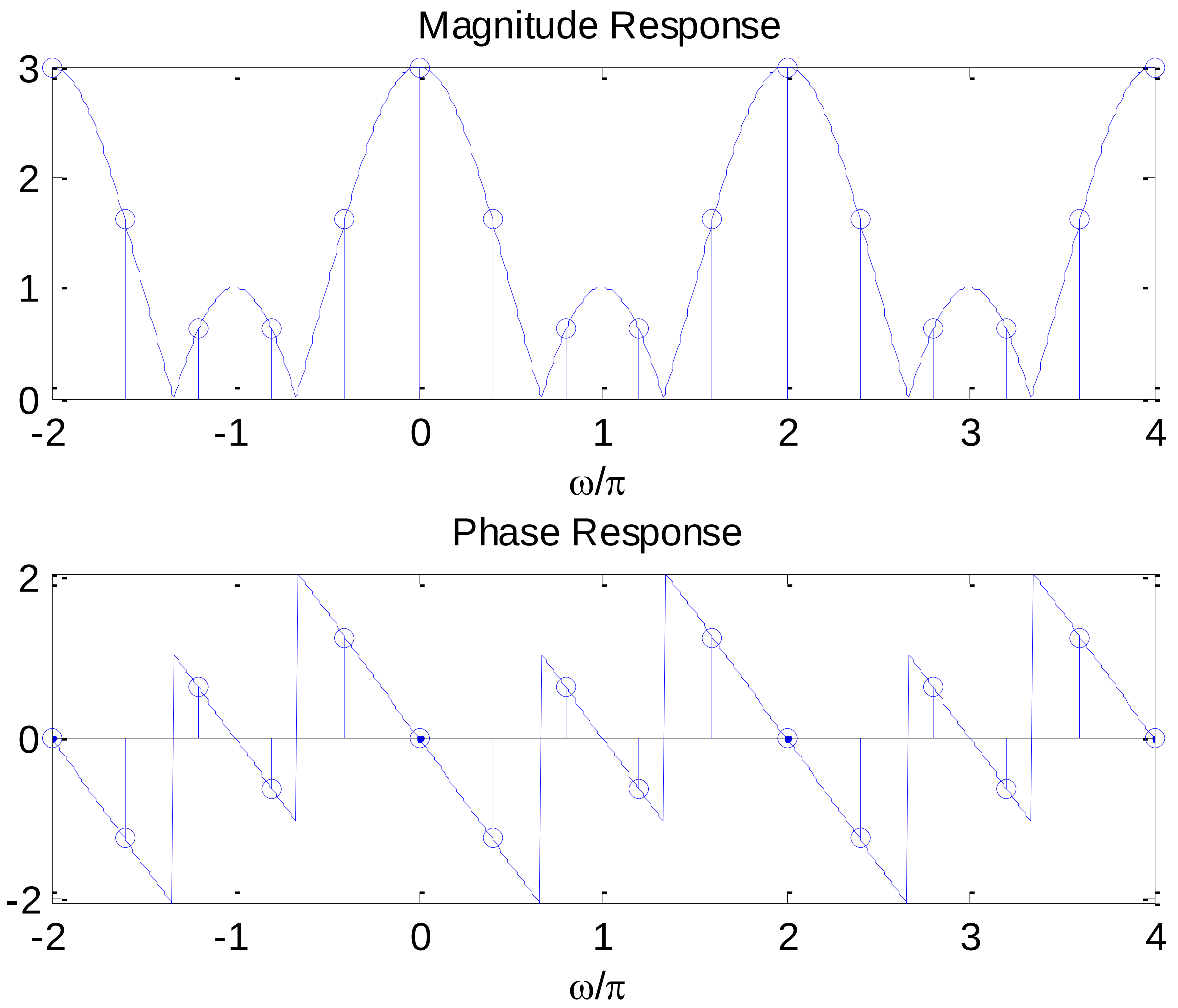


Fig.7.5: DFS and DTFT plots with $N = 5$

Suppose $\tilde{x}[n]$ in Example 7.1 is modified as:

$$\tilde{x}[n] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & n = 3, 4, \cdots, 9 \end{cases}$$

Via appending 5 zeros in each period, now we have $N = 10$.

What is the period of the DFS?

What is its relationship with that of Example 7.2?

How about if infinite zeros are appended?

The MATLAB programs are provided as `ex7_2.m`, `ex7_2_2.m` and `ex7_2_3.m`.

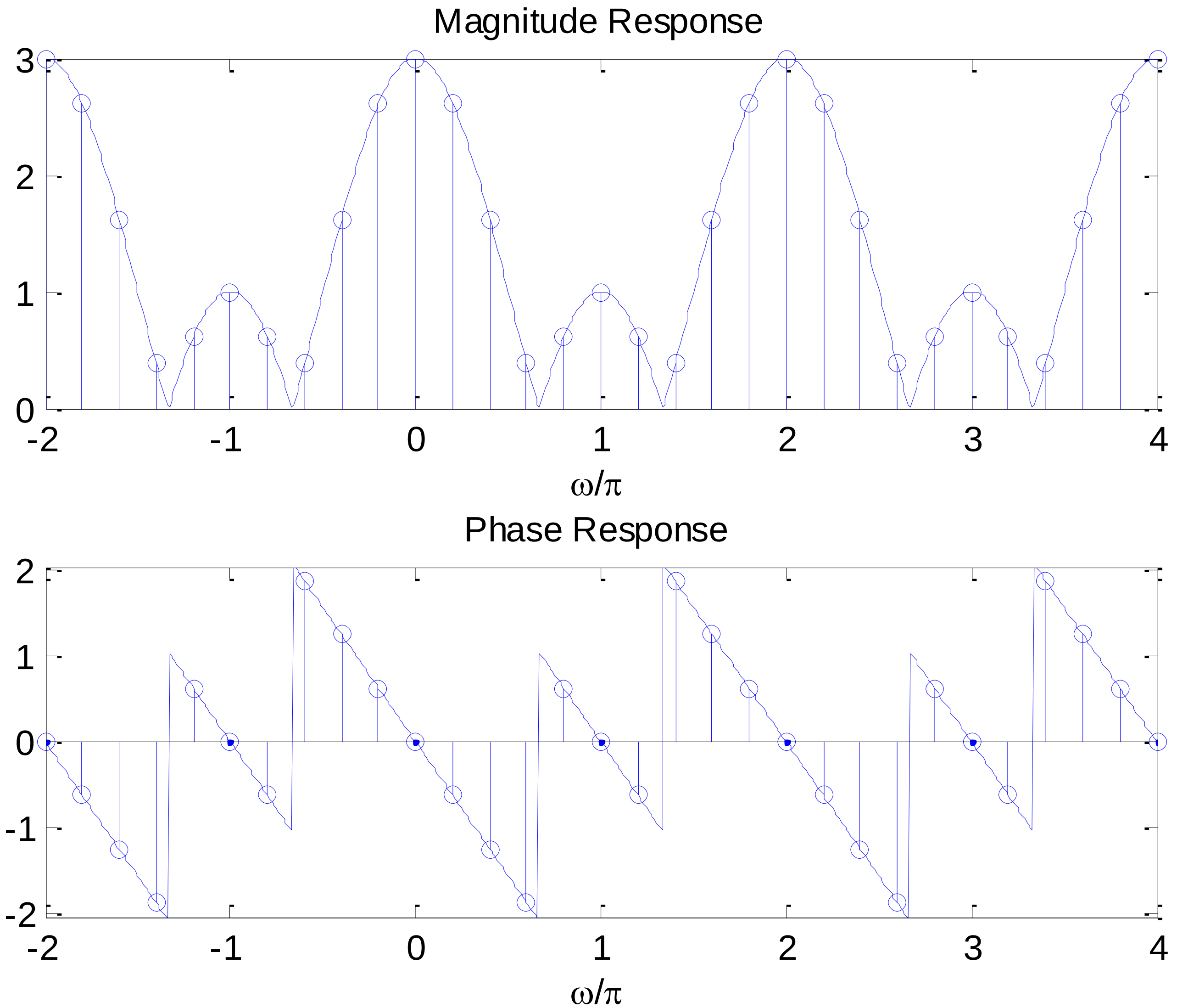


Fig.7.6: DFS and DTFT plots with $N = 10$

Properties of DFS

1. Periodicity

If $\tilde{x}[n]$ is a periodic sequence with period N, its DFS $\tilde{X}[k]$ is also periodic with period N:

$$\tilde{x}[n] = \tilde{x}[n+rN] \leftrightarrow \tilde{X}[k] = \tilde{X}[k+rN] \qquad (7.18)$$

where r is any integer. The proof is obtained with the use of (7.10) and $W_N^{rN} = e^{-j2\pi r} = 1$ as follows:

$$\begin{aligned} \tilde{X}[k+rN] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(k+rN)n} = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} W_N^{n(rN)} \\ &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} = \tilde{X}[k] \end{aligned} \qquad (7.19)$$

2. Linearity

Let $(\tilde{x}_1[n], \tilde{X}_1[k])$ and $(\tilde{x}_2[n], \tilde{X}_2[k])$ be two DFS pairs with the same period of N. We have:

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \leftrightarrow a\tilde{X}_1[k] + b\tilde{X}_2[k] \tag{7.20}$$

3. Shift of Sequence

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then

$$\tilde{x}[n-m] \leftrightarrow W_N^{km}\tilde{X}[k] \tag{7.21}$$

and

$$W_N^{-nl}\tilde{x}[n] \leftrightarrow \tilde{X}[k-l] \tag{7.22}$$

where N is the period while m and l are any integers. Note that (7.21) follows (6.10) by putting $\omega = 2\pi k/N$ and (7.22) follows (6.11) via the substitution of $\omega_0 = 2\pi l/N$.

4. Duality

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then

$$\tilde{X}[n] \leftrightarrow N\tilde{x}[-k] \tag{7.23}$$

5. Symmetry

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then

$$\tilde{x}^*[n] \leftrightarrow \tilde{X}^*[-k] \tag{7.24}$$

and

$$\tilde{x}^*[-n] \leftrightarrow \tilde{X}^*[k] \tag{7.25}$$

Note that (7.24) corresponds to the DTFT conjugation property in (6.14) while (7.25) is similar to the time reversal property in (6.15).

6. Periodic Convolution

Let $(\tilde{x}_1[n], \tilde{X}_1[k])$ and $(\tilde{x}_2[n], \tilde{X}_2[k])$ be two DFS pairs with the same period of N. We have

$$\tilde{x}_1[n] \tilde{\otimes} \tilde{x}_2[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{x}_2[n-m] \leftrightarrow \tilde{X}_1[k] \tilde{X}_2[k] \qquad (7.26)$$

Analogous to (6.18), $\tilde{\otimes}$ denotes discrete-time convolution within one period.

With the use of (7.11) and (7.21), the proof is given as follows:

$$
\begin{aligned}
\sum_{n=0}^{N-1}\left[\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]\right]W_N^{nk} &= \sum_{m=0}^{N-1}\tilde{x}_1[m]\left[\sum_{n=0}^{N-1}\tilde{x}_2[n-m]W_N^{nk}\right] \\
&= \sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{X}_2[k]W_N^{mk} \\
&= \tilde{X}_2[k]\left[\sum_{m=0}^{N-1}\tilde{x}_1[m]W_N^{mk}\right] \\
&= \tilde{X}_1[k]\tilde{X}_2[k] \qquad (7.27)
\end{aligned}
$$

To compute $\tilde{x}[n]\tilde{\otimes}\tilde{y}[n]$ where both $\tilde{x}[n]$ and $\tilde{y}[n]$ are of period N, we indeed only need the samples with $n = 0, 1, \cdots, N-1$.

Let $\tilde{z}[n] = \tilde{x}[n]\tilde{\otimes}\tilde{y}[n]$. Expanding (7.26), we have:

$$\tilde{z}[n] = \tilde{x}[0]\tilde{y}[n] + \cdots + \tilde{x}[N-2]\tilde{y}[n-(N-2)] + \tilde{x}[N-1]\tilde{y}[n-(N-1)] \quad (7.28)$$

For $n = 0$:

$$\begin{aligned}\tilde{z}[0] &= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[0-(N-2)] + \tilde{x}[N-1]\tilde{y}[0-(N-1)] \\ &= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[0-(N-2)+N] + \tilde{x}[N-1]\tilde{y}[0-(N-1)+N] \\ &= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[2] + \tilde{x}[N-1]\tilde{y}[1]\end{aligned} \quad (7.29)$$

For $n = 1$:

$$\begin{aligned}\tilde{z}[1] &= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[1-(N-2)] + \tilde{x}[N-1]\tilde{y}[1-(N-1)] \\ &= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[1-(N-2)+N] + \tilde{x}[N-1]\tilde{y}[1-(N-1)+N] \\ &= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[3] + \tilde{x}[N-1]\tilde{y}[2]\end{aligned} \quad (7.30)$$

A period of $\tilde{z}[n]$ can be computed in matrix form as:

$$\begin{bmatrix} \tilde{z}[0] \\ \tilde{z}[1] \\ \vdots \\ \tilde{z}[N-2] \\ \tilde{z}[N-1] \end{bmatrix} = \begin{bmatrix} \tilde{y}[0] & \tilde{y}[N-1] & \cdots & \tilde{y}[2] & \tilde{y}[1] \\ \tilde{y}[1] & \tilde{y}[0] & \cdots & \tilde{y}[3] & \tilde{y}[2] \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \tilde{y}[N-2] & \tilde{y}[N-3] & \cdots & \tilde{y}[0] & \tilde{y}[N-1] \\ \tilde{y}[N-1] & \tilde{y}[N-2] & \cdots & \tilde{y}[1] & \tilde{y}[0] \end{bmatrix} \begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \vdots \\ \tilde{x}[N-2] \\ \tilde{x}[N-1] \end{bmatrix} \quad (7.31)$$

<u>Example 7.3</u>

Given two periodic sequences $\tilde{x}[n]$ and $\tilde{y}[n]$ with period 4:

$$\left[\, \tilde{x}[0]\ \tilde{x}[1]\ \tilde{x}[2]\ \tilde{x}[3] \,\right] = \left[\, 4\ \ -3\ 2\ \ -1 \,\right]$$

and

$$\left[\, \tilde{y}[0]\ \tilde{y}[1]\ \tilde{y}[2]\ \tilde{y}[3] \,\right] = \left[\, 1\ 2\ 3\ 4 \,\right]$$

Compute $\tilde{z}[n] = \tilde{x}[n] \tilde{\otimes} \tilde{y}[n]$.

Using (7.31), $\tilde{z}[n]$ is computed as:

$$
\begin{bmatrix} \tilde{z}[0] \\ \tilde{z}[1] \\ \tilde{z}[2] \\ \tilde{z}[3] \end{bmatrix} = \begin{bmatrix} \tilde{y}[0] & \tilde{y}[3] & \tilde{y}[2] & \tilde{y}[1] \\ \tilde{y}[1] & \tilde{y}[0] & \tilde{y}[3] & \tilde{y}[2] \\ \tilde{y}[2] & \tilde{y}[1] & \tilde{y}[0] & \tilde{y}[3] \\ \tilde{y}[3] & \tilde{y}[2] & \tilde{y}[1] & \tilde{y}[0] \end{bmatrix} \begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \tilde{x}[2] \\ \tilde{x}[3] \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ 4 \\ 10 \end{bmatrix}
$$

The square matrix can be determined using the MATLAB command `toeplitz([1,2,3,4],[1,4,3,2])`. That is, we only need to know its first row and first column.

Periodic convolution can be utilized to compute convolution of finite-duration sequences in (3.17) as follows.

Let $x[n]$ and $y[n]$ be finite-duration sequences with lengths M and N, respectively, and $z[n] = x[n] \otimes y[n]$ which has a length of $(M + N - 1)$

We append $(N - 1)$ and $(M - 1)$ zeros at the ends of $x[n]$ and $y[n]$ for constructing periodic $\tilde{x}[n]$ and $\tilde{y}[n]$ where both are of period $(M + N - 1)$

$z[n]$ is then obtained from one period of $\tilde{x}[n]\tilde{\otimes}\tilde{y}[n]$.

<u>Example 7.4</u>

Compute the convolution of $x[n]$ and $y[n]$ with the use of periodic convolution. The lengths of $x[n]$ and $y[n]$ are 2 and 3 with $x[0] = 2$, $x[1] = 3$, $y[0] = 1$, $y[1] = -4$ and $y[2] = 5$.

The length of $x[n] \otimes y[n]$ is 4. As a result, we append two zeros and one zero in $x[n]$ and $y[n]$, respectively. According to (7.31), the MATLAB code is:

```
toeplitz([1,-4,5,0],[1,0,5,-4])*[2;3;0;0]
```

which gives

```
2     -5     -2     15
```

Note that the command `conv([2,3],[1,-4,5])` also produces the same result.

Discrete Fourier Transform

DFT is used for analyzing discrete-time finite-duration signals in the frequency domain

Let $x[n]$ be a finite-duration sequence of length N such that $x[n]=0$ outside $0 \leq n \leq N-1$. The DFT pair of $x[n]$ is:

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases} \qquad (7.32)$$

and

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \qquad (7.33)$$

If we extend $x[n]$ to a periodic sequence $\tilde{x}[n]$ with period N, the DFS pair for $\tilde{x}[n]$ is given by (7.10)-(7.11). Comparing (7.32) and (7.10), $X[k]=\tilde{X}[k]$ for $0 \le k \le N-1$. As a result, DFT and DFS are equivalent within the interval of $[0, N-1]$

<u>Example 7.5</u>

Find the DFT coefficients of a finite-duration sequence $x[n]$ which has the form of

$$x[n]=\begin{cases} 1, & n=0,1,2 \\ 0, & \text{otherwise} \end{cases}$$

Using (7.32) and Example 7.1 with $N=3$, we have:

$$\begin{aligned} X[k] &= \sum_{n=0}^{2} x[n] W_N^{kn} = W_3^0 + W_3^k + W_3^{2k} \\ &= e^{-\frac{j2\pi k}{3}}\left[1+2\cos\left(\frac{2\pi k}{3}\right)\right] = \begin{cases} 3, & k=0 \\ 0, & k=1,2 \end{cases} \end{aligned}$$

Together with $X[k]$ whose index is outside the interval of $0 \le k \le 2$, we finally have:

$$X[k] = \begin{cases} 3, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

If the length of $x[n]$ is considered as $N = 5$ such that $x[3] = x[4] = 0$, then we obtain:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn} = W_5^0 + W_5^k + W_5^{2k} \\ &= \begin{cases} e^{-\frac{j2\pi k}{5}} \left[1 + 2\cos\left(\frac{2\pi k}{5}\right)\right], & k = 0, 1, \cdots, 4 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The MATLAB command for DFT computation is `fft`. The MATLAB code to produce magnitudes and phases of $X[k]$ is:

```
N=5;
x=[1 1 1 0 0];                   %append 2 zeros
subplot(2,1,1);
stem([0:N-1],abs(fft(x))); %plot magnitude response
title('Magnitude Response');
subplot(2,1,2);
stem([0:N-1],angle(fft(x)));%plot phase response
title('Phase Response');
```

According to Example 7.2 and the relationship between DFT and DFS, the DFT will approach the DTFT when we append infinite zeros at the end of $x[n]$

The MATLAB program is provided as `ex7_5.m`.

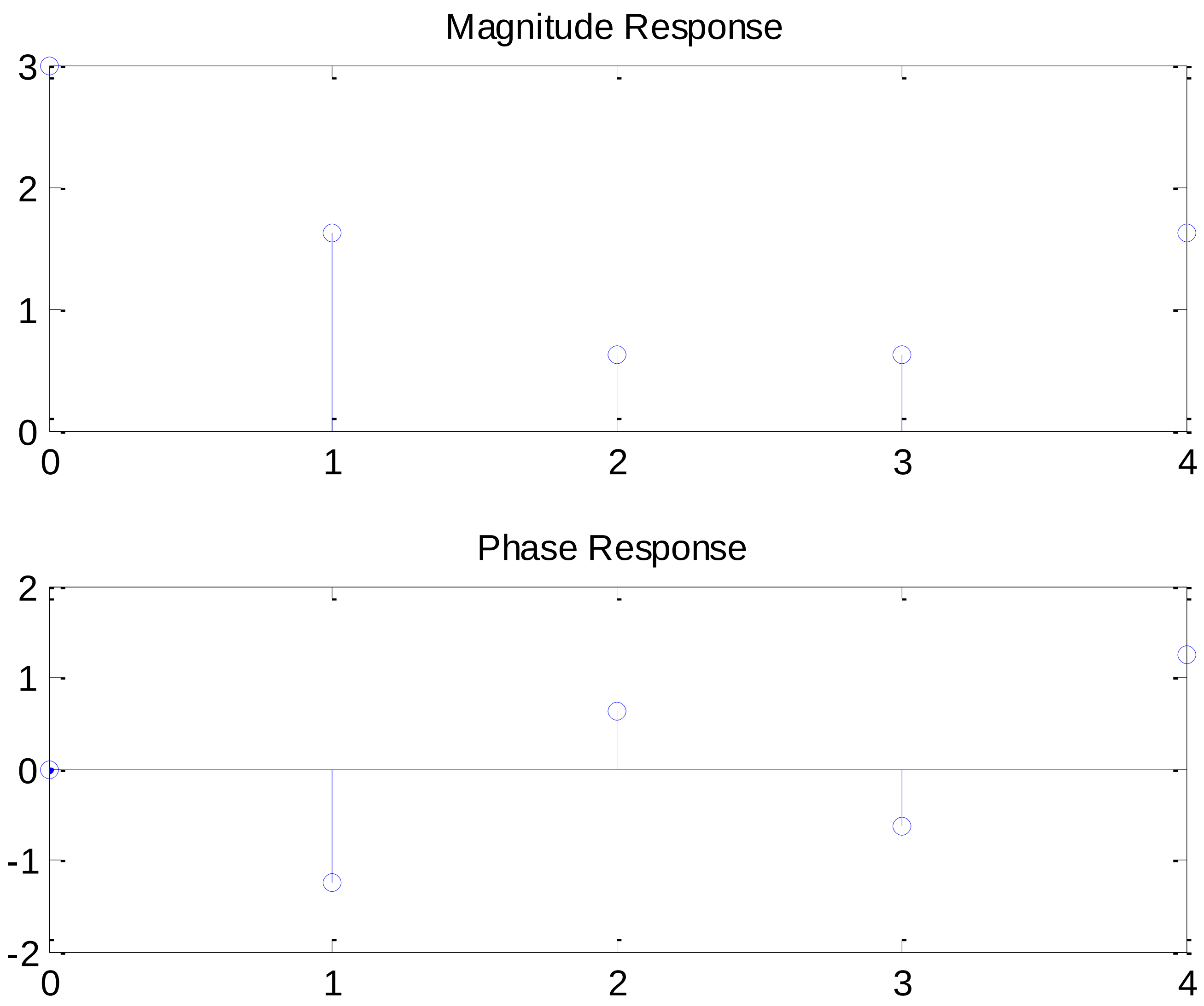


Fig.7.7: DFT plots with $N = 5$

Example 7.6

Given a discrete-time finite-duration sinusoid:

$$x[n] = 2\cos(0.7\pi n + 1), \quad n = 0, 1, \cdots, 20$$

Estimate the tone frequency using DFT.

Consider the continuous-time case first. According to (2.16), Fourier transform pair for a complex tone of frequency Ω_0 is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

That is, Ω_0 can be found by locating the peak of the Fourier transform. Moreover, a real-valued tone $\cos(\Omega_0 t)$ is:

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

From the Fourier transform of $\cos(\Omega_0 t)$, Ω_0 and $-\Omega_0$ are located from the two impulses.

Analogously, there will be two peaks which correspond to frequencies 0.7π and -0.7π in the DFT for $x[n]$.

The MATLAB code is

```
N=21;                  %number of samples is 21
A=2;                   %tone amplitude is 2
w=0.7*pi;              %frequency is 0.7*pi
p=1;                   %phase is 1
n=0:N-1;               %define a vector of size N
x=A*cos(w*n+p);        %generate tone
X=fft(x);              %compute DFT
subplot(2,1,1);
stem(n,abs(X));       %plot magnitude response
subplot(2,1,2);
stem(n,angle(X));     %plot phase response
```

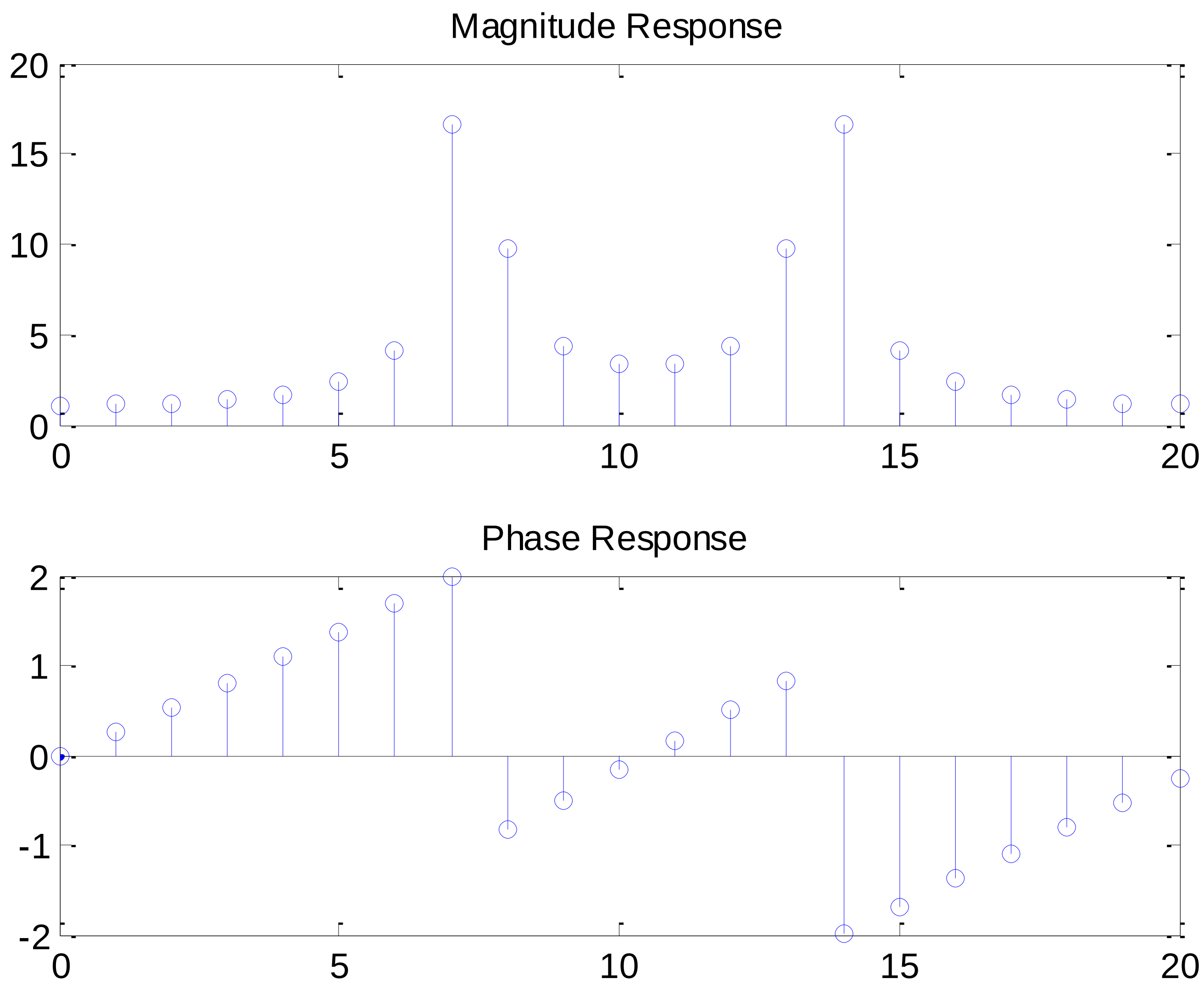


Fig.7.8: DFT plots for a real tone

```
X =
 1.0806              1.0674+0.2939i   1.0243+0.6130i
 0.9382+0.9931i   0.7756+1.5027i   0.4409+2.3159i
-0.4524+4.1068i  -6.7461+15.1792i  6.5451-7.2043i
 3.8608-2.1316i   3.3521-0.5718i   3.3521+0.5718i
 3.8608+2.1316i   6.5451+7.2043i  -6.7461-15.1792i
-0.4524-4.1068i   0.4409-2.3159i   0.7756-1.5027i
 0.9382-0.9931i   1.0243-0.6130i   1.0674-0.2939i
```

Interestingly, we observe that $\Re\{X[k]\} = \Re\{X[N-k]\}$ and $\Im\{X[k]\} = -\Im\{X[N-k]\}$. In fact, all real-valued sequences possess these properties so that we only have to compute around half of the DFT coefficients.

As the DFT coefficients are complex-valued, we search the frequency according to the magnitude plot.

There are two peaks, one at $k=7$ and the other at $k=14$ which correspond to $\omega=0.7\pi$ and $\omega=-0.7\pi$, respectively. **Why?**

From Example 7.2, it is clear that the index k refers to $\omega=2\pi k/N$. As a result, an estimate of ω_0 is:

$$\hat{\omega}_0=\frac{2\pi\cdot 7}{21}\approx 0.6667\pi$$

To improve the accuracy, we append a large number of zeros to $x[n]$. The MATLAB code for $x[n]$ is now modified as:

```
x=[A*cos(w.*n+p) zeros(1,1980)];
```

where 1980 zeros are appended.

The MATLAB code is provided as `ex7_6.m` and `ex7_6_2.m`.

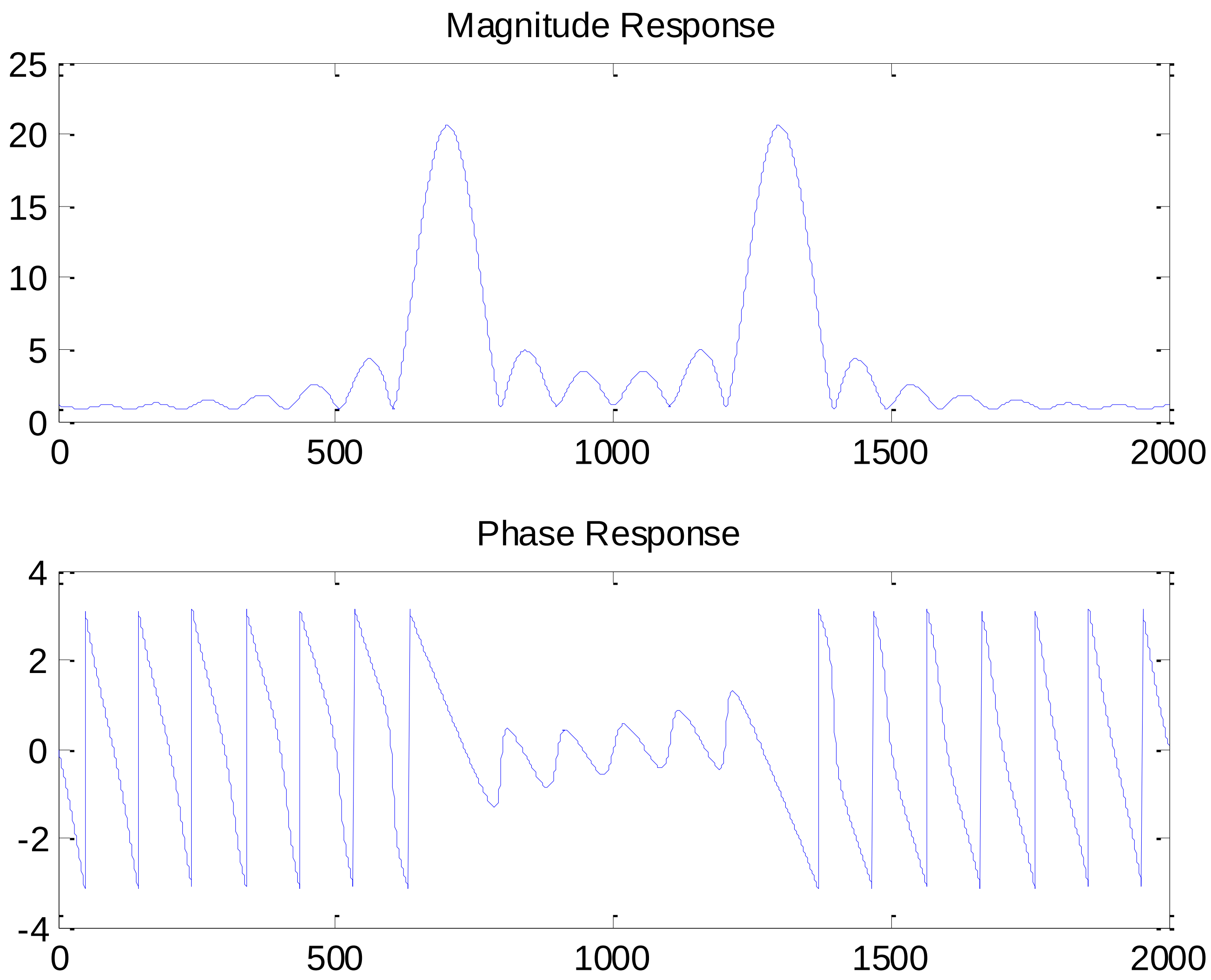


Fig.7.9: DFT plots for a real tone with zero padding

The peak index is found to be $k = 702$ with $N = 2001$. Thus

$$\hat{\omega}_0 = \frac{2\pi \cdot 702}{2001} \approx 0.7016\pi$$

Example 7.7

Find the inverse DFT coefficients for $X[k]$ which has a length of $N = 5$ and has the form of

$$X[k] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & n = 3, 4 \end{cases}$$

Plot $x[n]$.

Using (7.33) and Example 7.5, we have:

$$
\begin{aligned}
x[n] &= \frac{1}{N}\sum_{n=0}^{N-1} X[k]W_N^{-kn} = \frac{1}{5}\left(W_5^0 + W_5^{-n} + W_5^{-2n}\right) \\
&= \begin{cases} \frac{1}{5}e^{\frac{j2\pi n}{5}}\left[1 + 2\cos\left(\frac{2\pi n}{5}\right)\right], & n = 0,1,\cdots,4 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

The main MATLAB code is:

```
N=5;
X=[1 1 1 0 0];
subplot(2,1,1);
stem([0:N-1],abs(ifft(X)));
subplot(2,1,2);
stem([0:N-1],angle(ifft(X)));
```

The MATLAB program is provided as `ex7_7.m`.

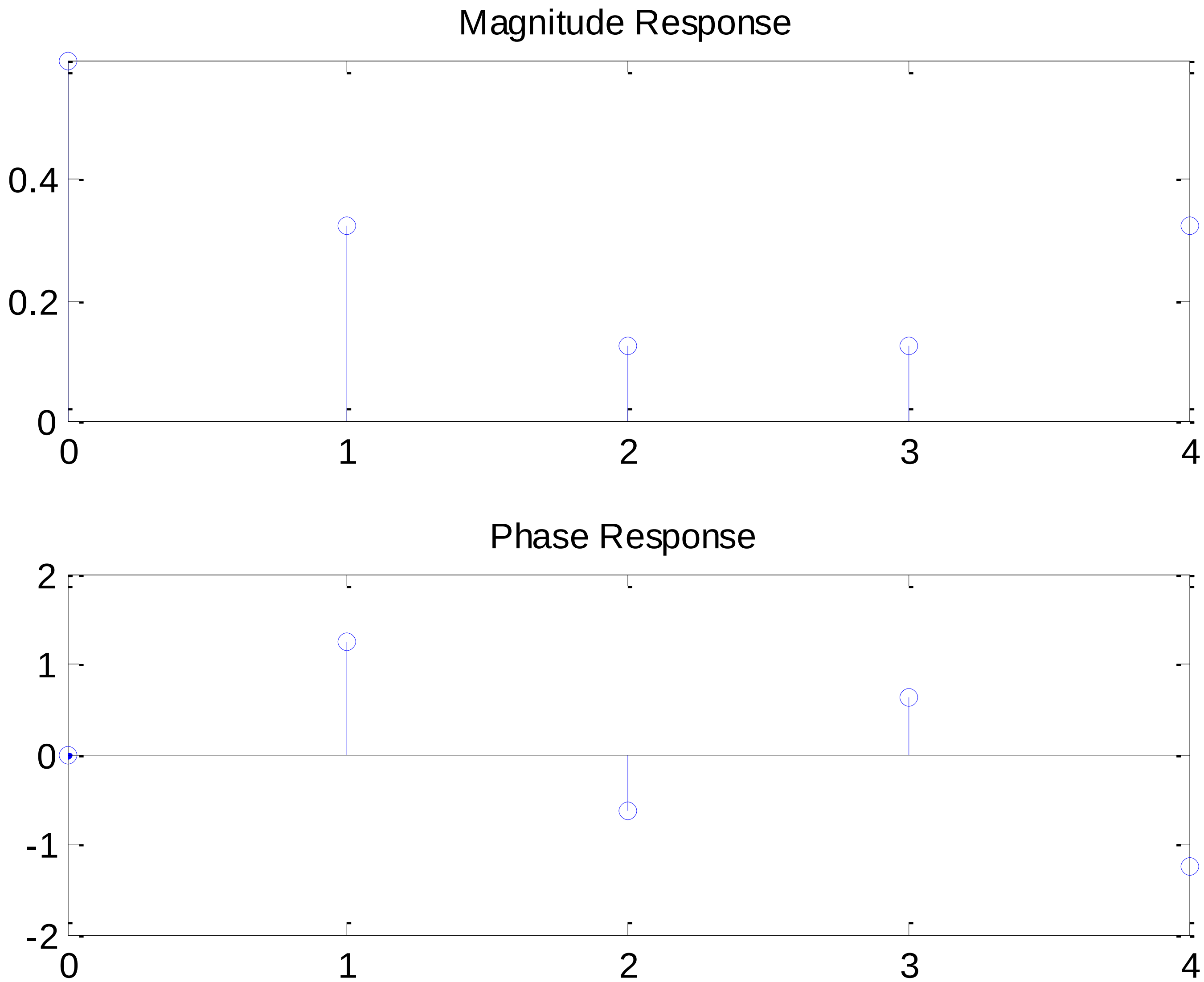


Fig.7.10: Inverse DFT plots

Properties of DFT

Since DFT pair is equal to DFS pair within $[0, N-1]$, their properties will be identical if we take care of the values of $x[n]$ and $X[k]$ when the indices are outside the interval

1. Linearity

Let $(x_1[n], X_1[k])$ and $(x_2[n], X_2[k])$ be two DFT pairs with the same duration of N. We have:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k] \quad (7.34)$$

Note that if $x_1[n]$ and $x_2[n]$ are of different lengths, we can properly append zero(s) to the shorter sequence to make them with the same duration.

2. Circular Shift of Sequence

If $x[n] \leftrightarrow X[k]$, then

$$x[(n-m) \mod (N)] \leftrightarrow W_N^{km} X[k] \quad (7.35)$$

Note that in order to make sure that the resultant time index is within the interval of $[0, N-1]$, we need circular shift, which is defined as

$$(n-m) \mod (N) = n - m + r \cdot N \quad (7.36)$$

where the integer r is chosen such that

$$0 \leq n - m + r \cdot N \leq N - 1 \quad (7.37)$$

Example 7.8

Determine $x_1[n] = x[(n-2) \mod (4)]$ where $x[n]$ is of length 4 and has the form of:

$$x[n] = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \\ 2, & n = 2 \\ 4, & n = 3 \end{cases}$$

According to (7.36)-(7.37) with $N = 4$, $x_1[n]$ is determined as:

$$\begin{aligned}
x_1[0] &= x[(0-2) \mod (4)] = x[2] = 2, \quad r = 1 \\
x_1[1] &= x[(1-2) \mod (4)] = x[3] = 4, \quad r = 1 \\
x_1[2] &= x[(2-2) \mod (4)] = x[0] = 1, \quad r = 0 \\
x_1[3] &= x[(3-2) \mod (4)] = x[1] = 3, \quad r = 0
\end{aligned}$$

3. Duality

If $x[n] \leftrightarrow X[k]$, then

$$X[n] \leftrightarrow Nx[(-k) \mod (N)] \tag{7.38}$$

4. Symmetry

If $x[n] \leftrightarrow X[k]$, then

$$x^*[n] \leftrightarrow X^*[(-k) \mod (N)] \tag{7.39}$$

and

$$x^*[(-n) \mod (N)] \leftrightarrow X^*[k] \tag{7.40}$$

5. Circular Convolution

Let $(x_1[n], X_1[k])$ and $(x_2[n], X_2[k])$ be two DFT pairs with the same duration of N. We have

$$x_1[n] \otimes_N x_2[n] = \sum_{m=0}^{N-1} x_1[m] x_2[(n-m) \mod (N)] \leftrightarrow X_1[k] X_2[k] \quad (7.41)$$

where $\otimes_N$ is the circular convolution operator.

Fast Fourier Transform

FFT is a fast algorithm for DFT and inverse DFT computation.

Recall (7.32):

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1 \tag{7.42}$$

Each $X[k]$ involves N and $(N-1)$ complex multiplications and additions, respectively.

Computing all DFT coefficients requires N^2 complex multiplications and $N(N-1)$ complex additions.

Assuming that $N = 2^v$, the corresponding computational requirements for FFT are $0.5N\log_2(N)$ complex multiplications and $N\log_2(N)$ complex additions.

N	Direct Computation		FFT	
	Multiplication N^2	Addition $N(N-1)$	Multiplication $0.5N\log_2(N)$	Addition $N\log_2(N)$
2	4	2	1	2
8	64	56	12	24
32	1024	922	80	160
64	4096	4022	192	384
2^{10}	1048576	1048576	5120	10240
2^{20}	~10^{12}	~10^{12}	~10^{7}	~2×10^{7}

Table 7.1: Complexities of direct DFT computation and FFT

Basically, FFT makes use of two ideas in its development:

- Decompose the DFT computation of a sequence into successively smaller DFTs

- Utilize two properties of $W_N^k = e^{-j2\pi k/N}$:

  - complex conjugate symmetry property:

$$W_N^{k(N-n)} = W_N^{-kn} = \left(W_N^{kn}\right)^* \tag{7.43}$$

  - periodicity in n and k:

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{n(k+N)} \tag{7.44}$$

Decimation-in-Time Algorithm

The basic idea is to compute (7.42) according to

$$X[k] = \sum_{n=\text{even}}^{N-1} x[n]W_N^{kn} + \sum_{n=\text{odd}}^{N-1} x[n]W_N^{kn} \qquad (7.45)$$

Substituting $n = 2r$ and $n = 2r+1$ for the first and second summation terms:

$$\begin{aligned} X[k] &= \sum_{r=0}^{N/2-1} x[2r]W_N^{2rk} + \sum_{r=0}^{N/2-1} x[2r+1]W_N^{(2r+1)k} \\ &= \sum_{r=0}^{N/2-1} x[2r]\left(W_N^2\right)^{rk} + W_N^k \sum_{r=0}^{N/2-1} x[2r+1]\left(W_N^2\right)^{rk} \end{aligned} \qquad (7.46)$$

Using $W_N^2 = W_{N/2}$ since $W_N^2 = e^{-j2\pi/N\cdot 2} = e^{-j2\pi/(N/2)}$, we have:

$$\begin{aligned} X[k] &= \sum_{r=0}^{N/2-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{N/2-1} x[2r+1] W_{N/2}^{rk} \\ &= G[k] + W_N^k \cdot H[k], \quad k = 0, 1, \cdots, N-1 \end{aligned} \tag{7.47}$$

where $G[k]$ and $H[k]$ are the DFTs of the even-index and odd-index elements of $x[n]$, respectively. That is, $X[k]$ can be constructed from two $N/2$-point DFTs, namely, $G[k]$ and $H[k]$.

Further simplifications can be achieved by writing the N equations as 2 groups of $N/2$ equations as follows:

$$X[k] = G[k] + W_N^k \cdot H[k], \quad k = 0, 1, \cdots, N/2-1 \tag{7.48}$$

and

$$
\begin{aligned}
X[k+N/2] &= \sum_{r=0}^{N/2-1} x[2r]W_{N/2}^{r(k+N/2)} + W_N^{k+N/2} \sum_{r=0}^{N/2-1} x[2r+1]W_{N/2}^{r(k+N/2)} \\
&= \sum_{r=0}^{N/2-1} x[2r]W_{N/2}^{rk} - W_N^k \sum_{r=0}^{N/2-1} x[2r+1]W_{N/2}^{rk} \\
&= G[k] - W_N^k \cdot H[k], \quad k = 0, 1, \cdots, N/2-1 \qquad (7.49)
\end{aligned}
$$

with the use of $W_{N/2}^{N/2} = 1$ and $W_N^{N/2} = -1$. Equations (7.48) and (7.49) are known as the butterfly merging equations.

Noting that $N/2$ multiplications are also needed to calculate $W_N^k H[k]$, the number of multiplications is reduced from N^2 to $2(N/2)^2 + N/2 = N(N+1)/2$.

The decomposition step of (7.48)-(7.49) is repeated v times until 1-point DFT is reached.

Decimation-in-Frequency Algorithm

The basic idea is to decompose the frequency-domain sequence $X[k]$ into successively smaller subsequences.

Recall (7.42) and employing $W_N^{2r(n+N/2)} = W_N^{2nr} \cdot W_N^{rN} = W_N^{2nr}$ and $W_N^2 = W_{N/2}$, the even-index DFT coefficients are:

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)} = \sum_{n=0}^{N/2-1} x[n] W_N^{2nr} + \sum_{n=N/2}^{N-1} x[n] W_N^{2nr}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{2nr} + \sum_{n=0}^{N/2-1} x[n+N/2] W_N^{2r(n+N/2)}$$

$$= \sum_{n=0}^{N/2-1} (x[n] + x[n+N/2]) \cdot W_{N/2}^{nr}, \quad r = 0, 1, \cdots, N/2-1 \quad (7.50)$$

Using $W_N^{Nr}=1$ and $W_N^{N/2}=-1$, the odd-index coefficients are:

$$\begin{aligned}
X[2r+1] &= \sum_{n=0}^{N/2-1} x[n]W_N^{n(2r+1)} + \sum_{n=N/2}^{N-1} x[n]W_N^{n(2r+1)} \\
&= \sum_{n=0}^{N/2-1} x[n]W_N^{n}W_{N/2}^{nr} + \sum_{n=0}^{N/2-1} x[n+N/2]W_N^{(n+N/2)(2r+1)} \\
&= \sum_{n=0}^{N/2-1} x[n]W_N^{n}W_{N/2}^{nr} + W_N^{N/2(2r+1)}\sum_{n=0}^{N/2-1} x[n+N/2]W_N^{n(2r+1)} \\
&= \sum_{n=0}^{N/2-1} \left(x[n]-x[n+N/2]\right)W_N^{n}\cdot W_{N/2}^{nr}, \quad r=0,1,\cdots,N/2-1
\end{aligned} \tag{7.51}$$

$X[2r]$ and $X[2r+1]$ are equal to $N/2$-point DFTs of $(x[n]+x[n+N/2])$ and $(x[n]-x[n+N/2])\,W_N^n$, respectively. The decomposition step of (7.50)-(7.51) is repeated v times until 1-point DFT is reached

Fast Convolution with FFT

The convolution of two finite-duration sequences

$$y[n] = x_1[n] \otimes x_2[n]$$

where $x_1[n]$ is of length N_1 and $x_2[n]$ is of length N_2 requires computation of ($N_1 + N_2$-1) samples which corresponds to $N_1 N_2 - \min\{N_1, N_2\}$ complex multiplications

An alternate approach is to use FFT:

$$y[n] = \mathrm{IFFT}\left\{\mathrm{FFT}\{x_1[n]\} \times \mathrm{FFT}\{x_2[n]\}\right\}$$

In practice:

- Choose the minimum $N \geq N_1 + N_2 - 1$ and is power of 2
- Zero-pad $x_1[n]$ and $x_2[n]$ to length N, say, $\breve{x}_1[n]$ and $\breve{x}_2[n]$
- $\breve{y}[n] = \mathrm{IFFT}\left\{\mathrm{FFT}\{\breve{x}_1[n]\} \times \mathrm{FFT}\{\breve{x}_2[n]\}\right\}$

From (7.33), the inverse DFT has a factor of $1/N$, the IFFT thus requires $N+(N/2)\log_2(N)$ multiplications. As a result, the total multiplications for $\breve{y}[n]$ is $2N+(3N/2)\log_2(N)$

Using FFT is more computationally efficient than direct convolution computation for longer data lengths:

N_1	N_2	N	$N_1 N_2 - \min\{N_1, N_2\}$	$2N+(3N/2)\log_2(N)$
2	5	8	8	52
10	15	32	140	304
50	80	256	3950	3584
50	1000	2048	49950	37888
512	10000	16384	4119488	376832

MATLAB and C source codes for FFT can be found at:

http://www.ece.rutgers.edu/~orfanidi/intro2sp/#progs