**Z Transform**

Chapter Intended Learning Outcomes:

(i) Understanding the relationship between $z$ transform and the Fourier transform for discrete-time signals

(ii) Understanding the characteristics and properties of $z$ transform

(iii) Ability to compute $z$ transform and inverse $z$ transform

(iv) Ability to apply $z$ transform for analyzing linear time-invariant (LTI) systems
**Definition**

The $z$ transform of $x[n]$, denoted by $X(z)$, is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$  \hspace{1cm} (5.1)

where $z$ is a continuous complex variable.

**Is $X(z)$ real-valued or complex-valued?**

**Relationship with Fourier Transform**

Employing (4.2), we construct the continuous-time sampled signal $x_s(t)$ with a sampling interval of $T$ from $x[n]$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$  \hspace{1cm} (5.2)
Taking Fourier transform of $x_s(t)$ with using properties of $\delta(t)$:

\[ X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t)e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT)e^{-j\Omega t} dt \]

\[ = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT} \tag{5.3} \]

Defining $\omega = \Omega T$ as the discrete-time frequency parameter and writing $X_s(j\Omega)$ as $X(e^{j\omega})$, (5.3) becomes

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{5.4} \]

which is known as discrete-time Fourier transform (DTFT) or Fourier transform of discrete-time signals.
$X(e^{j\omega})$ is periodic with period $2\pi$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = X(e^{j(\omega+2k\pi)}) \quad (5.5)$$

where $k$ is any integer. Since $z$ is a continuous complex variable, we can write

$$z = re^{j\omega} \quad (5.6)$$

where $r = |z| > 0$ is magnitude and $\omega = \angle(z)$ is angle of $z$. Employing (5.6), the $z$ transform is:

$$X(z)|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n}) e^{-j\omega n} \quad (5.7)$$

which is equal to the DTFT of $x[n]r^{-n}$. When $r = 1$ or $z = e^{j\omega}$, (5.7) and (5.4) are identical:
\[ X(z) \big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \] (5.8)

**Fig. 5.1: Relationship between \( X(z) \) and \( X(e^{j\omega}) \) on the \( z \)-plane**
Region of Convergence (ROC)

ROC indicates when $z$ transform of a sequence converges

Generally there exists some $z$ such that

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \to \infty$$  \hspace{1cm} (5.9)

where the $z$ transform does not converge

The set of values of $z$ for which $X(z)$ converges or

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$  \hspace{1cm} (5.10)

is called the ROC, which must be specified along with $X(z)$ in order for the $z$ transform to be complete
Assuming that $x[n]$ is of infinite length, we decompose $X(z)$:

$$X(z) = X_- (z) + X_+ (z)$$  \hspace{1cm} (5.11)

where

$$X_- (z) = \sum_{n=\infty}^{-1} x[n] z^{-n} = \sum_{m=1}^{\infty} x[-m] z^{m}$$  \hspace{1cm} (5.12)

and

$$X_+ (z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$  \hspace{1cm} (5.13)

Let $f_n(z) = x[n] z^{-n}$, $X_+(z)$ is expanded as:

$$X_+(z) = x[0]z^{-0} + x[1]z^{-1} + \cdots + x[n]z^{-n} + \cdots$$

$$= f_0(z) + f_1(z) + \cdots + f_n(z) + \cdots$$  \hspace{1cm} (5.14)
According to the ratio test, convergence of $X_+(z)$ requires

$$\lim_{n\to\infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| < 1$$ (5.15)

Let $\lim_{n\to\infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+ > 0$. $X_+(z)$ converges if

$$\lim_{n\to\infty} \left| \frac{x[n+1]z^{-n-1}}{x[n]z^{-n}} \right| = \lim_{n\to\infty} \left| \frac{x[n+1]}{x[n]} \right| |z^{-1}| < 1$$

$$\Rightarrow |z| > \lim_{n\to\infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+$$ (5.16)

That is, the ROC for $X_+(z)$ is $|z| > R_+$. 
Let \( \lim_{m \to \infty} \left| \frac{x[-m]}{x[-m-1]} \right| = R_- > 0 \). \( X_-(z) \) converges if

\[
\lim_{m \to \infty} \left| \frac{x[-m-1]z^{m+1}}{x[-m]z^m} \right| = \lim_{m \to \infty} \left| \frac{x[-m-1]}{x[-m]} \right| |z| < 1
\]

\[
\Rightarrow |z| < \lim_{m \to \infty} \left| \frac{x[-m]}{x[-m-1]} \right| = R_-
\]

(5.17)

As a result, the ROC for \( X_-(z) \) is \( |z| < R_- \)

Combining the results, the ROC for \( X(z) \) is \( R_+ < |z| < R_- \):

- ROC is a ring when \( R_+ < R_- \)
- No ROC if \( R_- < R_+ \) and \( X(z) \) does not exist
Fig. 5.2: ROCs for $X_+(z)$, $X_-(z)$ and $X(z)$

**Poles and Zeros**

Values of $z$ for which $X(z) = 0$ are the zeros of $X(z)$

Values of $z$ for which $X(z) = \infty$ are the poles of $X(z)$
In many real-world applications, $X(z)$ is represented as a rational function:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{M} b_k z^k}{\sum_{k=0}^{N} a_k z^k} \quad (5.18)$$

Factorizing $P(z)$ and $Q(z)$, (5.18) can be written as

$$X(z) = \frac{b_0(z - d_1)(z - d_2) \cdots (z - d_M)}{a_0(z - c_1)(z - c_2) \cdots (z - c_N)} \quad (5.19)$$

How many poles and zeros in (5.18)? What are they?
Example 5.1
Determine the $z$ transform of $x[n] = a^n u[n]$ where $u[n]$ is the unit step function. Then determine the condition when the DTFT of $x[n]$ exists.

Using (5.1) and (3.3), we have

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

According to (5.10), $X(z)$ converges if

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

Applying the ratio test, the convergence condition is

$$|az^{-1}| < 1 \iff |z| > |a|$$
Note that we cannot write $|z| > a$ because $a$ may be complex

For $|z| > |a|$, $X(z)$ is computed as

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1 - (az^{-1})^\infty}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Together with the ROC, the $z$ transform of $x[n] = a^n u[n]$ is:

$$X(z) = \frac{z}{z - a}, \quad |z| > |a|$$

It is clear that $X(z)$ has a zero at $z = 0$ and a pole at $z = a$. Using (5.8), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 > |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| < 1$. 
Otherwise, its DTFT does not exist. In general, the DTFT $X(e^{j\omega})$ exists if its ROC includes the unit circle. If $|z| > |a|$ includes $|z| = 1$, $|a| < 1$ is required.

![ROC diagrams for $|a| < 1$ and $|a| > 1$](image)

**Fig.5.3:** ROCs for $|a| < 1$ and $|a| > 1$ when $x[n] = a^n u[n]$
Example 5.2
Determine the $z$ transform of $x[n] = -a^n u[-n - 1]$. Then determine the condition when the DTFT of $x[n]$ exists.

Using (5.1) and (3.3), we have

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = - \sum_{m=1}^{\infty} a^{-m} z^m = - \sum_{m=1}^{\infty} (a^{-1} z)^m$$

Similar to Example 5.1, $X(z)$ converges if $|a^{-1} z| < 1$ or $|z| < |a|$, which aligns with the ROC for $X_{-}(z)$ in (5.17). This gives

$$X(z) = - \sum_{m=1}^{\infty} (a^{-1} z)^m = - \frac{a^{-1} z \left(1 - (a^{-1} z)^\infty\right)}{1 - a^{-1} z} = - \frac{a^{-1} z}{1 - a^{-1} z} = \frac{z}{z - a}$$

Together with ROC, the $z$ transform of $x[n] = -a^n u[-n - 1]$ is:

$$X(z) = \frac{z}{z - a}, \quad |z| < |a|$$
Using (5.8), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 < |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| > 1$.

Fig.5.4: ROCs for $|a| < 1$ and $|a| > 1$ when $x[n] = -a^n u[-n-1]$
Example 5.3
Determine the $z$ transform of $x[n] = a^n u[n] + b^n u[-n - 1]$ where $|a| < |b|$.

Employing the results in Examples 5.1 and 5.2, we have

$$X(z) = \frac{1}{1 - az^{-1}} + \left(-\frac{1}{1 - bz^{-1}}\right), \quad |z| > |a| \quad \text{and} \quad |z| < |b|$$

$$= \frac{(a - b)z^{-1}}{(1 - az^{-1})(1 - bz^{-1})}$$

$$= \frac{(a - b)z}{(z - a)(z - b)}, \quad |a| < |z| < |b|$$

Note that its ROC agrees with Fig.5.2.

What are the pole(s) and zero(s) of $X(z)$?
Example 5.4
Determine the $z$ transform of $x[n] = \delta[n + 1]$.

Using (5.1) and (3.2), we have

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n + 1]z^{-n} = z$$

Example 5.5
Determine the $z$ transform of $x[n]$ which has the form of:

$$x[n] = \begin{cases} 
a^n, & 0 \leq n \leq N - 1 \\
0, & \text{otherwise}
\end{cases}$$

Using (5.1), we have

$$X(z) = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

What are the ROCs in Examples 5.4 and 5.5?
Finite-Duration and Infinite-Duration Sequences

Finite-duration sequence: values of $x[n]$ are nonzero only for a finite time interval

Otherwise, $x[n]$ is called an infinite-duration sequence:

- **Right-sided**: if $x[n] = 0$ for $n < N_+ < \infty$ where $N_+$ is an integer (e.g., $x[n] = a^n u[n]$ with $N_+ = 0$; $x[n] = a^n u[n - 10]$ with $N_+ = 10$; $x[n] = a^n u[n + 10]$ with $N_+ = -10$)

- **Left-sided**: if $x[n] = 0$ for $n > N_- > -\infty$ where $N_-$ is an integer (e.g., $x[n] = -a^n u[-n - 1]$ with $N_- = -1$)

- **Two-sided**: neither right-sided nor left-sided (e.g., Example 5.3)
Fig. 5.5: Finite-duration sequences
Figure 5.6: Infinite-duration sequences
<table>
<thead>
<tr>
<th>Sequence</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta[n]$</td>
<td>$1$</td>
<td>$\text{All } z$</td>
</tr>
<tr>
<td>$\delta[n-m]$</td>
<td>$z^{-m}$</td>
<td>$</td>
</tr>
<tr>
<td>$a^n u[n]$</td>
<td>$\frac{1}{1 - az^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$-a^n u[-n-1]$</td>
<td>$\frac{1}{1 - az^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$na^n u[n]$</td>
<td>$\frac{az^{-1}}{(1 - az^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>$-na^n u[-n-1]$</td>
<td>$\frac{az^{-1}}{(1 - az^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>$a^n \cos(bn) u[n]$</td>
<td>$\frac{1 - a \cos(b)z^{-1}}{1 - 2a \cos(b)z^{-1} + a^2 z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>$a^n \sin(bn) u[n]$</td>
<td>$\frac{a \sin(b)z^{-1}}{1 - 2a \cos(b)z^{-1} + a^2 z^{-2}}$</td>
<td>$</td>
</tr>
</tbody>
</table>

Table 5.1: $z$ transforms for common sequences
Eight ROC properties are:

P1. There are four possible shapes for ROC, namely, the entire region except possibly \( z = 0 \) and/or \( z = \infty \), a ring, or inside or outside a circle in the \( z \)-plane centered at the origin (e.g., Figures 5.5 and 5.6)

P2. The DTFT of a sequence \( x[n] \) exists if and only if the ROC of the \( z \) transform of \( x[n] \) includes the unit circle (e.g., Examples 5.1 and 5.2)

P3: The ROC cannot contain any poles (e.g., Examples 5.1 to 5.5)

P4: When \( x[n] \) is a finite-duration sequence, the ROC is the entire \( z \)-plane except possibly \( z = 0 \) and/or \( z = \infty \) (e.g., Examples 5.4 and 5.5)
P5: When \( x[n] \) is a right-sided sequence, the ROC is of the form \(|z| > |p_{\text{max}}|\) where \( p_{\text{max}} \) is the pole with the largest magnitude in \( X(z) \) (e.g., Example 5.1)

P6: When \( x[n] \) is a left-sided sequence, the ROC is of the form \(|z| < |p_{\text{min}}|\) where \( p_{\text{min}} \) is the pole with the smallest magnitude in \( X(z) \) (e.g., Example 5.2)

P7: When \( x[n] \) is a two-sided sequence, the ROC is of the form \(|p_a| < |z| < |p_b|\) where \( p_a \) and \( p_b \) are two poles with the successive magnitudes in \( X(z) \) such that \(|p_a| < |p_b|\) (e.g., Example 5.3)

P8: The ROC must be a connected region

Example 5.6
A \( z \) transform \( X(z) \) contains three poles, namely, \( a, b \) and \( c \) with \(|a| < |b| < |c|\). Determine all possible ROCs.
Fig. 5.7: ROC possibilities for three poles
What are other possible ROCs?

**Inverse z Transform**

Inverse $z$ transform corresponds to finding $x[n]$ given $X(z)$ and its ROC.

The $z$ transform and inverse $z$ transform are one-to-one mapping provided that the ROC is given:

$$x[n] \leftrightarrow X(z) \quad (5.20)$$

There are 4 commonly used techniques to evaluate the inverse $z$ transform. They are

1. **Inspection**
2. **Partial Fraction Expansion**
3. **Power Series Expansion**
4. **Cauchy Integral Theorem**
**Inspection**

When we are familiar with certain transform pairs, we can do the inverse \( z \) transform by inspection.

**Example 5.7**

Determine the inverse \( z \) transform of \( X(z) \) which is expressed as:

\[
X(z) = \frac{z}{2z - 1}, \quad |z| > 0.5
\]

We first rewrite \( X(z) \) as:

\[
X(z) = \frac{0.5}{1 - 0.5z^{-1}}
\]
Making use of the following transform pair in Table 5.1:

\[ a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a| \]

and putting \( a = 0.5 \), we have:

\[
\frac{0.5}{1 - 0.5z^{-1}} \leftrightarrow 0.5(0.5)^n u[n]
\]

By inspection, the inverse \( z \) transform is:

\[ x[n] = (0.5)^{n+1} u[n] \]
Partial Fraction Expansion

It is useful when \( X(z) \) is a rational function in \( z^{-1} \):

\[
X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{5.21}
\]

For pole and zero determination, it is advantageous to multiply \( z^{M+N} \) to both numerator and denominator:

\[
X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} \tag{5.22}
\]
- When \( M > N \), there are \((M - N)\) pole(s) at \( z = 0 \)
- When \( M < N \), there are \((N - M)\) zero(s) at \( z = 0 \)

To obtain the partial fraction expansion from (5.21), the first step is to determine the \( N \) nonzero poles, \( c_1, c_2, \cdots, c_N \)

There are 4 cases to be considered:

Case 1: \( M < N \) and all poles are of **first order**

For first-order poles, all \( \{c_k\} \) are distinct. \( X(z) \) is:

\[
X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \tag{5.23}
\]

For each first-order term of \( A_k / (1 - c_k z^{-1}) \), its inverse \( z \) transform can be easily obtained by inspection
Multiplying both sides by \((1 - c_k z^{-1})\) and evaluating for \(z = c_k\)

\[
A_k = (1 - c_k z^{-1}) X(z) \bigg|_{z=c_k} \quad (5.24)
\]

An illustration for computing \(A_1\) with \(N = 2 > M\) is:

\[
X(z) = \frac{A_1}{1 - c_1 z^{-1}} + \frac{A_2}{1 - c_2 z^{-1}}
\]

\[
\Rightarrow (1 - c_1 z^{-1}) X(z) = A_1 + \frac{A_2 (1 - c_1 z^{-1})}{1 - c_2 z^{-1}} \quad (5.25)
\]

Substituting \(z = c_1\), we get \(A_1\)

In summary, three steps are:

- Find poles
- Find \(\{A_k\}\)
- Perform inverse \(z\) transform for the fractions by inspection
Example 5.8
Find the pole and zero locations of $H(z)$:

$$H(z) = -\frac{1 + 0.1z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$

Then determine the inverse $z$ transform of $H(z)$.

We first multiply $z^2$ to both numerator and denominator polynomials to obtain:

$$H(z) = -\frac{z(z + 0.1)}{z^2 - 2.05z + 1}$$

Apparently, there are two zeros at $z = 0$ and $z = -0.1$. On the other hand, by solving the quadratic equation at the denominator polynomial, the poles are determined as $z = 0.8$ and $z = 1.25$. 
According to (5.23), we have:

\[ H(z) = \frac{A_1}{1 - 0.8z^{-1}} + \frac{A_2}{1 - 1.25z^{-1}} \]

Employing (5.24), \( A_1 \) is calculated as:

\[ A_1 = (1 - 0.8z^{-1}) H(z) \bigg|_{z=0.8} = -\frac{1 + 0.1z^{-1}}{1 - 1.25z^{-1}} \bigg|_{z=0.8} = 2 \]

Similarly, \( A_2 \) is found to be \(-3\). As a result, the partial fraction expansion for \( H(z) \) is

\[ H(z) = \frac{2}{1 - 0.8z^{-1}} - \frac{3}{1 - 1.25z^{-1}} \]

As the ROC is not specified, we investigate all possible scenarios, namely, \( |z| > 1.25 \), \( 0.8 < |z| < 1.25 \), and \( |z| < 0.8 \).
For $|z| > 1.25$, we notice that

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1 - 0.8z^{-1}}, \quad |z| > 0.8$$

and

$$(1.25)^n u[n] \leftrightarrow \frac{1}{1 - 1.25z^{-1}}, \quad |z| > 1.25$$

where both ROCs agree with $|z| > 1.25$. Combining the results, the inverse $z$ transform $h[n]$ is:

$$h[n] = (2(0.8)^n - 3(1.25)^n) u[n]$$

which is a right-sided sequence and aligns with P5.

For $0.8 < |z| < 1.25$, we make use of

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1 - 0.8z^{-1}}, \quad |z| > 0.8$$

and
\[-(1.25)^n u[-n - 1] \leftrightarrow \frac{1}{1 - 1.25z^{-1}}, \quad |z| < 1.25\]

where both ROCs agree with \(0.8 < |z| < 1.25\). This implies:

\[h[n] = 2(0.8)^n u[n] + 3(1.25)^n u[-n - 1]\]

which is a two-sided sequence and aligns with P7.

Finally, for \(|z| < 0.8\):

\[-(0.8)^n u[-n - 1] \leftrightarrow \frac{1}{1 - 0.8z^{-1}}, \quad |z| < 0.8\]

and

\[-(1.25)^n u[-n - 1] \leftrightarrow \frac{1}{1 - 1.25z^{-1}}, \quad |z| < 1.25\]

where both ROCs agree with \(|z| < 0.8\). As a result, we have:

\[h[n] = (-2(0.8)^n + 3(1.25)^n) u[-n - 1]\]

which is a left-sided sequence and aligns with P6.
Suppose $h[n]$ is the impulse response of a discrete-time LTI system. Recall (3.15) and (3.16):

$$h[n] = 0, \quad n < 0$$

and

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

The three possible impulse responses:

- $h[n] = (2(0.8)^n - (1.25)^n) u[n]$ is the impulse response of a causal but unstable system

- $h[n] = 2(0.8)^n u[n] + (1.25)^n u[-n - 1]$ corresponds to a noncausal but stable system

- $h[n] = (-2(0.8)^n + (1.25)^n) u[-n - 1]$ is noncausal and unstable

Which of the $h[n]$ has/have DTFT?
Case 2: $M \geq N$ and all poles are of first order

In this case, $X(z)$ can be expressed as:

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}}$$  \hspace{2cm} (5.26)

- $B_l$ are obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator.

- $A_k$ can be obtained using (5.24).

Example 5.9
Determine $x[n]$ which has $z$ transform of the form:

$$X(z) = \frac{4 - 2z^{-1} + z^{-2}}{1 - 1.5z^{-1} + 0.5z^{-2}}, \quad |z| > 1$$
The poles are easily determined as $z = 0.5$ and $z = 1$

According to (5.26) with $M = N = 2$:

$$X(z) = B_0 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

The value of $B_0$ is found by dividing the numerator polynomial by the denominator polynomial as follows:

$$\frac{2}{0.5z^{-2} - 1.5z^{-1} + 1} \frac{2}{z^{-2} - 2z^{-1} + 4} \frac{1}{z^{-2} - 3z^{-1} + 2} \frac{1}{z^{-1} + 2}$$

That is, $B_0 = 2$. Thus $X(z)$ is expressed as

$$X(z) = 2 + \frac{2 + z^{-1}}{(1 - 0.5z^{-1})(1 - z^{-1})} = 2 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 - z^{-1}}$$
According to (5.24), $A_1$ and $A_2$ are calculated as

$$A_1 = \left. \frac{4 - 2z^{-1} + z^{-2}}{1 - z^{-1}} \right|_{z=0.5} = -4$$

and

$$A_2 = \left. \frac{4 - 2z^{-1} + z^{-2}}{1 - 0.5z^{-1}} \right|_{z=1} = 6$$

With $|z| > 1$:

$$\delta[n] \leftrightarrow 1$$

$$(0.5)^n u[n] \leftrightarrow \frac{1}{1 - 0.5z^{-1}}, \quad |z| > 0.5$$

and

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

the inverse $z$ transform $x[n]$ is:

$$x[n] = 2\delta[n] - 4(0.5)^n u[n] + 6u[n]$$
Case 3: \( M < N \) with multiple-order pole(s)

If \( X(z) \) has a \( s \)-order pole at \( z = c_i \) with \( s \geq 2 \), this means that there are \( s \) repeated poles with the same value of \( c_i \). \( X(z) \) is:

\[
X(z) = \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - c_i z^{-1})^m} \tag{5.27}
\]

- When there are two or more multiple-order poles, we include a component like the second term for each corresponding pole
- \( A_k \) can be computed according to (5.24)
- \( C_m \) can be calculated from:

\[
C_m = \frac{1}{(s - m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}} \bigg|_{w=c_i^{-1}} \left[ (1 - c_i w)^s X(w^{-1}) \right] \tag{5.28}
\]
Example 5.10
Determine the partial fraction expansion for $X(z)$:

$$X(z) = \frac{4}{(1 + z^{-1})(1 - z^{-1})^2}$$

It is clear that $X(z)$ corresponds to Case 3 with $N = 3 > M$ and one second-order pole at $z = 1$. Hence $X(z)$ is:

$$X(z) = \frac{A_1}{1 + z^{-1}} + \frac{C_1}{1 - z^{-1}} + \frac{C_2}{(1 - z^{-1})^2}$$

Employing (5.24), $A_1$ is:

$$A_1 = \frac{4}{(1 - z^{-1})^2} \bigg|_{z=-1} = 1$$
Applying (5.28), $C_1$ is:

\[
C_1 = \frac{1}{(2 - 1)!(-1)^{2-1}} \cdot \frac{d}{dw} \left[ (1 - 1 \cdot w)^2 \frac{4}{(1 + w)(1 - w)^2} \right] \bigg|_{w=1}
\]

\[
= - \frac{d}{dw} \frac{4}{1 + w} \bigg|_{w=1}
\]

\[
= \frac{4}{(1 + w)^2} \bigg|_{w=1}
\]

\[
= 1
\]

and

\[
C_2 = \frac{1}{(2 - 2)!(-1)^{2-2}} \cdot \left[ (1 - 1 \cdot w)^2 \frac{4}{(1 + w)(1 - w)^2} \right] \bigg|_{w=1}
\]

\[
= \frac{4}{1 + w} \bigg|_{w=1}
\]

\[
= 2
\]
Therefore, the partial fraction expansion for $X(z)$ is

$$X(z) = \frac{1}{1 + z^{-1}} + \frac{1}{1 - z^{-1}} + \frac{2}{(1 - z^{-1})^2}$$

Case 4: $M \geq N$ with multiple-order pole(s)

This is the most general case and the partial fraction expansion of $X(z)$ is

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - c_i z^{-1})^m} \quad (5.29)$$

assuming that there is only one multiple-order pole of order $s \geq 2$ at $z = c_i$. It is easily extended to the scenarios when there are two or more multiple-order poles as in Case 3. The $A_k$, $B_l$ and $C_m$ can be calculated as in Cases 1, 2 and 3.
Power Series Expansion

When \( X(z) \) is expanded as power series according to (5.1):

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \tag{5.30}
\]

any particular value of \( x[n] \) can be determined by finding the coefficient of the appropriate power of \( z^{-1} \)

**Example 5.11**

Determine \( x[n] \) which has \( z \) transform of the form:

\[
X(z) = 2z^2 (1 - 0.5z^{-1}) (1 + z^{-1}) (1 - z^{-1}), \quad 0 < |z| < \infty
\]

Expanding \( X(z) \) yields

\[
X(z) = 2z^2 - z - 2 + z^{-1}
\]

From (5.30), \( x[n] \) is deduced as:

\[
x[n] = 2\delta[n+2] - \delta[n+1] - 2\delta[n] + \delta[n-1]
\]
Example 5.12

Determine $x[n]$ whose $z$ transform is given as:

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

With the use of the power series expansion for $\log(1 + \lambda)$:

$$\log(1 + \lambda) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \lambda^n}{n}, \quad |\lambda| < 1$$

$X(z)$ with $|az^{-1}| < 1$ can be expressed as

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

From (5.30), $x[n]$ is deduced as:

$$x[n] = \frac{(-1)^{n+1} a^n}{n} u[n - 1]$$
Example 5.13
Determine \( x[n] \) whose \( z \) transform has the form of:

\[
X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|
\]

With the use of

\[
\frac{1}{1 - \lambda} = 1 + \lambda + \lambda^2 + \cdots, \quad |
\lambda| < 1
\]

Carrying out long division in \( X(z) \) with \( |az^{-1}| < 1 \):

\[
X(z) = 1 + az^{-1} + (az^{-1})^2 + \cdots
\]

From (5.30), \( x[n] \) is deduced as:

\[
x[n] = a^n u[n]
\]

which agrees with Example 5.1 and Table 5.1
Example 5.14
Determine $x[n]$ whose $z$ transform has the form of:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

We first express $X(z)$ as:

$$X(z) = \frac{-a^{-1}z}{-a^{-1}z} \cdot \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z}$$

Carrying out long division in $X(z)$ with $|a^{-1}z| < 1$:

$$X(z) = -a^{-1}z \left(1 + a^{-1}z + (a^{-1}z)^2 + \cdots\right)$$

From (5.30), $x[n]$ is deduced as:

$$x[n] = -a^n u[-n - 1]$$

which agrees with Example 5.2 and Table 5.1
Properties of z Transform

1. Linearity

Let \((x_1[n], X_1(z))\) and \((x_2[n], X_2(z))\) be two \(z\) transform pairs with ROCs \(\mathcal{R}_{x_1}\) and \(\mathcal{R}_{x_2}\), respectively, we have

\[
a x_1[n] + b x_2[n] \leftrightarrow a X_1(z) + b X_2(z)
\]  

(5.31)

Its ROC is denoted by \(\mathcal{R}\), which includes \(\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}\) where \(\cap\) is the intersection operator. That is, \(\mathcal{R}\) contains at least the intersection of \(\mathcal{R}_{x_1}\) and \(\mathcal{R}_{x_2}\).

Example 5.15
Determine the \(z\) transform of \(y[n]\) which is expressed as:

\[
y[n] = x_1[n] + x_2[n]
\]

where \(x_1[n] = (0.2)^n u[n]\) and \(x_2[n] = (-0.3)^n u[n]\). By inspection,
the $z$ transforms of $x_1[n]$ and $x_2[n]$ are:

$$x_1[n] = (0.2)^n u[n] \leftrightarrow \frac{1}{1 - 0.2z^{-1}}, \quad |z| > 0.2$$

and

$$x_2[n] = (-0.3)^n u[n] \leftrightarrow \frac{1}{1 + 0.3z^{-1}}, \quad |z| > 0.3$$

According to the linearity property, the $z$ transform of $y[n]$ is

$$Y(z) = \frac{1}{1 - 0.2z^{-1}} + \frac{1}{1 + 0.3z^{-1}}, \quad |z| > 0.3$$

2. Time Shifting

A time-shift of $n_0$ in $x[n]$ causes a multiplication of $z^{-n_0}$ in $X(z)$

$$x[n - n_0] \leftrightarrow z^{-n_0}X(z) \quad (5.32)$$

The ROC for $x[n - n_0]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z = 0$ or $z = \infty$
Example 5.16
Find the $z$ transform of $x[n]$ which has the form of:

$$x[n] = a^{n-1}u[n - 1]$$

Employing the time-shifting property with $n_0 = 1$ and:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

we easily obtain

$$a^{n-1}u[n - 1] \leftrightarrow z^{-1} \cdot \frac{1}{1 - az^{-1}} = \frac{z^{-1}}{1 - az^{-1}}, \quad |z| > |a|$$

Note that using (5.1) with $|z| > |a|$ also produces the same result but this approach is less efficient:

$$X(z) = \sum_{n=1}^{\infty} a^{n-1}z^{-n} = a^{-1} \sum_{n=1}^{\infty} (az^{-1})^n = a^{-1} \frac{az^{-1} \left[1 - (az^{-1})^\infty\right]}{1 - az^{-1}} = \frac{z^{-1}}{1 - az^{-1}}$$
3. Multiplication by an Exponential Sequence (Modulation)

If we multiply $x[n]$ by $z_0^n$ in the time domain, the variable $z$ will be changed to $z/z_0$ in the $z$ transform domain. That is:

$$z_0^n x[n] \leftrightarrow X(z/z_0) \quad (5.33)$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, the ROC for $z_0^n x[n]$ is $|z_0|R_+ < |z| < |z_0|R_-$

**Example 5.17**

With the use of the following $z$ transform pair:

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

Find the $z$ transform of $x[n]$ which has the form of:

$$x[n] = a^n \cos(bn)u[n]$$
Noting that $\cos(bn) = (e^{jbn} + e^{-jbn})/2$, $x[n]$ can be written as:

$$x[n] = \frac{1}{2} (ae^{j b})^n u[n] + \frac{1}{2} (ae^{-j b})^n u[n]$$

By means of the modulation property of (5.33) with the substitution of $z_0 = ae^{j b}$ and $z_0 = ae^{-j b}$, we obtain:

$$\frac{1}{2} (ae^{j b})^n u[n] \leftrightarrow \frac{1}{21 - (z/(ae^{j b}))^{-1}} = \frac{1}{21 - ae^{j b}z^{-1}}, \quad |z| > |a|$$

and

$$\frac{1}{2} (ae^{-j b})^n u[n] \leftrightarrow \frac{1}{21 - (z/(ae^{-j b}))^{-1}} = \frac{1}{21 - ae^{-j b}z^{-1}}, \quad |z| > |a|$$

By means of the linearity property, it follows that

$$X(z) = \frac{1}{21 - ae^{j b}z^{-1}} + \frac{1}{21 - ae^{-j b}z^{-1}} = \frac{1 - a \cos(b)z^{-1}}{1 - 2a \cos(b)z^{-1} + a^2 z^{-2}}, \quad |z| > |a|$$

which agrees with Table 5.1.
4. Differentiation

Differentiating $X(z)$ with respect to $z$ corresponds to multiplying $x[n]$ by $n$ in the time domain:

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \quad (5.34)$$

The ROC for $nx[n]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z = 0$ or $z = \infty$.

Example 5.18
Determine the $z$ transform of $x[n] = na^n u[n]$.

Since

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

and
By means of the differentiation property, we have

\[ \frac{d}{dz} \left( \frac{1}{1 - az^{-1}} \right) = \frac{d}{d(1 - az^{-1})} \cdot \frac{d(1 - az^{-1})}{dz} = -\frac{az^{-2}}{(1 - az^{-1})^2} \]

which agrees with Table 5.1.

5. Conjugation

The \( \mathcal{Z} \) transform pair for \( x^*[n] \) is:

\[ x^*[n] \leftrightarrow X^*(\bar{z}^*) \quad (5.35) \]

The ROC for \( x^*[n] \) is identical to that of \( x[n] \)
6. Time Reversal

The $z$ transform pair for $x[-n]$ is:

$$x[-n] \leftrightarrow X(z^{-1})$$  \hspace{1cm} (5.36)

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, the ROC for $x[-n]$ is $1/R_- < |z| < 1/R_+$

Example 5.19
Determine the $z$ transform of $x[n] = -na^{-n}u[-n]$

Using Example 5.18:

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1 - az^{-1})^2}, \hspace{0.5cm} |z| > |a|$$

and from the time reversal property:

$$X(z) = \frac{az}{(1 - az)^2} = \frac{a^{-1}z^{-1}}{(1 - a^{-1}z^{-1})^2}, \hspace{0.5cm} |z| < |a^{-1}|$$
7. Convolution

Let \((x_1[n], X_1(z))\) and \((x_2[n], X_2(z))\) be two \(z\) transform pairs with ROCs \(\mathcal{R}_{x_1}\) and \(\mathcal{R}_{x_2}\), respectively. Then we have:

\[
x_1[n] \otimes x_2[n] \leftrightarrow X_1(z)X_2(z)
\]  
(5.37)

and its ROC includes \(\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}\).

The proof is given as follows.

Let

\[
y[n] = x_1[n] \otimes x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n - k]
\]  
(5.38)

With the use of the time shifting property, \(Y(z)\) is:
Transfer Function of Linear Time-Invariant System

A LTI system can be characterized by the transfer function, which is a $z$ transform expression.
Starting with:

\[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]  \hspace{1cm} (5.40)

Applying \( z \) transform on \( (5.40) \) with the use of the linearity and time shifting properties, we have

\[ Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k} \]  \hspace{1cm} (5.41)

The transfer function, denoted by \( H(z) \), is defined as:

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \]  \hspace{1cm} (5.42)
The system impulse response $h[n]$ is given by the inverse $z$ transform of $H(z)$ with an appropriate ROC, that is, $h[n] \leftrightarrow H(z)$, such that $y[n] = x[n] \otimes h[n]$. This suggests that we can first take the $z$ transforms for $x[n]$ and $h[n]$, then multiply $X(z)$ by $H(z)$, and finally perform the inverse $z$ transform of $X(z)H(z)$.

Example 5.20
Determine the transfer function for a LTI system whose input $x[n]$ and output $y[n]$ are related by:

$$y[n] = 0.1y[n - 1] + x[n] + x[n - 1]$$

Applying $z$ transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z) \left(1 - 0.1z^{-1}\right) = X(z) \left(1 + z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - 0.1z^{-1}}$$
Note that there are two ROC possibilities, namely, $|z| > 0.1$ and $|z| < 0.1$ and we cannot uniquely determine $h[n]$

**Example 5.21**
Find the difference equation of a LTI system whose transfer function is given by

$$H(z) = \frac{(1 + z^{-1})(1 - 2z^{-1})}{(1 - 0.5z^{-1})(1 + 2z^{-1})}$$

Let $H(z) = Y(z)/X(z)$. Performing cross-multiplication and inverse $z$ transform, we obtain:

$$(1 - 0.5z^{-1})(1 + 2z^{-1}) Y(z) = (1 + z^{-1})(1 - 2z^{-1}) X(z)$$

$\Rightarrow (1 + 1.5z^{-1} - z^{-2}) Y(z) = (1 - z^{-1} - 2z^{-2}) X(z)$


Examples 5.20 and 5.21 imply the equivalence between the difference equation and transfer function
Example 5.22
Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n - 1]$$

Applying $z$ transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z) = X(z) \left(1 - z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}$$

There is only one ROC possibility, namely, $|z| > 0$. Taking the inverse $z$ transform on $H(z)$, we get:

$$h[n] = \delta[n] - \delta[n - 1]$$

which agrees with Example 3.12
Example 5.23
Determine the output $y[n]$ if the input is $x[n] = u[n]$ and the LTI system impulse response is $h[n] = \delta[n] + 0.5\delta[n - 1]$

The $z$ transforms for $x[n]$ and $h[n]$ are

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

and

$$H(z) = 1 + 0.5z^{-1} \quad |z| > 0$$

As a result, we have:

$$Y(z) = X(z)H(z) = \frac{1}{1 - z^{-1}} + 0.5\frac{z^{-1}}{1 - z^{-1}}, \quad |z| > 1$$

Taking the inverse $z$ transform of $Y(z)$ with the use of the time shifting property yields:

$$y[n] = u[n] + 0.5u[n - 1]$$