COMPARISON OF VARIOUS PERIODOGRAMS FOR SINGLE TONE DETECTION AND FREQUENCY ESTIMATION

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ABSTRACT

Periodogram is a simple and efficient way to detect and estimate parameters of sinusoidal signals. In this paper, we attempt to determine the optimum periodogram implementation for a single real tone detection in the presence of white Gaussian noise, among its variants such as the Bartlett’s procedure and Welch method. The possibility of delaying threshold effect using different realizations for frequency estimation is also investigated. Simulation results show that the standard periodogram generally gives the best detection performance and it provides the minimum mean square frequency estimation error for all signal-to-noise ratio conditions.

1 INTRODUCTION

Detection and frequency estimation of sinusoidal signals from a finite number of noisy discrete-time observations have applications in many fields. It has been widely used in sonar and radar where we need to detect a moving target. Estimation of Doppler shift in this case is usually required in order to find its position and speed. A recent and interesting application is in the search for extraterrestrial intelligence (SETI) [1]. Other well known examples that makes use of tone detection and estimation include demodulation of frequency-shift keying signals, wind profiling [2], geolocation, and identification and tracking of an emergency location transmitter [3].

The periodogram and variations of it [4] are standards for spectrum analysis because the computation is especially efficient if FFT is employed. Samples of the periodogram of an N-point input sequence x(n), at the frequencies f_k = k/N, are given by

\[ P_x(k/N) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} \right|^2, \quad k = 0, 1, ..., N - 1 \]  

(1)

The values at other frequencies can be evaluated by either zero padding or interpolation. In the Bartlett method, the sequence x(n) is divided into K nonoverlapping segments, where each segment has length M. For each segment, the periodogram is computed and the Bartlett power spectrum estimate is obtained by averaging the periodograms for the K segments. By so doing, the variance in the periodogram estimate is reduced by a factor K but at the expense of reducing the frequency resolution by K. Welch had modified the Bartlett’s procedure by allowing the data segments to overlap and at the same time to be multiplied by a window function prior to computing the periodogram. The overlapping is used for further reducing the periodogram variance while the windowing is applied to reduce the spectral leakage [5] associated with finite observation intervals.

Although the many forms of periodogram have been derived for more than two decades, there is not a comprehensive study of these approaches in tone detection and frequency estimation. In this paper, the standard periodogram with and without windowing, Bartlett’s procedure and Welch method are compared in order to determine the best detector for a sinusoid in the presence of white Gaussian noise. We would also like to check if these modifications can delay the threshold effect of the conventional periodogram in frequency estimation of a noisy sinusoid, even though the latter provides the maximum likelihood estimate. Theoretical development for single tone detection and frequency estimation are given. In particular, the detection probability, the false alarm probability and the mean square frequency error of the periodogram and the Bartlett method are derived. Simulation results are presented to corroborate the analytical derivations and to evaluate the detection and estimation performance of different periodograms.

2 DETECTION

The single sinusoid detection problem is formulated as follows. Given N samples of a received signal x(n), it is required to decide between the hypotheses:

\[ H_0 : \quad x(n) = q(n), \quad n = 0, 1, ..., N - 1 \]

\[ H_1 : \quad x(n) = \alpha \sin(2\pi f_o n + \phi) + q(n) \]  

(2)

where q(n) is a white Gaussian process while \( \alpha \), f_o and \( \phi \) are unknown constants which represent the tone
amplitude, frequency and phase respectively. For simplicity, we assume that \( f_o \in (0, 0.5) \) and corresponds exactly to one of the FFT bins, \( k_o \), so that \( f_o = k_o/N \).

The periodogram detector consists of three steps, namely, compute the DFT power spectrum as given by (1), select the largest coefficient, and compare it with a threshold \( V_T \). If the peak coefficient is larger than \( V_T \), \( H_1 \) is accepted and \( H_0 \) is chosen otherwise. Here, we only need to compute the spectral coefficients for \( k = 1, 2, ..., N/2 - 1 \), because of the symmetric power spectrum. To study the detection performance, the probability density functions (PDFs) of \( P_z \) are required in order to calculate the false alarm probability \( P_{FA} = P(D_1|H_0) \) and detection probability \( P_D = P(D_1|H_1) \), where \( D_i \) represents the decision of hypothesis \( H_i \). Denote \( P_{z_1}(k/N) \) and \( P_{z_2}(k/N) \) as the spectral coefficients for the noise only case and the signal present case respectively. The value of \( P_{z_1}(k/N) \) is computed as

\[
P_{z_1}(k_o/N) = A^2 + B^2
\]

where

\[
A = \sqrt{N} \sum_{n=0}^{N-1} \left( \alpha \sin(2\pi f_o n + \phi) + q(n) \right) \cos\left(\frac{2\pi n k_o}{N}\right),
\]

\[
B = \sqrt{N} \sum_{n=0}^{N-1} \left( \alpha \sin(2\pi f_o n + \phi) + q(n) \right) \sin\left(\frac{2\pi n k_o}{N}\right)
\]

are real Gaussian variables. It can be shown [6] that \( A \) and \( B \) are independent and of identical variances equal to \( \sigma^2 = \sigma^2_0/2 \) where \( \sigma^2_0 \) represents the power of \( q(n) \). The PDF of \( P_{z_1}(k_o/N) \), denoted by \( p_{z_1}(u) \), is noncentral Chi-square and is of the form [7]

\[
p_{z_1}(u) = \frac{1}{2\sigma^2 e^{-\frac{u}{2\sigma^2}}} I_0\left(\sqrt{\frac{u s^2}{\sigma^2}}\right)
\]

where \( s^2 \) equals the squared sum of the mean of \( A \) and \( B \), that is, \( \alpha^2 N/4 \), and \( I_r \) is the \( r \)th-order modified Bessel function.

Similarly, it can be shown that the remaining \( P_{z_2}(k/N) \) and all \( P_{z_0}(k/N) \) are of central Chi-square distribution and have the same PDF function, \( p_{z_0}(u) \), given by

\[
p_{z_0}(u) = \frac{1}{2\sigma^2 e^{-\frac{u}{2\sigma^2}}}
\]

Note that in [8], the PDFs of the Fourier transform coefficients of a complex tone signal have been derived and they are similar to (4) and (5).

Assume that the DFT spectral coefficients are independent, the false alarm rate \( P_{FA} \) is calculated as

\[
P_{FA} = 1 - \left(1 - e^{-\frac{V_T}{\sigma}}\right)^{N/2-1}
\]

and the probability of detection \( P_D \) is given by

\[
P_D = 1 - \left(1 - e^{-\frac{V_T}{\sigma}}\right)^{N/2-2}(1 - Q_1\left(\frac{\sqrt{V_T}}{\sigma}\right))
\]

where \( Q \) denotes the generalized \( Q \) function.

When the Bartlett method is used instead of the standard periodogram, the number of the power coefficients are reduced to \( M \) because of averaging. By further assuming \( f_o = k_o/M \) where \( k_o \in [1, M/2 - 1] \) and following the above derivations, it can be shown that when \( H_1 \) is true, the PDF of the spectral coefficients at \( k = k_o \), denoted by \( p_{z_1}(u) \), is of the form

\[
p_{z_1}(u) = \frac{1}{2\sigma^2_0} \left(\frac{u}{s^2} \right)^{k_o-1/2} e^{-\frac{u^2}{2\sigma^2}} I_{K-1}\left(\sqrt{u s^2}/\sigma_0\right)
\]

where \( \sigma^2_0 = \sigma^2/(2K) \) and \( s^2 = \alpha\sqrt{M}/2 \). The PDF of all other coefficients for hypotheses \( H_0 \) and \( H_1 \) are equal to

\[
p_{z_2}(u) = \frac{1}{\sigma^2 K 2\pi^2 \Gamma(K)} u^{K-1} e^{-\frac{u^2}{2\sigma^2}}
\]

where \( \Gamma \) is the gamma function. As a result, the false alarm and detection probability of single tone detection using the Bartlett method, \( P_{FA} \) and \( P_D \), are given by

\[
P_{FA} = 1 - \left[\int_0^{V_T} p_{z_2}(u) du\right]^{N/2-1}
\]

and

\[
P_D = 1 - \left[\int_0^{V_T} p_{z_2}(u) du\right]^{N/2-2}\int_0^{V_T} p_{z_1}(u) du
\]

3 FREQUENCY ESTIMATION

In this section, the mean square frequency error using the periodogram and the Bartlett method is studied. Suppose \( N \) samples of the noisy sinusoid \( x(n) = \alpha \sin(2\pi f_o n + \phi) + q(n) \) are received. It is well known that the maximum likelihood frequency estimate is given by the location of the peak of the periodogram [8]. This estimator will attain the Cramer-Rao lower bound (CRLB) for frequency [9],

\[
var_f(N) = \frac{6\sigma^4}{\alpha^2 \pi^2 N(N^2 - 1)}
\]

when the signal-to-noise ratio (SNR) is greater than a threshold \( T_H \). When \( SNR < T_H \), however, the mean square frequency error rises very rapidly above the CRLB. It is because the excessive noise, in this case, has caused other false peaks to occur and as a result the true peak can hardly be located and this is known as occurrence of outliers. Assuming \( f_o = k_o/N \), the probability of this anomaly is derived as

\[
q = P(P_{z_1}(k_o/N) \leq \text{at least one of the remaining } P_{z_2}(k/N))
\]

\[
= \int_0^\infty P(P_{z_1}(k_o/N) = u)
\]
\[
\int_0^\infty \left[ 1 - \frac{2^k}{k^2} \prod_{k=1, k\neq k_0} p_x(k/N < u) \right] du
\]

which can be computed numerically by using Mathematica. Assuming the frequency \( f_o \) is uniformly distributed between 0 and 0.5, the overall mean square frequency error is thus given by

\[
msfe \triangleq (1-q) \text{var}_f(N) + q \int_0^{0.5} 2(u-f_o)^2 du \tag{14}
\]

Similarly, the mean square error in the frequency estimate using the Bartlett method, \( msfe_o \), equals

\[
msfe_o \triangleq \frac{1-q_o}{K} \text{var}_f(M) + q_o \int_0^{0.5} 2(u-f_o)^2 du \tag{15}
\]

where \( q_o \) denotes the probability of the occurrence of an outlier in the averaged periodogram, which is calculated using \( p_{sx}(u) \) and \( p_{ox}(u) \) as in the periodogram.

4 EXPERIMENTATION & DISCUSSION

Computer experiments were conducted to verify the theoretical derivations. The performance of different periodograms, including the windowed periodogram, the Bartlett and Welch method in single tone detection and frequency estimation were also evaluated. The variance of the white noise was fixed to unity while different SNRs were produced by properly scaling the tone amplitude \( \alpha \). Without loss of generality, the phase was set to 0. The total sample size \( N \) was 256 and the segment length \( M \) had a value of 64. The results for detection were averages of 12000 independent runs while those of frequency estimation were based on 500 independent trials.

Figure 1 shows the experimental and theoretical receiver operating characteristics (ROCs) in detecting a real sinusoid in the presence of white Gaussian noise using the standard periodogram. The simulation results were obtained by using the method suggested in [10]. Four values of SNR, namely, \(-9\) dB, \(-12\) dB, \(-15\) dB and \(-18\) dB were tried. The frequency \( f_o \) was chosen as 0.25. It can be seen that the simulation results agreed very well with the theoretical derivation. The test was repeated for the Bartlett method and Figure 2 illustrates, again, that the experimental and theoretical results were very similar. By comparing Figure 1 and 2, it can be seen that the periodogram provides a better detection performance than the Bartlett method.

Figure 3 compares the detection performance of different forms of periodograms for \( f_o = 0.25 \) and SNR \( = -12 \) dB. In the windowed periodogram and Welch method, the Hanning window function was used. It can be seen that the standard periodogram gave the best performance because it provided the largest SNR at \( f_o = 0.25 \) with no spectral leakage. The Welch method with 50% overlap was superior to the one without overlap and the windowed periodogram, but was inferior to the Bartlett’s procedure. Another similar experiment was conducted for a tone of frequency midway between two adjacent bins and the results are plotted in Figure 4. In this test, the amount of spectral leakage was maximized. It can be seen that the detectability of the 50% overlapped Welch method was comparable to the periodogram and they gave the best results. While the windowed periodogram and the Bartlett method performed fairly similar and were superior to the 0% overlapped Welch method.

Figure 5 plots the mean square frequency errors of the periodogram and the Bartlett method, together with the CRLB. The frequency \( f_o \) was assigned to 0.25. It can be observed that the theoretical values agreed with the simulation results in both methods except that they had earlier threshold effects. This may due to some of the simplifying assumptions made in obtaining (14) and (15), particularly that of a uniform
distribution of anomalies between 0 and 0.5. Moreover, at $\text{SNR} \geq -7 \text{ dB}$, the periodogram met the CRLB and had a mean square frequency error which was approximately one-tenth of the Bartlett method. Comparison of the five methods for frequency located in the middle of 2 bins is depicted in Figure 6. We see that the standard periodogram still provided the minimum mean square error, although the 50% overlapped Welch method had the smallest threshold. The second best was the windowed periodogram, the third were Welch method with 50% overlap, and Bartlett method and the 0% overlapped Welch method were the poorest frequency estimators.

To conclude, the conventional periodogram generally provides the best performance for single tone detection. Moreover, it gives the smallest mean square frequency error even when the threshold in frequency estimation is delayed by the 50% overlapped Welch method with the Hanning window.

References

Fig. 3 Comparison of ROCs for exact-bin frequency

Fig. 4 Comparison of ROCs for half-bin frequency

Fig. 5 Variances of frequency for exact-bin frequency

Fig. 6 Variances of frequency for half-bin frequency