A Closed Form Frequency Estimator for a Noisy Sinusoid

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Abstract - Based on the linear prediction property of sinusoidal signals, a closed form unbiased frequency estimator for a real sinusoid in white noise is proposed. The frequency estimate derives from minimizing a constrained least squares cost function and it is simple to implement either in batch or recursive mode. The relationship of the proposed method with the well-known Pisarenko harmonic decomposer (PHD) is also examined. Computer simulations are included to contrast the estimator performance with the PHD as well as the Cramér-Rao lower bound.

I. INTRODUCTION

Frequency estimation of sinusoidal signals in noise is a frequently addressed problem in the signal processing literature and it has a wide variety of applications such as angle of arrival estimation, demodulation of frequency-shift keying (FSK) signals, speech analysis and Doppler rate estimation [1]-[4].

For a noisy complex sinusoid, it is well known that [1] the maximum likelihood (ML) estimate of frequency is obtained from the periodogram maximum. Kenefic and Nuttall [5] have extended the problem to ML frequency estimation of a real tone and the optimum estimator maximizes a highly nonlinear and multimodal cost function. For both cases, ML methods involve extensive computations and this will be prohibited in applications where rapid frequency estimation is required. Apart from the ML approach, other frequency estimation techniques [3] include notch filtering, Capon methods, linear prediction, Yule-Walker methods and subspace-based approach. Among the subspace-based methods, the Pisarenko harmonic decomposer (PHD) [6] is of historical interest because it was the first to exploit the eigenstructure of the covariance matrix, and its performance has been extensively studied [6]-[10]. Interestingly, the PHD for a single real sinusoid can be implemented in a very simple way [11]-[12].

In this paper, we will focus on estimating the frequency of a real-valued tone in white noise. In Section II, the linear prediction approach for single sinusoidal

frequency estimation using least squares (LS) is first reviewed. It is observed that the standard LS cost function for a real-valued tone should be minimized subject to constraints in order to attain unbiased frequency estimation. We then develop a modified LS estimator which leads to unbiased as well as closed form frequency estimation. The proposed estimator is computationally attractive and on-line computation is allowed. Its relationship with the Pisarenko's method is investigated in Section III. Numerical examples are presented in Section IV to evaluate the performance of the proposed algorithm by comparing with the PHD. It is also shown that the estimator variance can approach the Cramér-Rao lower bound (CRLB) when the observation length is short and noise level is small. Finally, conclusions are drawn in Section V.

II. ESTIMATOR DEVELOPMENT

The noisy discrete-time measurements of the sinusoid are represented as

$$x(n) = \alpha \cos(\omega_0 n + \phi) + q(n), \quad n = 0, 1, \dots, N-1 \quad (1)$$

where the noise $q(n)$ is assumed to be a zero-mean white Gaussian process with variance σ_q^2 while $\alpha, \omega_0 \in (0, \pi)$ and $\phi \in [0, 2\pi)$ are unknown constants which denote the tone amplitude, frequency and phase, respectively. The integer N represents the number of available measurements. Without loss of generality, the sampling period is normalized to be unity. The task is to estimate ω_0 from the N samples of $x(n)$.

In the absence of noise, $x(n)$ is perfectly predictable from its past samples as:

$$x(n) = 2 \cos(\omega_0) x(n-1) - x(n-2) \quad (2)$$

Based on this linear prediction property, we can easily develop a LS estimator for ω_0 as follows,

$$\hat{\omega}_0 = \arg \min_{\hat{\omega}_0} \left\{ \sum_{n=2}^{N-1} e^2(n) \right\} \quad (3)$$

where $e(n)$ is an error function of the form

$$e(n) = x(n) - 2 \cos(\hat{\omega}_0) x(n-1) + x(n-2) \quad (4)$$

and $\hat{\omega}_0$ denotes the estimate of ω_0 .

However, this estimator is biased and it can be easily verified from the mean square value of $e(n)$, $E\{e^2(n)\}$, which is calculated as

$$E\{e^2(n)\} = 4(\cos(\hat{\omega}_0) - \cos(\omega_0))^2 \sigma_s^2 + 2(2 + \cos(2\hat{\omega}_0)) \sigma_q^2 \quad (5)$$

where $\sigma_s^2 = \alpha^2/2$ denotes the tone power, because the noise component is a function of $\hat{\omega}_0$. To remove the effect of noise, a number of methods [12]-[14] has been proposed to minimize $\sum_{n=2}^{N-1} e^2(n)$ or $E\{e^2(n)\}$ subject to the noise component of (5) equals a constant and they can be referred to as the unit-norm constraint approach.

Based on the unit-norm technique, we employ a modified error function, $\zeta(n)$, which has the form:

$$\zeta(n) = \frac{e(n)}{\sqrt{2(2 + \cos(2\hat{\omega}_0))}} \quad (6)$$

Using (5) and (6), it can be shown that the performance surface $E\{\zeta^2(n)\}$ has the minimum value of σ_q^2 at $\hat{\omega}_0 = \omega_0$. As a result, we form a new LS cost function for unbiased frequency estimation:

$$J_N(\hat{\omega}_0) = \sum_{n=2}^{N-1} \frac{e^2(n)}{2(2 + \cos(2\hat{\omega}_0))} \quad (7)$$

It is worthy to note that (7) can be considered as an alternate form of the modified least squares cost function suggested in [15] but a key difference is made by constraining the parameter vector as a function of the estimated frequency. In so doing, unlike [15], the solution can be directly derived.

Differentiating $J_N(\hat{\omega}_0)$ with respect to $\hat{\omega}_0$ and then setting the resultant expression to zero, we get a very nice equation

$$\begin{aligned} \sum_{n=2}^{N-1} e(n) ((x(n) + x(n-2)) \cos(\hat{\omega}_0) + x(n-1)) &= 0 \\ \Rightarrow 2A_N \cos^2(\hat{\omega}_0) - B_N \cos(\hat{\omega}_0) - A_N &= 0 \end{aligned} \quad (8)$$

where

$$A_N = \sum_{n=2}^{N-1} (x(n) + x(n-2)) x(n-1) \quad (9)$$

and

$$\begin{aligned} B_N &= x^2(N-1) - x^2(N-2) - x^2(1) + x^2(0) \\ &\quad + 2 \sum_{n=2}^{N-1} x(n) x(n-2) \end{aligned} \quad (10)$$

The frequency estimate is then computed from one of the roots of (8):

$$\hat{\omega}_0 = \cos^{-1} \left(\frac{B_N + \sqrt{B_N^2 + 8A_N^2}}{4A_N} \right) \quad (11)$$

Substituting the expected values of A_N and B_N , namely, $(N-2)\alpha^2 \cos(\omega_0)$ and $(N-2)\alpha^2 \cos(2\omega_0)$, respectively, into (11), we get $\hat{\omega}_0 = \omega_0$ and this implies that the proposed method gives an unbiased estimate of ω_0 .

Apart from the batch formula of (11), it is often desirable to compute the frequency estimate on a sample-by-sample basis particularly for real-time processing. From (9) and (10), it can easily be shown that A_N and B_N can be computed recursively using

$$A_N = A_{N-1} + x(N-2)(x(N-3) + x(N-1)) \quad (12)$$

and

$$\begin{aligned} B_N &= B_{N-1} + x^2(N-3) - 2x^2(N-2) \\ &\quad + x^2(N-1) + 2x(N-3)x(N-1) \end{aligned} \quad (13)$$

This on-line realization is simple to implement and is computationally attractive because only 8 additions, 7 multiplications, 1 division, 1 root operation and 1 arccos function are needed for each sampling interval.

III. COMPARISON WITH PHD

It is well known that the estimate of ω_0 in the PHD is found from the eigenvector corresponding to the smallest eigenvalue of the following 3×3 covariance matrix [7]-[8]:

$$\mathbf{R} = \begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{bmatrix} \quad (14)$$

where r_k is the sample covariance which has the form

$$r_k = \frac{1}{N-k} \sum_{n=0}^{N-k-1} x(n)x(n+k), \quad k = 0, 1, 2 \quad (15)$$

The eigenvector that corresponds to the smallest eigenvalue of $\mathbf{R}$ is given by

$$\mathbf{v} = \begin{bmatrix} 1 & \frac{-r_2 - \sqrt{r_2^2 + 8r_1^2}}{2r_1} & 1 \end{bmatrix} \quad (16)$$

and the PHD frequency estimate, denoted by $\hat{\omega}_0^o$, is computed as [10]-[11]

$$\hat{\omega}_0^o = \cos^{-1} \left(\frac{r_2 + \sqrt{r_2^2 + 8r_1^2}}{4r_1} \right) \quad (17)$$

Expressing A_N and B_N in terms of r_1 and r_2 yields

$$A_N = 2(N-2) \cdot \left(r_1 + \frac{2r_1 - x(0)x(1) - x(N-2)x(N-1)}{2(N-2)} \right) \quad (18)$$

$$\text{and} \quad B_N = 2(N-2) \cdot \left(r_2 + \frac{x^2(0) - x^2(1) - x^2(N-2) + x^2(N-1)}{2(N-2)} \right) \quad (19)$$

Comparing (11) and (17)-(19), it is found that the second terms of (18) and (19) contribute to the difference between the proposed algorithm and the original PHD. An obvious advantage of (11) observed from (15) and (17) is that on-line implementation of the PHD will be much more complicated than the proposed technique. Since (11) is very similar to the Pisarenko's method, although a novel derivation is used, we call this estimator as reformed Pisarenko harmonic decomposer (RPHD).

IV. SIMULATION RESULTS

Computer simulations had been carried out to evaluate the frequency estimation performance for a single real sinusoid of the proposed algorithm. We compared its accuracy with those of the original PHD [10]-[11] as well as the CRLB. All simulation results provided were averages of 500 independent runs. The tone amplitude was set to $\sqrt{2}$ and ϕ was a constant uniformly distributed between $[0, 2\pi)$ at each trial, while different signal-to-noise ratios (SNRs), defined as σ_s^2/σ_q^2 , were obtained by proper scaling the white Gaussian noise sequence.

Figure 1 shows the frequency variances of the RPHD and PHD as well the CRLB, for $\omega_0 \in [0.05\pi, 0.985\pi]$ at SNR = 20 dB and $N = 10$. We observe that the RPHD outperformed the PHD in spite of the similarity in their algorithms. Moreover, the variances of $\hat{\omega}_0$ exceeded the CRLB by less than 4 dB for $\omega_0 \in (0.2\pi, 0.8\pi)$ and it had the best performance at $\omega_0 \approx 0.5\pi$ where the degradation from the CRLB was approximately 1 dB. This means that the proposed method is a sub-optimum frequency estimator for a wide range of frequency when the SNR is high and observation interval is short.

The above test was repeated for $N = 400$ at SNR = 20 dB and the results are shown in Figures 2. It is seen that the RPHD was superior to the PHD even for a large sample size and the RPHD variances were less than those of the PHD by approximately 10 dB for $\omega_0 \in (0.2\pi, 0.8\pi)$. Note that the curve of RPHD variance differed from that of Figure 1 and it had minimum values at $\omega_0 \approx 0.28\pi$ and $\omega_0 \approx 0.72\pi$. Figures

3 plots the frequency variances of the RPHD and PHD versus SNR at $N = 10$ for $\omega_0 = 0.5\pi$. We observe that the superiority of the RPHD over the PHD increased with SNR. In addition, the deviation between the RPHD variances and CRLB had minimum values of around 1 dB for SNR ≥ 20 dB. The frequency variances of the RPHD and PHD versus N at SNR = 20 dB for $\omega_0 = 0.5\pi$ are shown in Figure 4. We see that the variances of the proposed method approached the CRLB for $N \in [4, 10]$ while the PHD was inferior to the RPHD by at least 11 dB for $N \geq 4$.

V. CONCLUSIONS

Linear prediction based least squares approach has been investigated for estimating the frequency of a single sinusoid embedded in white noise. A new closed form frequency estimator for a noisy real-valued tone has been devised via minimizing a constrained least square cost function. The frequency estimator is computationally attractive and is simple to implement either in batch or recursive mode. The relationship of the devised algorithm with the Pisarenko's method is also explored. It is shown that the proposed method is generally superior to the Pisarenko harmonic decomposer and can give a very good estimation performance when the observation length is short and noise level is small.

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Figure 1: Frequency variances versus ω_0 at SNR = 20 dB & $N = 10$

Figure 2: Frequency variances versus ω_0 at SNR = 20 dB & $N = 400$

Figure 3: Frequency variances versus SNR at $\omega_0 = 0.5\pi$ & $N = 10$

Figure 4: Frequency variances versus N at $\omega_0 = 0.5\pi$ & SNR = 20 dB