A Comparative Study of Three Recursive Least Squares Algorithms for Single-Tone Frequency Tracking

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May 30, 2003

Keywords : Adaptive Frequency Tracking, Real Sinusoid, Recursive Least Squares Algorithm

Abstract : Modified covariance, Pisarenko harmonic decomposition and reformed Pisarenko harmonic decomposition methods are three closed form frequency estimators, which are derived from the linear prediction property of sinusoidal signals. In this paper, we develop the recursive least squares type realizations of these estimators for a single real tone, and their frequency tracking performances are contrasted via computer simulations.

1 Introduction

Frequency estimation and tracking of sinusoidal signals in noise have applications in many areas [1]-[3] such as carrier and clock synchronization, angle-of-arrival estimation, demodulation of frequency-shift keying (FSK) signals, Doppler estimation of sonar wave return and speech analysis.

The modified covariance (MC) [4]-[5], Pisarenko harmonic decomposition (PHD) [6]-[7], and reformed Pisarenko harmonic decomposition (RPHD) [8] methods are three batch mode frequency estimators for a real-valued tone in white noise. Basically, the three methods have the following common characteristics: (i) Their derivation is based on the linear prediction (LP) property of sinusoidal signals; (ii) They provide closed form formulae for frequency estimation; and (iii) They are computationally simple. In this paper, we will focus on developing adaptive algorithms for frequency tracking of a real sinusoid via modifying the MC, PHD and RPHD methods.

The rest of the paper is organized as follows. Recursive least squares (RLS) type realizations of the MC, PHD and RPHD methods are developed in Section 2. Simulation results are presented in Section 3 to contrast the frequency tracking performances of the three modified estimators. Finally, conclusions are drawn in Section 4.

2 Recursive Least Squares Algorithms for Frequency Tracking

The signal model for single real tone frequency estimation is

$$x(n) = s(n) + q(n) \tag{1}$$

where

$$s(n) = \alpha \cos(\omega n + \phi) \tag{2}$$

The noise $q(n)$ is a zero-mean white process with unknown variance while α, $\omega \in (0, \pi)$ and $\phi \in [0, 2\pi)$ denote the unknown tone amplitude, frequency and phase, respectively. The task is to estimate the frequency, which is assumed time-varying, from the received signal $x(n)$.

The LP approach makes use of the simple recurrence of the sinusoidal signal:

$$s(n) = 2\cos(\omega)s(n-1) - s(n-2) \tag{3}$$

In the following, three closed form LP based frequency estimators, namely, the MC, PHD and RPHD methods, are modified in order to track the changing frequency.

A. Modified Covariance

From (3), we can generate a prediction error function of the form

$$e(n) = x(n) - 2\cos(\gamma)x(n-1) + x(n-2) \tag{4}$$

where γ is the parameter to be determined. The idea of the MC method is to minimize the sum of squares of $e(n)$ and the corresponding frequency estimate, denoted by $\hat{\omega}_{\mathrm{MC}}$, is given by [4]-[5]

$$\hat{\omega}_{\mathrm{MC}} = \arg\min_{\gamma} \left\{ \sum_{n=3}^{k} e^2(n) \right\} = \cos^{-1}\left(\frac{A_k}{2B_k}\right) \tag{5}$$

where

$$A_k = \sum_{n=3}^{k} x(n-1)[x(n) + x(n-2)] \tag{6}$$

$$B_k = \sum_{n=3}^{k} x^2(n-1) \tag{7}$$

and k denotes the number of available samples. In our study, the MC method is realized adaptively by computing A_k and B_k on a sample-by-sample basis as follows,

$$A_k = \lambda A_{k-1} + x(k-1)\left[x(k-2) + x(k)\right] \tag{8}$$

and

$$B_k = \lambda B_{k-1} + x^2(k-1) \tag{9}$$

where $0 < \lambda < 1$ is the forgetting factor which is responsible for parameter tracking. It is easy to see that when $\lambda = 1$, the recursive equations of (8) and (9) will be identical to the batch formulae of (6) and (7), respectively [5],[8]. In so doing, the RLS type version of the

MC method requires 3 additions, 5 multiplications, 1 division and 1 arccosine function for each iteration.

B. Pisarenko Harmonic Decomposition

Pisarenko [6] was the first who has exploited the eigenstructure of the covariance matrix in frequency estimation. The PHD estimate for a single real tone utilizes (3) in the sample covariance measurements with a unit-norm constraint for providing unbiased frequency estimates, and it is given by [7]:

$$\hat{\omega}_{\text{PHD}} = \cos^{-1}\left(\frac{r_{2,k} + \sqrt{r_{2,k}^2 + 8r_{1,k}^2}}{4r_{1,k}}\right) \tag{10}$$

where the sample covariances $r_{1,k}$ and $r_{2,k}$ are defined as

$$r_{i,k} = \frac{1}{k-i}\sum_{n=1}^{k-i} x(n)x(n+i), \qquad i = 1,2 \tag{11}$$

The derivation of the adaptive version of the PHD method is similar to the MC algorithm. We first express $\{r_{i,k}\}$ in terms of $\{r_{i,k-1}\}$ and then multiply the forgetting factor to the latter. As a result, $\hat{\omega}_{\text{PHD}}$ is updated iteratively via

$$r_{1,k} = \lambda\frac{k-2}{k-1}r_{1,k-1} + \frac{x(k-1)x(k)}{k-1} \tag{12}$$

and

$$r_{2,k} = \lambda\frac{k-3}{k-2}r_{2,k-1} + \frac{x(k-2)x(k)}{k-2} \tag{13}$$

At each iteration, 7 additions, 16 multiplications, 3 division, 1 root operation and 1 arccosine function are needed. It is noteworthy that a number of RLS type algorithms, such as [9]-[10], have been devised for multiple sinusoidal frequency estimation, while our proposed adaptive scheme is simpler but only suitable for single-tone applications.

B. Reformed Pisarenko Harmonic Decomposition

Similar to Pisarenko's method, the idea of the RPHD method is to minimize the sum of squares of $e(n)$ subject to another similar constraint, so that unbiased frequency estimation can be attained. The key distinction between the PHD and RPHD methods is that the former deals with the sample covariances while the latter works on the data measurements directly, although the resultant algorithms have differences only in incorporating the signal samples at the beginning and at the end in the calculation. The RPHD frequency estimate, denoted by $\hat{\omega}_{\text{RPHD}}$, is given by [8]

$$\hat{\omega}_{\text{RPHD}} = \cos^{-1}\left(\frac{C_k + \sqrt{C_k^2 + 8A_k^2}}{4A_k}\right) \tag{14}$$

where

$$C_k = x^2(k) - x^2(k-1) - x^2(2) + x^2(1) + 2\sum_{n=3}^{k} x(n)x(n-2) \tag{15}$$

and A_k is already defined in (6). In the adaptive version of the RPHD method, A_k is updated according to (8) and C_k is adjusted iteratively in a similar manner as follows [8],

$$C_k = \lambda C_{k-1} + x^2(k-3) - 2x^2(k-2) + x^2(k-1) + 2x(k-3)x(k-1) \tag{16}$$

The computational requirement of the adaptive realization per iteration is 8 additions, 9 multiplications, 1 division, 1 root operation and 1 arccosine function.

3 Simulation Results

Computer simulations had been conducted to compare the sinusoidal frequency tracking performances of the adaptive realizations of the MC, PHD and RPHD methods in white Gaussian noise. The tone amplitude was set to $\sqrt{2}$ and ϕ was a constant uniformly distributed between $[0, 2\pi)$ at each trial. High and moderate signal-to-noise ratio (SNR) conditions were investigated and they were obtained by properly scaling the noise variance. The value of λ was assigned as 0.95 in all methods. All results provided were averages of 1000 independent runs.

Figure 1 shows the trajectories for the frequency estimates of the three adaptive algorithms for a step change in ω at a high SNR condition of 10 dB. The actual frequency had a value of 1.0 during the first 200 iterations and then changed instantaneously to 2.0 afterwards. It can be seen that the learning curves of the PHD and RPHD methods were almost identical and their frequency estimates converged to the desired values of 1.0 and 2.0 at approximately the 20th and 300th iteration, respectively. Furthermore, the MC algorithm had similar convergence speed but it provided biased frequency estimates of 1.05 and 1.95 upon convergence. The corresponding mean square frequency errors (MSFEs) are plotted in Figure 2. We observe that the RPHD algorithm had the smallest MSFEs for both frequencies and the improvement over the PHD method was around 2 to 3 dB. On the other hand, the MC algorithm had much larger MSFEs, which were mainly due to the frequency bias.

The above test was repeated for a moderate SNR condition of 0 dB and the results are shown in Figures 3 and 4. We see that the trajectories for the frequency estimates as well as MSFEs of the PHD and RPHD algorithms were very similar, and they provided unbiased frequency tracking. On the contrary, the MC method gave biased frequency estimates of 1.3 and 1.8, and yielding the largest MSFEs.

4 Conclusions

In this paper, we have developed recursive least squares type realizations for the modified covariance (MC), Pisarenko harmonic decomposition (PHD) and reformed Pisarenko harmonic decomposition (RPHD) methods in tracking single real tone frequency. It is shown that the frequency tracking performance of the adaptive RHPD algorithm generally outperforms those of the MC and PHD methods, and its computational requirement is moderate among the three methods.

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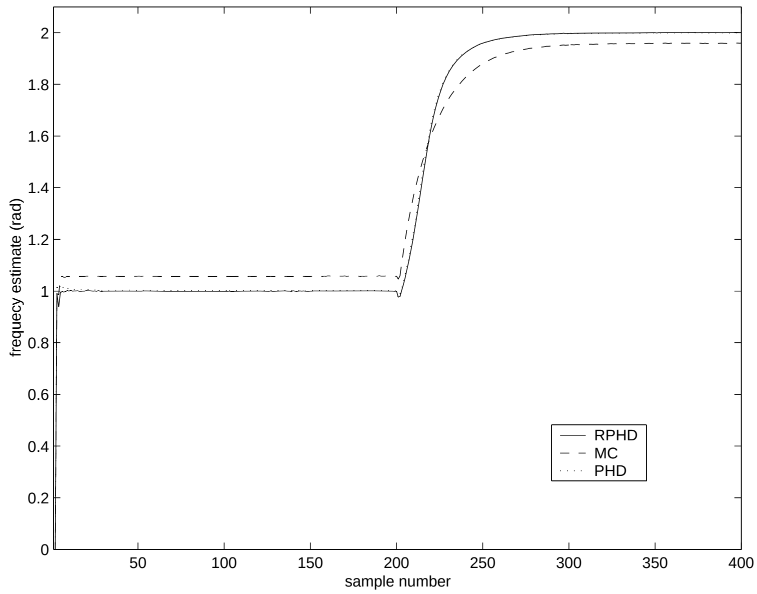

Figure 1: Frequency estimates at SNR = 10 dB

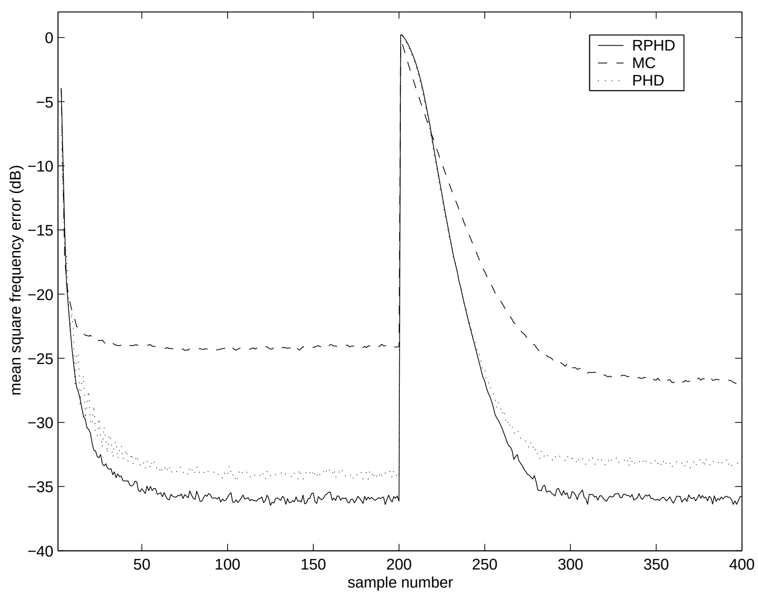

Figure 2: Mean square frequency errors at SNR = 10 dB

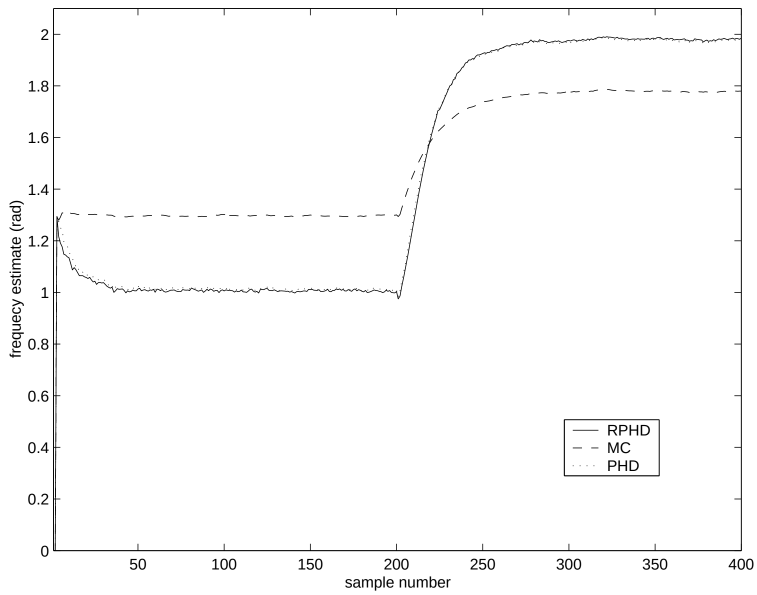

Figure 3: Frequency estimates at SNR = 0 dB

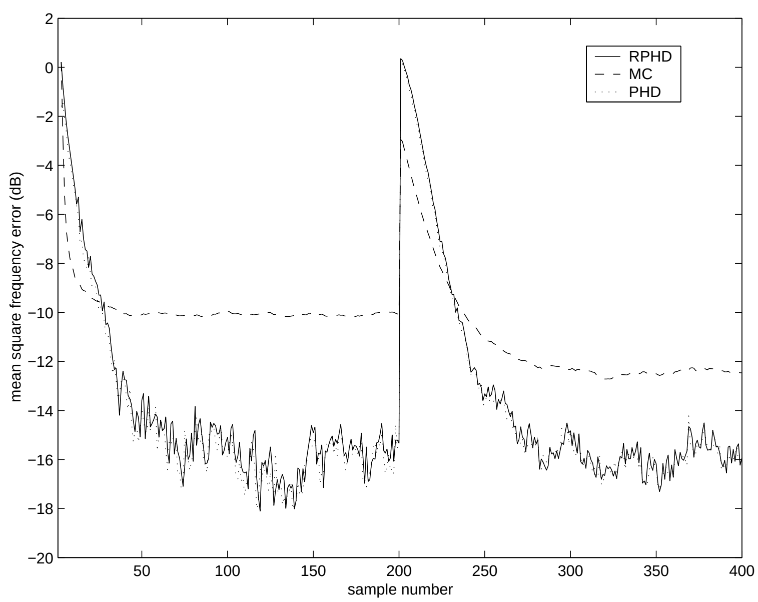

Figure 4: Mean square frequency errors at SNR = 0 dB