Correspondence

Subspace-Based Algorithm for Parameter Estimation of Polynomial Phase Signals

Yuntao Wu, Hing Cheung So, and Hongqing Liu

***Abstract*—In this correspondence, parameter estimation of a polynomial phase signal (PPS) in additive white Gaussian noise is addressed. Assuming that the order of the PPS is at least 3, the basic idea is first to separate its phase parameters into two sets by a novel signal transformation procedure, and then the multiple signal classification (MUSIC) method is utilized for joint estimating the phase parameters with second-order and above. In doing so, the parameter search dimension is reduced by a half as compared to the maximum likelihood and nonlinear least squares approaches. In particular, the problem of cubic phase signal estimation is studied in detail and its simplification for a chirp signal is given. The effectiveness of the proposed approach is also demonstrated by comparing with several conventional techniques via computer simulations.**

***Index Terms*—Parameter estimation, polynomial phase signal, subspace method.**

I. Introduction

In many application areas such as radar, speech processing, wireless communications, seismology and neuroethology, the received signals have continuous instantaneous phase. According to Weierstrass' theorem [1], the instantaneous phase can be well approximated by a finite-order polynomial in time within a finite-duration interval. As a result, polynomial phase signal (PPS) is a proper model for these real-world signals and its parameter estimation has received considerable attention in the field of signal processing [2]–[12], [14]–[16].

For a mono-component complex-valued constant-amplitude PPS with order P of the form $A\exp\left\{j\sum_{p=0}^{P} a_p t^p\right\}$, the maximum-likelihood (ML) [2], [3] and nonlinear instantaneous least squares (NILS) [4] estimators can provide very high estimation accuracy. In fact, the ML estimator and NILS method with maximum size windows are equivalent and both are statistically efficient in the sense that the estimator variances achieve Cramér–Rao lower bound (CRLB) asymptotically under additive white Gaussian noise. However, their computational requirements are extremely demanding because a P-dimensional maximization/minimization is required and global convergence is not guaranteed due to the non-convexity of their corresponding cost functions, which make them not practically useful in most situations. As a computationally efficient alternative, the polynomial-phase transform [5], which is also referred to as the high-order ambiguity function (HAF) [6], can estimate the PPS parameters with P one-dimensional searches via multiple nonlinear operations on the received signal. Founding on [5] and [6], Barbarossa *et al.* [7] have generalized the HAF to product high-order ambiguity function (PHAF). Similar to HAF, polynomial Wigner–Ville distribution (PWVD) [8], [9] is another high-order multiple transform technique for dealing with PPSs. In addition, Benidir *et al.* [10] have developed the generalized ambiguity function and generalized Wigner distribution which are similar to the HAF and PWVD, respectively. Although the HAF and PWVD approaches require much less computation than the ML and NILS techniques, they have a higher signal-to-noise ratio (SNR) threshold and their estimation performance is suboptimal. Note that all methods for nonlinear parameter estimation will suffer from a threshold effect, meaning that their estimation performance degrades considerably when the SNR falls below a certain threshold value, which is referred to as the SNR threshold. Recently, a bilinear transform technique known as cubic phase function (CPF) [11] for third-order PPS parameter estimation is proposed which can provide approximately optimum performance with smaller SNR threshold. Following [11], extensions to CPF for estimating the parameters of PPSs of order greater than 3, which are referred to as higher-order phase functions (HPFs), are developed in [12]. Nevertheless, high-order multiple transform corresponds to sequential multiple one-dimensional search computations and thus joint parameter estimation is not allowed. A major problem for these computationally simpler methods [5]–[12] is that the estimation accuracy of the lower-order phase parameters is dependent on that of the higher-order phase parameters, which leads to the so-called error propagation effect.

In this correspondence, a novel subspace-based method for PPS parameter estimation is devised. The observed PPS is first converted to another sequence by a novel signal transformation procedure. Assuming that $P \geq 3$, the multiple signal classification (MUSIC) algorithm [13] is then utilized to jointly estimate the higher-order phase parameters, namely, $a_2, a_3, \ldots, a_P$. After obtaining these estimates, the problem is then reduced to a single complex tone estimation problem, where A, a_0 and a_1, correspond to the sinusoidal amplitude, phase and frequency, respectively, which can be easily solved. The main advantage of the proposed methodology is that almost joint PPS parameter estimation is achieved and thus the error propagation effect is greatly reduced. Note that the subspace approach has already been suggested for joint PPS parameter estimation in the literature [14]–[16]. PPS estimation with time-varying amplitudes is addressed in [14] where the HAF is employed to transform the PPS prior to applying the subspace-based techniques. In [15], a Capon's form of Wigner distribution is developed for estimating PPS parameters in the presence of interference, while [16] presents a robust algorithm for operating in impulsive noise environments. Comparing with [14]–[16], which need P-dimensional search computation, our method is more computationally attractive because the corresponding parameter search dimension is reduced to $\lceil (P-1)/2 \rceil$ where $\lceil\ \rceil$ denotes the ceiling operator.

The rest of the correspondence is organized as follows. In Section II, the proposed MUSIC estimator for a cubic phase signal is devised and the theoretical performance of parameter estimates is also analyzed. In Section III, generalization of the proposed methodology to higher-order PPSs is presented. Simulation results are included in Section IV

Manuscript received November 21, 2007; revised May 16, 2008. First published June 20, 2007; current version published September 17, 2008. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Mark J. Coates. The work described in this correspondence was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 119605).

Y. Wu was with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong. He is now with the School of Computer Science and Engineering, Wuhan Institute of Technology, Wuhan 430073, China (e-mail: ytwu@sina.com).

H. C. So and H. Liu are with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong (e-mail: hcso@ee.cityu.edu.hk; hongqiliu2@student.cityu.edu.hk).

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Digital Object Identifier 10.1109/TSP.2008.927457

to evaluate the performance of the proposed method by comparing with the HPF, PHAF and NILS methods as well as the CRLB. Finally, conclusions are drawn in Section V.

II. Parameter Estimation of Cubic Phase Signal

In this section, we focus on estimating the parameters of a typical PPS, namely, cubic phase signal [11], in additive white Gaussian noise. The proposed method first converts the observed signal to another sequence which is suitable for parameter estimation with the MUSIC approach.

A. Signal Conversion

The noisy cubic phase signal model is

$$x(t) = Ae^{j(a_0+a_1t+a_2t^2+a_3t^3)} + n(t), \quad t = \frac{-N}{2}, \ldots, 0, \ldots, \frac{N}{2} \tag{1}$$

where $A > 0$ is the signal amplitude and a_i, $i = 0, 1, 2, 3$, are the signal phases, and all of them are real and unknown. The $n(t)$ is the additive complex white Gaussian noise with zero-mean and unknown variance σ_n^2 and the sample number $(N+1)$ is assumed an odd integer. The task is to estimate the unknown deterministic parameters, namely, a_i, $i = 0$, 1, 2, 3, and A, from the $(N+1)$ samples of $\{x(t)\}$.

Motivating by the symmetry of the sample interval, we first define the following correlation sequences, $x_1(t)$ and $x_2(t)$:

$$x_1(t) = x(t)x(-t), \quad x_2(t) = x(t)x^*(-t), \quad t = 1, \ldots, \frac{N}{2} \tag{2}$$

where $*$ denotes the conjugate operator and the sample $x(0)$ is not used.

With the use of (1), (2) can be expressed as

$$\begin{aligned} x_1(t) &= A^2e^{2j(a_0+a_2t^2)} + n_1(t), \\ x_2(t) &= A^2e^{2j(a_1t+a_3t^3)} + n_2(t), \quad t = 1, \ldots, \frac{N}{2} \end{aligned} \tag{3}$$

where

$$\begin{aligned} n_1(t) =& Ae^{j(a_0+a_1t+a_2t^2+a_3t^3)}n(-t) \\ &+ Ae^{j(a_0-a_1t+a_2t^2-a_3t^3)}n(t) + n(t)n(-t) \end{aligned}$$

and

$$\begin{aligned} n_2(t) =& Ae^{-j(a_0-a_1t+a_2t^2-a_3t^3)}n(t) \\ &+ Ae^{j(a_0+a_1t+a_2t^2+a_3t^3)}n^*(-t) + n(t)n^*(-t) \end{aligned}$$

are the corresponding noise components. We see that the parameters to be estimated are now separated into two independent groups, namely, (a_0, a_2) and (a_1, a_3), through the correlation computation.

As $x_1(t)$ and $x_2(t)$ are quadratic phase and cubic phase signals, respectively, our next step is to transform $x_2(t)$ to another signal $x_3(t)$ such that the resultant order is identical to that of $x_1(t)$. The $x_3(t)$ has the form of

$$\begin{aligned} x_3(t) &= x_2(t+1)x_2^*(t-1) \\ &= A^4e^{j(4(a_3+a_1)+12a_3t^2)} + n_3(t), \\ t &= 2, \ldots, \frac{N}{2} - 1 \end{aligned} \tag{4}$$

where

$$\begin{aligned} n_3(t) =& A^2e^{2j(a_1(t+1)+a_3(t+1)^3)}n_2^*(t-1) \\ &+A^2e^{-2j(a_1(t-1)+a_3(t-1)^3)}n_2(t+1) + n_2(t+1)n_2^*(t-1) \end{aligned}$$

is the noise component in $x_3(t)$ and the value of t starts at $t = 2$. It is worthy to mention that $x_1(t)$ and $x_3(t)$ are similar to the bilinear kernels in the CPF [11] and PWVD [8], respectively.

Analogous to the signal model for parameter estimation of multiple complex tones, we add $x_1(t)$ and $x_3(t)$ together to construct a noisy two-component chirp signal with constant amplitudes, denoted by $y(t)$

$$\begin{aligned} y(t) &= x_1(t) + x_3(t) \\ &= A^2e^{2j(a_0+a_2t^2)} + A^4e^{j(4(a_3+a_1)+12a_3t^2)} + \varepsilon(t), \\ t &= 2, \ldots, \frac{N}{2} - 1 \end{aligned} \tag{5}$$

where $\varepsilon(t) = n_1(t) + n_3(t)$. Grouping all $\{y(t)\}$ together, we have the following matrix representation:

$$\mathbf{Y} = \mathbf{Vs} + \mathbf{N} \tag{6}$$

where $\mathbf{Y} = [y(2), y(3), \ldots y(N/2-1)]^T$, $\mathbf{V} = [\mathbf{v}(\alpha_2) \quad \mathbf{v}(\alpha_3)]$, $\mathbf{v}(\alpha_i) = \left[e^{j\alpha_i 2^2}, e^{j\alpha_i 3^2}, \ldots e^{j\alpha_i(N/2-1)^2}\right]^T$, $i = 2, 3$, $\mathbf{N} = [\varepsilon(2), \varepsilon(3), \ldots \varepsilon(N/2-1)]^T$, and $\mathbf{s} = [A^2e^{j2a_0} \quad A^4e^{j4(a_1+a_3)}]^T$ with $\alpha_2 = 2a_2$, $\alpha_3 = 12a_3$ and T stands for the transpose operation. In the following, the well-known MUSIC method [13], which is a high resolution subspace-based algorithm, is utilized to jointly estimate a_2 and a_3 based on the data model of (6). The remaining parameters, namely, A, a_0 and a_1, can then be straightforwardly determined.

B. Proposed Music Method

Under the assumption that $n(t)$ a zero-mean white Gaussian process and $A^2/\sigma_n^2 \gg 1$, the covariance matrix for $\mathbf{Y}$, denoted by $\mathbf{R}_Y$, is calculated as

$$\begin{aligned} \mathbf{R}_Y &= E\{\mathbf{YY}^H\} \\ &= \mathbf{Vss}^H\mathbf{V}^H + \mathbf{V}E\{\mathbf{sN}^H + \mathbf{Ns}^H\}\mathbf{V}^H \\ &\quad + E\{\mathbf{NN}^H\} \\ &= \mathbf{Vss}^H\mathbf{V}^H + \mathbf{V}[\mathbf{s}E\{\mathbf{N}^H\} + E\{\mathbf{N}\}\mathbf{s}^H]\mathbf{V}^H \\ &\quad + E\{\mathbf{NN}^H\} \\ &\approx \mathbf{VR_sV}^H + \sigma^2\mathbf{I} \end{aligned} \tag{7}$$

where $\mathbf{R_s} = \mathbf{ss}^H$, $\sigma^2 = [(4A^6+2A^2)\sigma_n^2 + (6A^4+1)\sigma_n^4 + 4A^2\sigma_n^6 + \sigma_n^8]$ and $\mathbf{I}$ is the $(N/2-2)\times(N/2-2)$ identity matrix with E and H denote the expectation operator and conjugate transpose, respectively. Note that we have ignored the off-diagonal elements of $E\{\mathbf{NN}^H\}$ which is valid for sufficiently large SNR conditions.

As $\mathbf{R}_Y$ is of full rank and the rank of $\mathbf{R_s}$ is 1 while the dimension of the signal subspace should be 2, we utilize the smoothing technique in [17] to construct $\mathbf{R}$ from $\mathbf{R}_Y$:

$$\mathbf{R} = \mathbf{R}_Y - \mathbf{JR}_Y\mathbf{J} \tag{8}$$

where $\mathbf{J}$ is the exchange matrix with dimension $(N/2-2)\times(N/2-2)$. Assuming that $\mathbf{V}$ is full column rank and noting that $\mathbf{JIJ} = \mathbf{I}$, the rank of $\mathbf{R}$ is easily shown to be 2. That is, in doing so, the correct signal and noise subspaces are attained.

We can then obtain the noise subspace of $\mathbf{R}$, denoted by $\mathbf{E}_n$, from the $(N/2-4)$ eigenvectors corresponding to the $(N/2-4)$ zero eigenvalues of $\mathbf{R}$. Applying the MUSIC methodology, the estimates of α_2 and α_3, denoted by $\hat{\alpha}_2$ and $\hat{\alpha}_3$, are found from the two peaks of the following 1-D function:

$$\{\hat{\alpha}_2, \hat{\alpha}_3\} = \arg\max_{\alpha} \frac{1}{\mathbf{v}^H(\alpha)\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\mathbf{v}(\alpha)}. \tag{9}$$

The $\hat{\mathbf{E}}_n$ is the estimate of $\mathbf{E}_n$ based on the sample covariance matrix for $\mathbf{Y}$ which is computed from the finite-length $x(t)$. The estimates of a_2 and a_3, denoted by $\hat{a}_2$ and $\hat{a}_3$, are then determined as $\hat{a}_2 = \hat{\alpha}_2/2$ and $\hat{a}_3 = \hat{\alpha}_3/12$.

A simple matching procedure for associating the two estimated parameters $\hat{\alpha}_2$ and $\hat{\alpha}_3$ is given as follows. Define three vectors of length $(N/2 - 1)$, namely, $\mathbf{w}_0$ and $\mathbf{w}_i$, $i = 2, 3$, where the tth element of $\mathbf{w}_0$ is $x_1(t+1)x_1^*(t)$, $t = 2, 3, \ldots, N/2$, and $\mathbf{w}_i = \left[e^{j\hat{\alpha}_i 5}, e^{j\hat{\alpha}_i 7}, \ldots, e^{j\hat{\alpha}_i (N+1)}\right]$. The estimate of $\hat{a}_2$ is taken as the $\hat{\alpha}_i/2$ which corresponds to the smaller value of $\|\mathbf{w}_0 - \hat{\mathbf{w}}_i\|$.

After obtaining $\hat{a}_2$ and $\hat{a}_3$, the estimates of a_0 and a_1, namely, $\hat{a}_0$ and $\hat{a}_1$, can be determined with the use of (3) and (4)

$$\hat{a}_0 = \frac{1}{2}\angle\left(\sum_{t=1}^{N/2} x_1(t)e^{-j2\hat{a}_2 t^2}\right) \tag{10}$$

and

$$\hat{a}_1 = \frac{1}{4}\angle\left(\sum_{t=2}^{N/2-1} x_3(t)e^{-j12\hat{a}_3 t^2}\right) - \hat{a}_3 \tag{11}$$

where $\angle$ denotes the phase angle operator. When all phase coefficients are found, the estimate of A, denoted by $\hat{A}$, is computed as

$$\hat{A} = \left|\frac{1}{N+1}\sum_{t=-N/2}^{N/2} x(t)e^{-j(\hat{a}_0+\hat{a}_1 t+\hat{a}_2 t^2+\hat{a}_3 t^3)}\right|. \tag{12}$$

It is noteworthy that the estimation accuracy of $\hat{a}_0$, $\hat{a}_1$ and A is only dependent on that of the higher-order phase coefficients, $\hat{a}_2$ and $\hat{a}_3$, respectively. Furthermore, apart from (10) and (11), more accurate estimation approaches [18]–[20] are available by converting $x(t)$ to a noisy complex sinusoid with the use of $\hat{a}_2$ and $\hat{a}_3$.

C. Asymptotic Performance Analysis

The asymptotic mean square errors of the estimated parameters $\{\hat{a}_k\}$ are derived as follows. Expression (9) is equivalent to

$$\{\hat{\alpha}_2, \hat{\alpha}_3\} = \arg\min_{\alpha} f(\alpha) \tag{13}$$

where $f(\alpha) = \mathbf{v}^H(\alpha)\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\mathbf{v}(\alpha)$.

For sufficiently large N and SNR, the function $f(\alpha)$ has two local minima at $\alpha \approx \alpha_2$ and $\alpha \approx \alpha_3$. Using Taylor's series to expand $f(\hat{\alpha}_k)$ around α_k, $k = 2, 3$, up to second-order term, we get

$$f'(\alpha_k) + f''(\alpha_k)(\hat{\alpha}_k - \alpha_k) \approx 0. \tag{14}$$

When $f''(\alpha)$ is sufficiently smooth around $\alpha = \alpha_k$, it can be substituted with its expected value. Define $\text{MSE}(a_k) = E\left\{(\hat{a}_k - a_k)^2\right\}$, $k = 0, 1, 2, 3$. Exploiting the performance analysis result of MUSIC algorithm in [21], $\text{MSE}(a_k)$, $k = 2, 3$, are evaluated as

$$\text{MSE}(a_k) \approx \frac{E\left\{\left(\Re(\mathbf{v}' H(\alpha_k)\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\mathbf{v}(\alpha_k))\right)^2\right\}}{\left(\Re(\mathbf{v}' H(\alpha_k)\mathbf{E}_n\mathbf{E}_n^H\mathbf{v}'(\alpha_k))\right)^2} \tag{15}$$

where $\Re$ denotes the real part. According to (10) and (11), $\text{MSE}(a_0)$ and $\text{MSE}(a_1)$ can be derived easily from $\text{MSE}(a_2)$ and $\text{MSE}(a_3)$ as

$$\text{MSE}(a_0) \approx \frac{1}{144}(N+1)^2(N+2)^2\text{MSE}(a_2) \tag{16}$$

and

$$\text{MSE}(a_1) \approx \frac{(N^2-5N+9)^2}{9}\text{MSE}(a_3). \tag{17}$$

D. Simplification for a Chirp Signal

When $P = 2$, we construct $y(t)$ using

$$\begin{aligned} y(t) &= x_2(t) + x_1(t+1)x_1^*(t-1) \\ &= A^2 e^{j2a_1 t} + A^4 e^{j8a_2 t} + \varepsilon(t), \\ t &= 2, \ldots, \frac{N}{2} - 1. \end{aligned} \tag{18}$$

The signal model is reduced to the frequency estimation problem of two sinusoids. We can still employ the proposed MUSIC method to jointly estimate a_1 and a_2 via α_k, $k = 1, 2$, which are defined as $\alpha_1 = 2a_1$ and $\alpha_2 = 8a_2$ with the use of (9).

III. Extension to Higher-Order Polynomial Phase Signal

In the Section, we extend our proposed approach in Section II to higher-order PPS parameter estimation. Let $x(t)$ be the noisy PPS of order P:

$$x(t) = Ae^{j\sum_{p=0}^{P} a_p t^p} + n(t), \quad t = \frac{-N}{2}, \ldots, 0, 1, \ldots, \frac{N}{2} \tag{19}$$

where A and a_p, $p = 0, 1, \ldots, P$, are all deterministic but unknown. Without loss of generality, we assume that P is known [2], [3] and is odd, that is, $P = 2M + 1$ where M is a positive integer.

Following Section II, we define the following correlation sequences from $x(t)$:

$$x_1(t) = x(t)x(-t), \quad x_2(t) = x(t)x^*(-t), \quad t = 1, \ldots, \frac{N}{2}. \tag{20}$$

With the use of (19), (20) can be expressed as

$$x_1(t) = A^2 e^{2j\sum_{k=0}^{M} a_{2k}t^{2k}} + n_1(t) \tag{21}$$

and

$$x_2(t) = A^2 e^{2j\sum_{k=0}^{M} a_{2k+1}t^{2k+1}} + n_2(t) \tag{22}$$

where $n_1(t)$ and $n_2(t)$ are the corresponding noise components. From (21) and (22), it is seen that the unknown phase parameters are separated into two sets, namely, $\{a_0, a_2, \ldots, a_{2M}\}$ and $\{a_1, a_3, \ldots, a_{2M+1}\}$, respectively. Using the binomial formula, we further construct $x_3(t)$:

$$\begin{aligned} x_3(t) &= x_2(t+1)x_2^*(t-1) \\ &= A^4 e^{2j\sum_{k=0}^{M} a'_{2k}t^{2k}} + n_3(t), \qquad t = 2, \ldots, \frac{N}{2} - 1 \end{aligned} \tag{23}$$

where

$$a'_{2k} = 2\sum_{l=k}^{M} a_{2l+1}C_{2l+1}^{2l+1-2k}, \quad k = 0, 1, \ldots, M \tag{24}$$

with $C_N^K = N!/((N-K)!K!)$ and $n_3(t)$ represents the disturbance in $x_3(t)$. Note that there is no odd order phase term in (23). The problem of estimating a_{2k+1}, $k = 0, 1, \ldots, M$ in (22) is now transformed to estimation of a'_{2k}, $k = 0, 1, \ldots, M$ in (23).

Using (21) and (23), we form $y(t)$

$$\begin{aligned} y(t) &= x_1(t) + x_3(t) \\ &= \left[\, e^{2j\sum_{k=0}^{M} a_{2k}t^{2k}} \quad e^{2j\sum_{k=0}^{M} a'_{2k}t^{2k}} \,\right] \\ &\quad \times \begin{bmatrix} A^2 e^{j2a_0} \\ A^4 e^{j4\sum_{k=0}^{M} a_{2k+1}} \end{bmatrix} + \epsilon(t) \end{aligned} \tag{25}$$

where $y(t)$ can be regarded as a 2-component $2M$th-order PPS with constant amplitudes and $\epsilon(t) = n_1(t) + n_3(t)$. Writing $y(t)$ in matrix form yields

$$\mathbf{Y} = [\mathbf{v}(a_2, \ldots, a_{2M}) \quad \mathbf{v}(a'_2, \ldots, a'_{2M})]\,\mathbf{s} + \mathbf{N} \tag{26}$$

where

$\mathbf{v}(\alpha_2, \ldots, \alpha_{2M}) = \left[e^{j\sum_{k=1}^{M} 2^{2k}\alpha_{2k}}, \ldots, e^{j\sum_{k=1}^{M}(N/2-1)^{2k}\alpha_{2k}}\right]^T$, $\alpha_{2k} = 2a_{2k},\ 2a'_{2k}$, $k = 1, 2, \ldots, M$, and $\mathbf{s} = [A^2 e^{j2a_0} \quad A^4 e^{j2a'_0}]^T$ while $\mathbf{Y}$ and $\mathbf{N}$ have the same forms as in (6). Note that in (26), the noise covariance matrix, $E\{\mathbf{N}\mathbf{N}^H\}$, is also proportional to $\mathbf{I}$. The MUSIC method is applied to (26) to estimate the parameters a_{2k} and a'_{2k}, $k = 0, 1, \ldots, M$, as follows. We first obtain the noise subspace $\mathbf{E}_n$ of the modified covariance matrix of $\mathbf{Y}$

$$\mathbf{R} = E\{\mathbf{Y}\mathbf{Y}^H\} - \mathbf{J}E\{\mathbf{Y}\mathbf{Y}^H\}\mathbf{J}. \tag{27}$$

Defining two parameter vectors, namely, $\boldsymbol{\alpha}_e = 2[a_2, a_4, \ldots, a_{2M}]$ and $\boldsymbol{\alpha}'_o = 2[a'_2, a'_4, \ldots, a'_{2M}]$, their MUSIC estimates are found by searching the following M-dimensional function:

$$\{\hat{\boldsymbol{\alpha}}_e, \hat{\boldsymbol{\alpha}}'_o\} = \arg\max_{\boldsymbol{\alpha}} \frac{1}{\mathbf{v}(\boldsymbol{\alpha})^H \hat{\mathbf{E}}_n \hat{\mathbf{E}}_n^H \mathbf{v}(\boldsymbol{\alpha})} \tag{28}$$

where $\boldsymbol{\alpha} = [\alpha_2, \alpha_4, \ldots, \alpha_{2M}]$ and $\hat{\mathbf{E}}_n$ is the noisy version of $\mathbf{E}_n$ computed from the finite-length data. Using (24), the least squares (LS) estimates of $a_3, a_5, \ldots, a_{2M+1}$ are then determined from $a'_2, a'_4, \ldots, a'_{2M}$:

$$\begin{bmatrix} \hat{a}_3 \\ \vdots \\ \hat{a}_{2M+1} \end{bmatrix} = \begin{bmatrix} C_3^1 & \cdots & C_{2M+1}^{2M-1} \\ \vdots & \cdots & \vdots \\ 0 & \cdots & C_{2M+1}^1 \end{bmatrix}^{\#} \begin{bmatrix} \hat{a}'_2 \\ \vdots \\ \hat{a}'_{2M} \end{bmatrix} \tag{29}$$

where $\#$ denotes the pseudo-inverse. After estimating $a_2, a_3, \ldots, a_P$, the remaining parameters, namely, a_0, a_1 and A, can be obtained as follows. Employing the estimates of $\boldsymbol{\alpha}_e$ and $\boldsymbol{\alpha}'_o$, with the use of (21) and (23), $\hat{a}_0$, $\hat{a}_1$ and $\hat{A}$ are computed as

$$\hat{a}_0 = \frac{1}{2}\angle\left(\sum_{t=1}^{N/2} x_1(t) e^{-2j\sum_{k=1}^{M}\hat{a}_{2k}t^{2k}}\right) \tag{30}$$

$$\hat{a}_1 = \frac{1}{4}\angle\left(\sum_{t=2}^{N/2-1} x_3(t) e^{-2j\sum_{k=1}^{M}\hat{a}'_{2k}t^{2k}}\right) - (\hat{a}_3 + \cdots + \hat{a}_{2M+1}) \tag{31}$$

and

$$\hat{A} = \left|\frac{1}{N+1}\sum_{t=-N/2}^{N/2} x(t) e^{-j\sum_{p=0}^{P}\hat{a}_p t^p}\right|. \tag{32}$$

It is noteworthy that when $P = 3$, (28) and (30)–(32) can be shown to reduce to (9)–(12), respectively.

IV. Results and Discussion

Computer simulations have been carried out to evaluate the PPS parameter estimation performance of the proposed algorithm in the presence of complex white Gaussian noise by comparing with the HPF [12], PHAF [7] and NILS [4] methods as well as the CRLB. The signal amplitude is chosen as $A = 1$ and the noise sequence is scaled accordingly to achieve different SNR conditions where $\mathrm{SNR} = A^2/\sigma_n^2$.

In the first experiment, we compare the shapes of the one-dimensional cost functions for parameter search in the MUSIC, HPF and PHAF methods in cubic phase signal estimation. Note that NILS cost function does not correspond to one-dimensional search and thus it is not included in the comparison. The phase parameters are $a_0 = 0$, $a_1 = 0.3\pi$, $a_2 = -0.001\pi$ and $a_3 = 0.00001\pi$. Two different conditions of SNR and N, namely, $\mathrm{SNR} = 10$ dB and $N = 301$, and $\mathrm{SNR} = 5$ dB and $N = 501$, which correspond to a smaller data length with higher SNR and larger data length with a smaller SNR, respectively, are examined. The cost functions are plotted in Figs. 1–6 where each figure contains the results of ten independent runs. For the proposed scheme, the two true peaks are located at $2a_2$ and $12a_3$ while the true peaks are located at $2a_2$ and a_3 in the HPF and PHAF methods, respectively. Note that estimates of a_0 and a_1 in the MUSIC estimator are not shown as they depend on the values of $\hat{a}_2$ and $\hat{a}_3$. For the same reason, we only show the estimate of a_3 in the PHAF method. While the HPF cost function [12] is a function of $2(a_2 + 3a_3 n)$ where n is

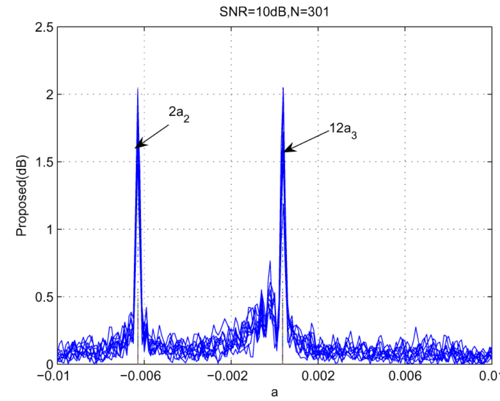

Fig. 1. Proposed estimator for a third-order PPS at $\mathrm{SNR} = 10$ dB and $N = 301$.

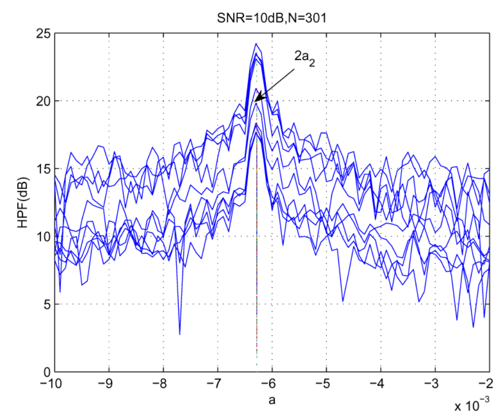

Fig. 2. HPF estimator for a third-order PPS at $\mathrm{SNR} = 10$ dB and $N = 301$.

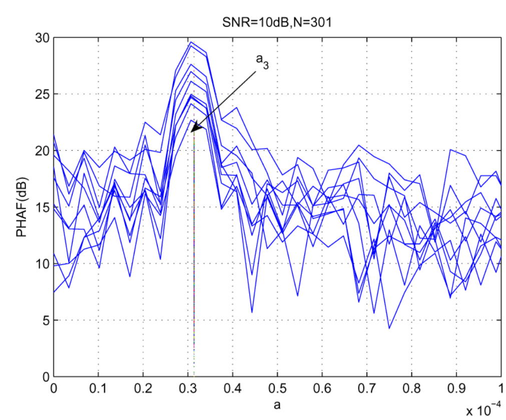

Fig. 3. PHAF estimator for a third-order PPS at $\mathrm{SNR} = 10$ dB and $N = 301$.

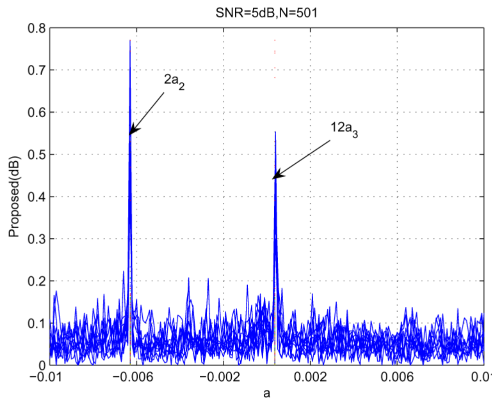

Fig. 4. Proposed estimator for a third-order PPS at $\mathrm{SNR} = 5$ dB and $N = 501$.

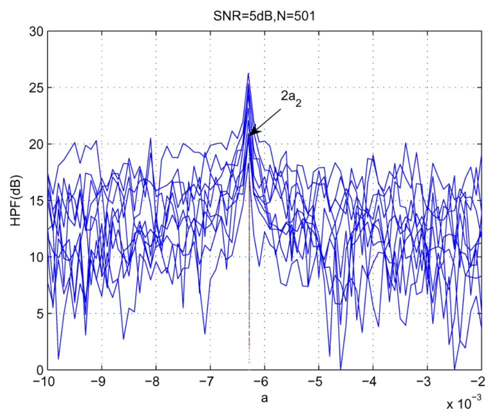

Fig. 5. HPF estimator for a third-order PPS at $\mathrm{SNR} = 5$ dB and $N = 501$.

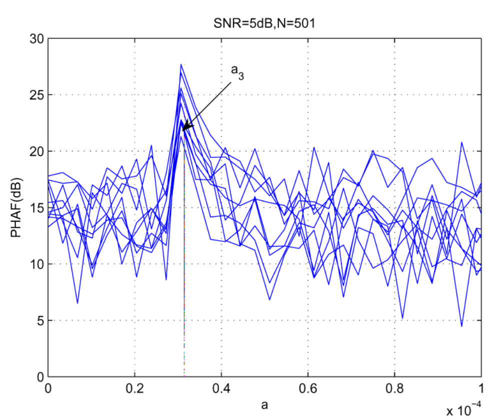

Fig. 6. PHAF estimator for a third-order PPS at $\mathrm{SNR} = 5$ dB and $N = 501$.

the corresponding time index and we simply set $n = 0$ to display the estimate of a_2. From the six figures, we see that the searching functions of the proposed scheme are less noisy than those of the HPF and PHAF approaches and their peaks are more distinguishable, which implies an easier peak search.

In the second experiment, we compare the root mean square error (RMSE) performance of the MUSIC, HPF, PHAF and NILS methods in parameter estimation of the above cubic phase signal. Their corresponding RMSEs versus SNR are shown in Figs. 7–10, respectively. The number of Monte Carlo simulations is 100 and there are 301 samples in the received data. From the figures, we see that the RMSEs of the

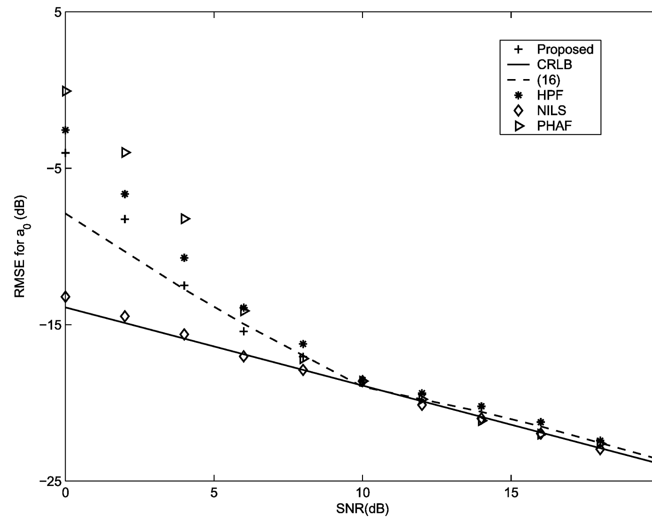

Fig. 7. RMSE versus SNR for a_0.

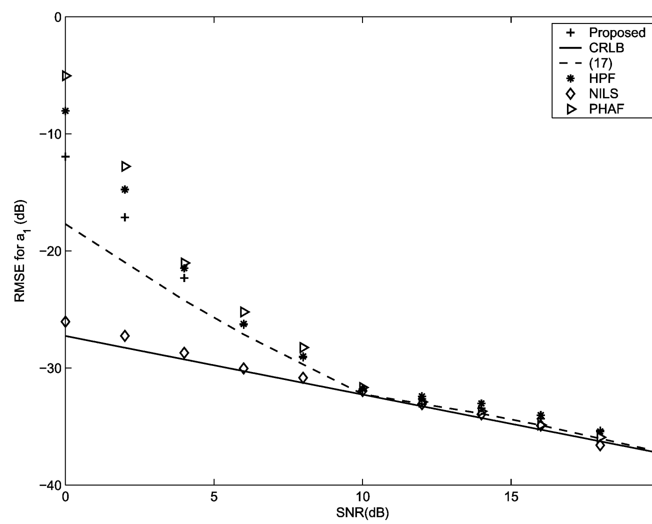

Fig. 8. RMSE versus SNR for a_1.

four parameter estimates in the proposed method are very close to the theoretical calculations and also attain the CRLB when $\mathrm{SNR} > 8$ dB. Note that the RMSEs of all methods exhibit monotonic behavior from a_0 to a_4 because the corresponding CRLBs for a_i are inversely proportional to N^{2i+1}. Since the estimation accuracy of the lower-order phase parameters is dependent on that of the higher-order phase parameters, the error propagation effect of parameter estimation in both the HPF and PHAF methods particularly for larger noise, cannot be avoided. As a result, the performance of the proposed method is superior to that of two computationally efficient estimators at lower SNR conditions. Although the estimation performance of the NILS method is the best among the four methods, it is the most computationally demanding because it requires $O(N^4 \log N)$ operations [4]. While our method only requires a one-dimensional search to find a_2 and a_3, corresponding to an $O(N \log N)$ complexity, and both the HPF and PHAF estimators need two one-dimensional search operations to estimate $\{a_2, a_3\}$.

In the last experiment, phase estimation of a fifth-order PPS using the proposed estimator is investigated for two scenarios of SNR and N, namely, $\mathrm{SNR} = 10$ dB and $N = 301$, and $\mathrm{SNR} = 5$ dB and $N = 501$. The phase parameters are $a_0 = 0$, $a_1 = 0.25\pi$, $a_2 = 0.25$, $a_3 = 0.025$, $a_4 = 0.02$ and $a_5 = 0.015$. According to (28), we need

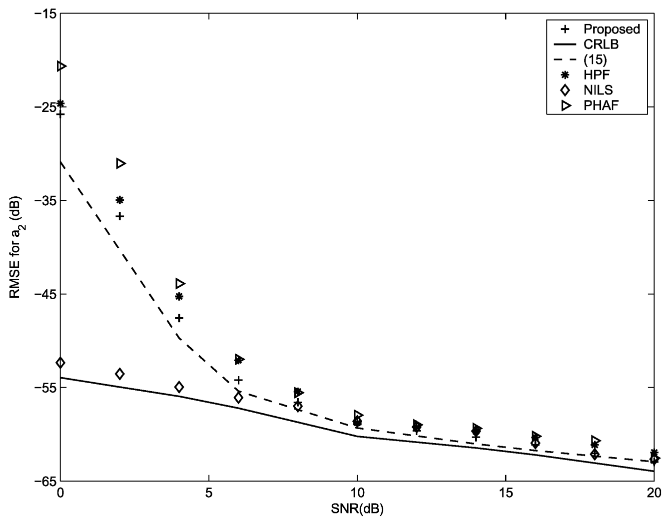

Fig. 9. RMSE versus SNR for a_2.

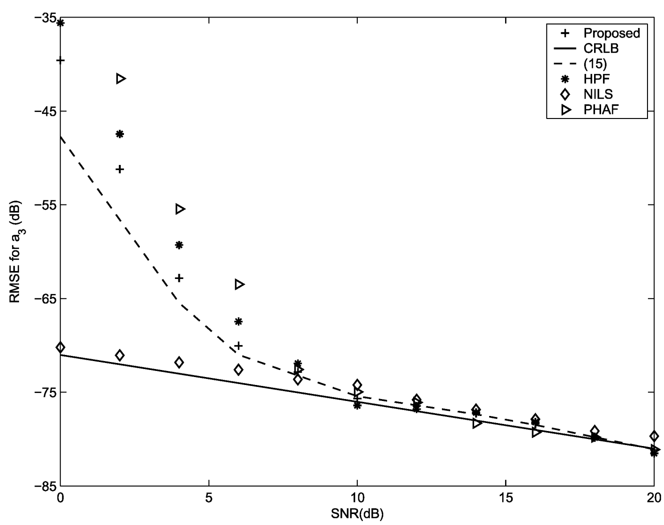

Fig. 10. RMSE versus SNR for a_3.

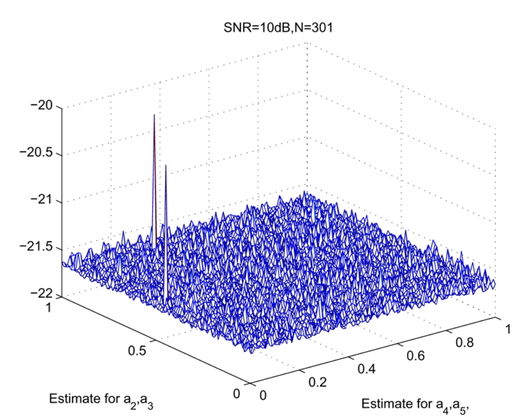

Fig. 11. Proposed estimator for a fifth-order PPS at $\mathrm{SNR} = 10$ dB and $N = 301$.

to perform a two-dimensional (2-D) search for the parameter pairs, namely, (a_2, a_4) and (a_3, a_5), which are obtained by the 2-D MUSIC

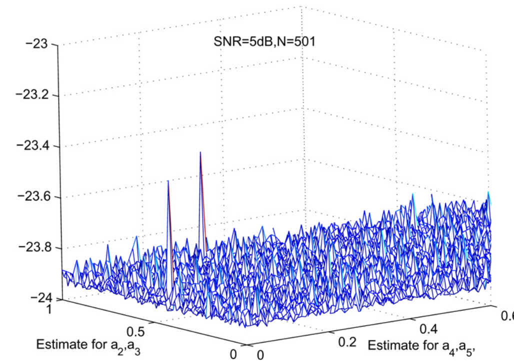

Fig. 12. Proposed estimator for a fifth-order PPS at $\mathrm{SNR} = 5$ dB and $N = 501$.

method. The two true peaks are located at $(2a_2, 2a_4)$ and $(12a_3, 40a_5)$. We see that the proposed MUSIC method can give accurate parameter estimation results for different SNRs and sample numbers.

V. Conclusion

A subspace-based approach for joint parameter estimation of polynomial phase signals (PPSs) in white Gaussian noise has been proposed. Prior to employing the multiple signal classification method, the observed signal is transformed to another sequence through a correlation computation procedure. In particular, parameter estimation of cubic phase signal is studied in detail and analyzed. It is shown that the proposed algorithm can give accurate phase estimation of PPSs at lower signal-to-noise ratio conditions. As a future work, we will extend our subspace-based parameter estimation approach for PPSs in impulsive noise environments.

Acknowledgment

The authors would like to thank the anonymous reviewers for their careful reading and constructive comments, which improved the clarity of this correspondence.

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Amendments to “Performance Analysis of Estimation Algorithms of Nonstationary ARMA Processes”

Feng Ding, Yang Shi, and Tongwen Chen

***Abstract*—In this correspondence, we add a condition in Theorem 1 and some explanations in the proof of Theorem 2 in** IEEE TRANSACTIONS ON SIGNAL PROCESSING**, vol. 54, no. 3, pp. 1041–1053, March 2006.**

I. AMENDMENTS TO THEOREM 1

The additional condition in Theorem 1 in [1] is stated in terms of the notation in [1]:

$$A4') \qquad [\ln r_0(t)]^\beta = o(\lambda_{\min}[P_0^{-1}(t)]), \text{ for any } \beta > 1+\varepsilon.$$

The additional steps in the proof briefly go as follows. Following [1, eq. (43)], for any vector $\omega \in \mathbb{R}^{n_0}$ with $\|\omega\| = 1$, we have

$$\begin{aligned}\sum_{i=1}^{t} \|\Phi_0^{\mathrm{T}}(i)\omega\|^2 &= \sum_{i=1}^{t} \|\Phi^{\mathrm{T}}(i)\omega - \tilde{\Phi}^{\mathrm{T}}(i)\omega\|^2 \\ &\le 2\sum_{i=1}^{t} \|\Phi^{\mathrm{T}}(i)\omega\|^2 + 2\sum_{i=1}^{t} \|\tilde{\Phi}(i)\|^2 \\ &= 2\sum_{i=1}^{t} \|\Phi^{\mathrm{T}}(i)\omega\|^2 + O([\ln r(t)]^\beta) \\ &= 2\sum_{i=1}^{t} \|\Phi^{\mathrm{T}}(i)\omega\|^2 + O([\ln r_0(t)]^\beta).\end{aligned}$$

Thus,

$$\lambda_{\min}[P_0^{-1}(t)] \le 2\lambda_{\min}[P^{-1}(t)] + O([\ln r_0(t)]^\beta).$$

Using A4’), it follows that

$$\lambda_{\min}[P_0^{-1}(t)] \le 2\lambda_{\min}[P^{-1}(t)] + o(\lambda_{\min}[P_0^{-1}(t)]).$$

Referring to [2, Ch. 4], we have

$$\lambda_{\min}[P_0^{-1}(t)] = O(\lambda_{\min}[P^{-1}(t)]), \text{ a.s.} \tag{44'}$$

Combining relations [1, eq. (43)] and (44’) above with the relation in [1, eq. (41)], we get

$$\|\hat{\theta}(t) - \theta\|^2 = O\left(\frac{[\ln r_0(t)]^\beta}{\lambda_{\min}[P_0^{-1}(t)]}\right), \text{ a.s., } \beta > 1+\varepsilon$$

as stated in Theorem 1 of [1]. □

Manuscript received February 7, 2007; revised April 28, 2008. First published June 6, 2008; current version published September 17, 2008. The associate editor coordinating the review of this paper and approving it for publication was Dr. Antonia Papandreou-Suppappola. This research was supported by the National Natural Science Foundation of China (No. 60574051) and by the Natural Science Foundation of Jiangsu Province (BK2007017, China) and by Program for Innovative Research Team of Jiangnan University.

F. Ding is with the Control Science and Engineering Research Center, Jiangnan University, Wuxi, 214122, China (e-mail: fding@jiangnan.edu.cn).

Y. Shi is with the Department of Mechanical Engineering, University of Saskatchewan, Saskatoon, SK S7N 5A9, Canada (e-mail: yang.shi@usask.ca).

T. Chen is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6G 2V4, Canada (e-mail: tchen@ece.ualberta.ca).

Digital Object Identifier 10.1109/TSP.2008.926687