work will involve analysis of retransmission strategies where more than one packet can be sent at a time.

REFERENCES

[1] N. E. Nahi, "Optimal recursive estimation with uncertain observation," *IEEE Trans. Inf. Theory*, vol. IT-15, no. 4, pp. 457–462, Jul. 1969.

[2] R. A. Monzingo, "Discrete optimal linear smoothing for systems with uncertain observations," *IEEE Trans. Inf. Theory*, vol. IT-21, no. 3, pp. 271–275, May 1975.

[3] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1453–1464, Sep. 2004.

[4] X. Liu and A. J. Goldsmith, "Kalman filtering with partial observation losses," in *Proc. IEEE Conf. Decision Control*, Bahamas, Dec. 2004, pp. 1413–1418.

[5] M. Huang and S. Dey, "Stability of Kalman filtering with Markovian packet losses," *Automatica*, vol. 43, no. 4, pp. 598–607, Apr. 2007.

[6] L. Schenato, "Optimal estimation in networked control systems subject to random delay and packet drop," in *Proc. IEEE Conf. Decision Control*, San Diego, CA, Dec. 2006, pp. 5615–5620.

[7] L. Shi, M. Epstein, A. Tiwari, and R. M. Murray, "Estimation with information loss: Asymptotic analysis and error bounds," in *Proc. IEEE Conf. Decision Control*, Seville, Spain, Dec. 2005, pp. 1215–1221.

[8] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.

[9] Y. Xu and J. P. Hespanha, "Estimation under uncontrolled and controlled communications in networked control systems," in *Proc. IEEE Conf. Decision Control*, Seville, Spain, Dec. 2005, pp. 842–847.

[10] A. Ribeiro, G. B. Giannakis, and S. I. Roumeliotis, "SOI-KF: Distributed Kalman filtering with low-cost communications using the sign of innovations," *IEEE Trans. Signal Process.*, vol. 54, no. 12, pp. 4782–4795, Dec. 2006.

[11] A. K. Fletcher, S. Rangan, and V. K. Goyal, "Estimation from lossy sensor data: Jump linear modelling and Kalman filtering," in *Proc. Int. Conf. Inf. Process. Sensor Netw.*, Berkeley, CA, Apr. 2004, pp. 251–258.

[12] A. S. Leong, S. Dey, and J. S. Evans, "Probability of error analysis for hidden Markov model filtering with random packet loss," *IEEE Trans. Signal Process.*, vol. 55, no. 3, pp. 809–821, Mar. 2007.

[13] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust H_∞ filtering for stochastic time-delay systems with missing measurements," *IEEE Trans. Signal Process.*, vol. 54, no. 7, pp. 2579–2587, Jul. 2006.

[14] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of control and estimation over lossy networks," *Proc. IEEE*, vol. 95, no. 1, pp. 163–187, Jan. 2007.

[15] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Englewood Cliffs, NJ: Prentice-Hall, 1979.

[16] S. Adlakha, "Kalman filtering in correlated losses," Stanford Univ., Stanford, CA, Tech. Rep., 2005.

[17] M. Epstein, L. Shi, A. Tiwari, and R. M. Murray, "Probabilistic performance of state estimation across a lossy network," *Automatica*, 2008, to be published.

Modified Pisarenko Harmonic Decomposition for Single-Tone Frequency Estimation

Kenneth Wing Kin Lui and Hing Cheung So

***Abstract*—In this correspondence, based on an alternative derivation of the Pisarenko harmonic decomposition (PHD) method, a new asymptotically unbiased estimator for the frequency of a single real tone in white noise is devised with the use of novel sample covariance expressions. Furthermore, extension to sample covariances with higher lags for performance enhancement is investigated while a simple and effective scheme is suggested to resolve the corresponding frequency ambiguity problem. The variance of the modified Pisarenko's method is also derived, which is then utilized to find the best estimate among all admissible solutions from various sets of sample covariances. Computer simulations are included to corroborate the theoretical development and to demonstrate that the proposed approach outperforms several existing low-complexity frequency estimators in terms of nearly uniform performance and estimation accuracy.**

***Index Terms*—Frequency estimation, Pisarenko's method, sample covariance, single real sinusoid.**

I. INTRODUCTION

Frequency estimation of sinusoidal signals in noise is a frequently addressed problem in the signal processing literature [1]–[5] because of its wide applicability in control theory, digital communications, biomedical engineering, instrumentation and measurement, and so on. In this work, we address the fundamental problem of single sinusoidal frequency estimation, and its discrete-time signal model is

$$x(n) = s(n) + q(n), \qquad n = 1, 2, \ldots N \tag{1}$$

where

$$s(n) = \alpha \cos(\omega n + \phi). \tag{2}$$

The α, $\omega \in (0, \pi)$, and $\phi \in [0, 2\pi)$ are unknown but deterministic constants that represent the tone amplitude, frequency, and phase, respectively, while the noise $q(n)$ is assumed to be a zero-mean white process with unknown variance σ^2. The task is to find ω given the N samples of $\{x(n)\}$.

Under Gaussian noise assumption, the maximum-likelihood (ML) estimate of frequency [6], with estimation accuracy of order $N^{-3/2}$ in standard error, is obtained by maximizing a highly nonlinear and multimodal cost function, and thus extensive computations are involved. Apart from the ML method, some relatively fast algorithms such as the discrete Fourier spectrum (DFS) interpolator [7], contraction mapping [8], [9], and weighted subspace fitting [10] can achieve this accuracy. It is worthy to note that the efficient methods for complex tone frequency estimation [11]–[14] generally cannot be employed to real-valued data. For applications where real-time estimation is required, more computationally efficient but suboptimal frequency estimators such as notch filtering, Capon methods, linear prediction [15]–[17], Yule-Walker methods [18] and subspace-based approaches [19] are widely used choices. In this correspondence, we focus on fast frequency estimation of a single real tone. Our main contributions are summarized as follows: 1) proposal of novel sample

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covariance expressions for single real-tone frequency estimation; 2) development and analysis of a set of new frequency estimators based on Pisarenko harmonic decomposition (PHD) [19]; and 3) design of a simple scheme to find the most accurate frequency estimate among all admissible solutions.

The rest of this correspondence is organized as follows. In Section II, we first review that the PHD method [19] for single-tone frequency estimation, which exploits the eigenstructure of the sample covariance matrix, can be derived in an alternative and simpler manner using the sample covariances of $x(n)$ with lags 1 and 2 [20]. Inspired by the modified covariance (MC) frequency estimator [15], [16] and reformed Pisarenko harmonic decomposer (RPHD) [17], novel sample covariance expressions are proposed in Pisarenko's frequency estimator. We then extend the idea using sample covariances with higher lags and an effective scheme is suggested to resolve the corresponding frequency ambiguity problem. In addition, the theoretical variance of the proposed approach is produced, which is utilized to find the best estimate among all admissible solutions based on various sample covariances. Numerical examples are presented in Section III to corroborate the analytical development and to evaluate the performance of the modified Pisarenko's estimator by comparing with the ML, MC, PHD, RPHD, and DFS interpolator algorithms as well as Cramér–Rao lower bound (CRLB). Finally, conclusions are drawn in Section IV.

II. Modified Pisarenko's Method

Denote the standard sample covariance of $x(n)$ with lag k by r_k, which is expressed as

$$r_k = \frac{1}{N-k}\sum_{n=1}^{N-k} x(n)x(n+k). \tag{3}$$

For sufficiently large N, it can be easily shown that the expected values of r_1 and r_2 are approximated as

$$E\{r_1\} \approx \frac{\alpha^2\cos(\omega)}{2} \tag{4}$$

and

$$E\{r_2\} \approx \frac{\alpha^2\cos(2\omega)}{2} = \alpha^2\left(\cos^2(\omega)+\frac{1}{2}\right) \tag{5}$$

where E denotes the expectation operator. Cross multiplying (4) and (5) and dropping the expectation operator, we obtain a quadratic equation for frequency that relates r_1 and r_2

$$2r_1\cos^2(\hat{\omega}) - r_2\cos(\hat{\omega}) - r_1 = 0 \tag{6}$$

where $\hat{\omega}$ denotes an estimate of ω. Only one root of (6) corresponds to the actual frequency and it can be verified that $\hat{\omega}$ has the form of

$$\hat{\omega} = \cos^{-1}\left(\frac{r_2+\sqrt{r_2^2+8r_1^2}}{4r_1}\right). \tag{7}$$

The solution of (7) is in fact identical to the PHD estimate [19], which is found from the eigenvector corresponding to the smallest eigenvalue of covariance matrix of $x(n)$. In fact, this alternative derivation has been presented in [20]. Due to the approximate relationships in (4) and (5), the PHD method is a biased frequency estimator and its bias generally decreases with N as illustrated in [17].

The key idea of this work is based on (4)–(6) and our major novelties include utilization of an alternative sample covariance function, as well as sample covariances with higher lags for improving frequency estimation performance. First, we propose to use the following scaled sample covariance expressions of r_1 and r_2:

$$r_1 = \sum_{n=4}^{N-1} x(n-1)[x(n)+x(n-2)] \tag{8}$$

and

$$r_2 = \sum_{n=5}^{N} x(n-2)[x(n)+x(n-4)]. \tag{9}$$

In fact, the choice of r_1 is not new and is found in the MC frequency estimator [15], [16] and reformed RPHD [17], while the proposed expression for r_2 straightforwardly follows that of r_1. Similar to [15], (8) can also be written as

$$r_1 = x(2)x(3) + 2x(3)x(4) + \cdots 2x(N-3)x(N-2) + x(N-2)x(N-1)$$

which is a covariance taper of the endpoints. It is noteworthy that not all available samples are used in (8), so that the numbers of terms in both r_1 and r_2 are identical. In doing so, the expected values of (8) and (9) then become

$$E\{r_1\} = 2\cos(\omega)\sum_{n=5}^{N} s^2(n-2) \tag{10}$$

and

$$E\{r_2\} = 2\cos(2\omega)\sum_{n=5}^{N} s^2(n-2). \tag{11}$$

From (6), (10), and (11), we expect that unbiased frequency estimation can be achieved and the proposed frequency estimate is given by (7) with the use of (8) and (9). In fact, a detailed proof regarding the asymptotically unbiasedness of our frequency estimator is provided in the Appendix. Since (4) and (5) are approximations while (10) and (11) are the exact counterparts, the proposed scheme will provide a higher estimation accuracy than the original PHD algorithm.

Our second direction for performance improvement [21] is to employ sample covariances with higher lags $\{r_k\}$ in estimating the frequency. This approach is analogous to employing a decimation in time domain [22]. Note that the idea of employing higher lags is also found in the high-order Yule–Walker method [18]. Extending (8) and (9), the general expressions for r_k and r_{2k} are

$$r_k = \sum_{n=4k}^{N-k} x(n-k)[x(n)+x(n-2k)] \tag{12}$$

and

$$r_{2k} = \sum_{n=5k}^{N} x(n-2k)[x(n)+x(n-4k)] \tag{13}$$

where we can observe that the maximum allowable value of k is $k_{\max} = \lfloor (N-1)/5 \rfloor$. The expected values of r_k and r_{2k} are easily shown to be

$$E\{r_k\} = 2\cos(k\omega)\sum_{n=5k}^{N} s^2(n-2k) \tag{14}$$

and

$$E\{r_{2k}\} = 2\cos(2k\omega)\sum_{n=5k}^{N} s^2(n-2k). \tag{15}$$

Following (4)–(6), the estimate of $\rho_k = \cos(k\omega)$, denoted by $\hat{\rho}_k$, is

$$\hat{\rho}_k = \frac{r_{2k}+\sqrt{r_{2k}^2+8r_k^2}}{4r_k}. \tag{16}$$

TABLE I
COMPUTATIONAL COMPLEXITY OF FREQUENCY ESTIMATORS

Method	Addition	Multiplication	Division	Square Root	$\cos^{-1}()$	$\tan^{-1}()$
Proposed with $k=1$	$4N-16$	$2N-4$	1	1	1	0
Proposed with $k=k^*$	$4k_{\max}(N+1)-$ $10k_{\max}(k_{\max}+1)+k^*+4$	$k_{\max}(2N+7)-$ $5k_{\max}(k_{\max}+1)+12$	$k_{\max}$	$k_{\max}$	k^*+1	0
ML (per frequency point)	$5N-2$	$5N+9$	1	0	0	0
MC	$3N-8$	$2N-3$	1	0	1	0
PHD	$2N-1$	$2N+3$	1	1	1	0
RPHD	$3N-3$	$2N+5$	1	1	1	0
DFS interpolator	$2N\log_2 N+54$	$2N\log_2 N+31$	0	0	0	1

However, $\hat{\rho}_k$ corresponds to k possible estimates of ω, denoted by $\hat{\omega}_{k,i}$, $i=1,2,\ldots,k$

$$\hat{\omega}_{k,i}=\frac{1}{k}\left[(-1)^{(i-1)}\cos^{-1}(\hat{\rho}_k)+\left\lfloor\frac{i}{2}\right\rfloor 2\pi\right]. \tag{17}$$

A simple way of finding ω from $\{\hat{\omega}_{k,i}\}$ is to compare each of them with $\omega_{1,1}$, that is, the frequency estimate computed from (8) and (9). The frequency estimate based on r_k and r_{2k} is then given by $\hat{\omega}_{k,i^*}$, where i^* is determined as

$$i^*=\arg\min_{i\in\{1,2,\ldots,k\}}|\hat{\omega}_{k,i}-\hat{\omega}_{1,1}|. \tag{18}$$

To derive the variance of the modified PHD estimator, we follow the work of [23]. Based on (6), we construct a quadratic function, namely, $f(\hat{\rho}_k)$

$$f(\hat{\rho}_k)=2r_k\hat{\rho}_k^2-r_{2k}\hat{\rho}_k-r_k. \tag{19}$$

For sufficiently large signal-to-noise ratio (SNR), one root of $f(\hat{\rho}_k)=0$ will be located in the proximity of $\cos(k\omega)$. The variance of $\hat{\rho}_k$, denoted by $\mathrm{var}(\hat{\rho}_k)$, is given by [23]

$$\mathrm{var}(\hat{\rho}_k)=E\{(\hat{\rho}_k-\rho_k)^2\}\approx\left.\frac{E\{f^2(\hat{\rho}_k)\}}{(E\{f'(\hat{\rho}_k)\})^2}\right|_{\hat{\rho}_k=\cos(k\omega)} \tag{20}$$

where $f'(\hat{\rho}_k)$ is the derivative of $f(\hat{\rho}_k)$ with respect to $\hat{\rho}_k$. Assuming that $q(n)$ is Gaussian distributed and N is sufficiently large, we have shown that (see the Appendix)

$$\mathrm{var}(\hat{\rho}_k)\approx\frac{k}{\mathrm{SNR}(N-5k+1)^2(\cos(2k\omega)+2)^2}+\frac{\cos^2(k\omega)(N-6k+1)+\cos^2(2k\omega)(N-5k+1)}{\mathrm{SNR}^2(N-5k+1)^2(\cos(2k\omega)+2)^2} \tag{21}$$

where $\mathrm{SNR}=\alpha^2/(2\sigma^2)$. The variance of the frequency estimate, namely, $\hat{\omega}_{k,i^*}$, denoted by $\mathrm{var}(\hat{\omega}_{k,i^*})$, is related to $\mathrm{var}(\hat{\rho}_k)$ as [23]

$$\mathrm{var}(\hat{\omega}_{k,i^*})=E\{(\hat{\omega}_{k,i^*}-\omega)^2\}\approx\frac{\mathrm{var}(\hat{\rho}_k)}{k^2\sin^2(k\omega)}. \tag{22}$$

For $\mathrm{SNR}\gg 1$, the second term of (21) can be ignored and a simplified expression for $\hat{\omega}_{k,i^*}$ is then

$$\mathrm{var}(\hat{\omega}_{k,i^*})\approx\frac{1}{\mathrm{SNR}k(N-5k+1)^2(\cos(2k\omega)+2)^2\sin^2(k\omega)}. \tag{23}$$

It is observed that for $k=1$, the standard error is of order N^{-1} whereas that of the ML approach is of order $N^{-3/2}$, which indicates the suboptimality of the proposed estimator. Nevertheless, the error is also a function of k for $k>1$, and Section III illustrates that (23) can approach the CRLB. Furthermore, the variance of the our approach is frequency dependant and we believe that the nonuniform performance results from the use of only two sample covariances, namely, r_k and r_{2k}, instead of all appropriate terms in the estimation process. Ignoring the trigonometric terms in (23), we see that the frequency variance generally decreases when sample covariances with higher lags are employed, which agrees with the intuitive observation in [21].

Expression (23) also indicates that there exists an optimal value of k for a particular frequency. The best choice from $\{\hat{\omega}_{k,i^*}\}$ can be determined by minimizing (23). Using (23), the ideal k, denoted by k^*, is then sought as

$$k^*=\arg\max_{k\in\{1,2,\ldots,k_{\max}\}}\left(k(N-5k+1)^2(\cos(2k\omega)+2)^2\sin^2(k\omega)\right). \tag{24}$$

Since ω is not available in (24), we will substitute $\cos(k\omega)$ with $\hat{\rho}_k$ in practice. After simple manipulations, the practical k is finally evaluated as

$$k^*=\arg\max_{k\in\{1,2,\ldots,k_{\max}\}}(k(N-5k+1)^2(2\hat{\rho}_k^2+1)^2(1-\hat{\rho}_k^2)). \tag{25}$$

In summary, the procedure of finding the best frequency estimate based on the proposed approach consists of the following three steps, namely: 1) compute all $\hat{\rho}_k$, $k=1,2,\ldots,k_{\max}$ using (12), (13), and (16); 2) determine k^* using (25); and 3) determine i^* using (17) and (18) and the optimal estimate is then given by $\hat{\omega}_{k^*,i^*}$. It is noteworthy to point out the our work is similar to [13], where a phase unwrapping rule analogous to (18) is suggested for single complex-tone frequency estimation with optimum k, namely, $k^*=2N/3$, which is frequency independent.

Finally, the computational complexity of the proposed approach with $k=1$ and $k=k^*$, together with that of the ML, MC, PHD, RPHD, and DFS five-term linear interpolator algorithms, is listed in Table I. It is seen that the modified Pisarenko's method with $k=1$, MC, PHD, and RPHD estimators involves similar operation requirements. When $k=k^*$ is employed, there is an increase in complexity of approximately $k_{\max}$ times compared with that of $k=1$, while the computational requirement of the DFS interpolator is of order $N\log_2 N$. As $k_{\max}$ is generally greater than $\log_2 N$, the DFS scheme is computationally simpler than the proposed method with $k=k^*$. Note that the ML implementation is the most computationally expensive because the operations in Table I only correspond to a single frequency point calculation in the ML cost function.

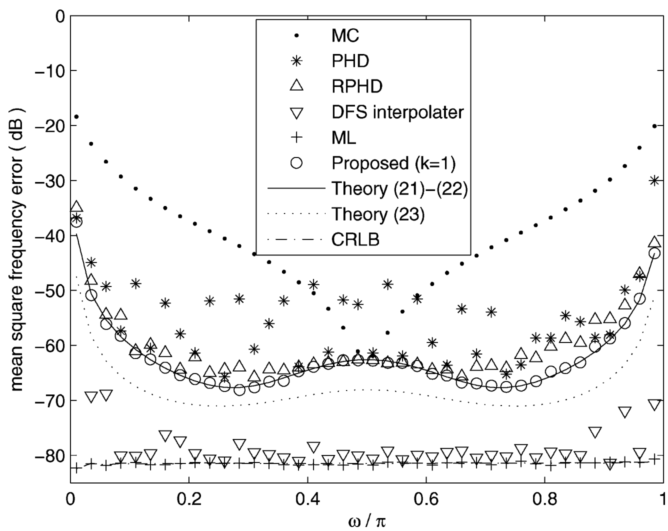

Fig. 1. Mean square frequency error versus ω at SNR = 20 dB and $N = 256$ with $k = 1$.

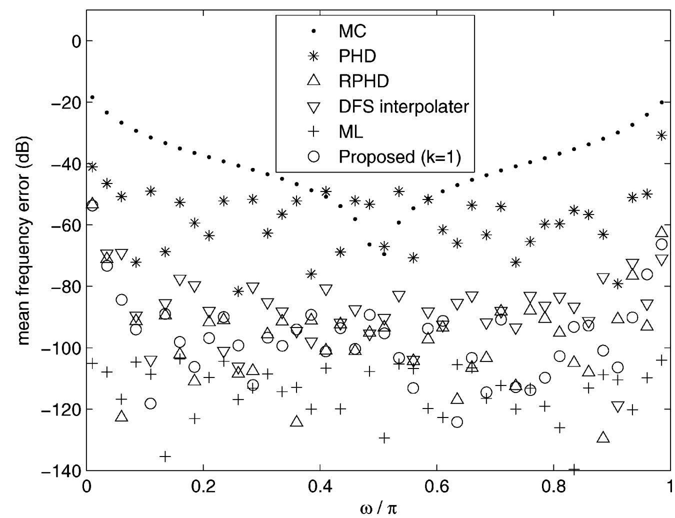

Fig. 2. Mean frequency error versus ω at SNR = 20 dB and $N = 256$ with $k = 1$.

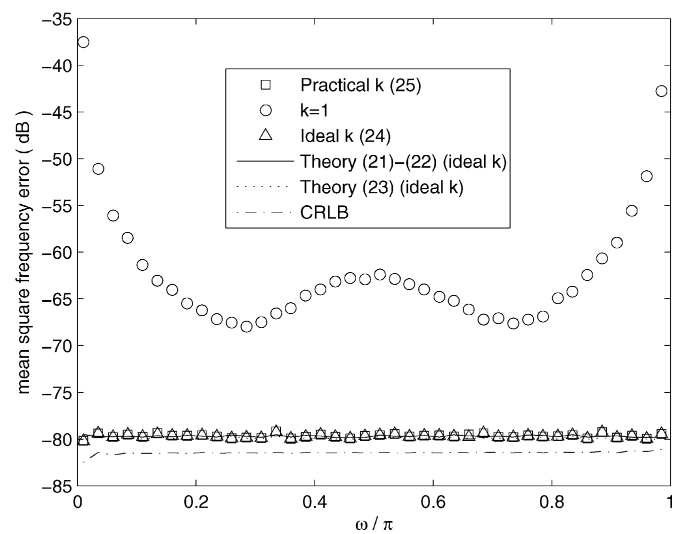

Fig. 3. Mean square frequency error versus ω at SNR = 20 dB and $N = 256$ with optimum k.

III. Simulation Results

Computer simulations have been carried out to evaluate the proposed approach for frequency estimation of a single real sinusoid in white Gaussian noise. We compared its performance with that of the ML [6], MC [15], [16], PHD [20], RPHD [17], and DFS interpolator [7] algorithms as well as CRLB for frequency estimation [24]. Note that the DFS interpolator can be employed for frequency estimation of multiple real or complex tones while the other estimators are specially designed for the signal model of (1) and (2) and thus they are only useful for real signals having a single frequency component. The tone amplitude was $\sqrt{2}$ and $\phi = 1$ was used, while different SNRs were obtained by proper scaling the noise variance σ^2. All simulation results provided were based on average of 1000 independent runs.

Fig. 1 shows the mean square frequency errors (MSFEs) of the six estimators as well as CRLB versus ω at SNR = 20 dB and $N = 256$. The proposed algorithm employed r_1 and r_2 of (8) and (9), which corresponded to the simple version of $k = 1$, so that the comparative performance with different forms of sample covariances can be observed. The variance expressions of the frequency estimate based on (21)–(23) are also plotted to check the validity of theoretical performance of the modified Pisarenko's method. We observe that the former predicted the performance of the proposed method very well while there was small discrepancy for the latter. Note that the calculated values of using (A.11) are not shown because the differences between (21) and (22) were negligible for $N = 256$. From Fig. 1, it is seen that the proposed scheme is superior to other three low-complexity algorithms particularly when ω was around 0.3π or 0.7π, although all four of them were suboptimal estimators because of their degradation from the CRLB. On the other hand, the most computationally expensive ML estimator could provide the best estimation performance while the accuracy of the DFS interpolator was nearly optimum except when ω was close to 0 or π, which resulted from the DFS frequency resolution limitation. The mean absolute frequency errors of the six methods, which were the absolute differences between ω and the corresponding mean frequency estimates, are shown in Fig. 2. It is observed that the biases of the modified Pisarenko's estimator with $k = 1$, DFS interpolator, ML, and RPHD methods were much smaller than those of the PHD and MC methods for the whole frequency range, which demonstrates the asymptotically unbiasedness of the proposed algorithm.

Fig. 3 illustrates the performance improvement of the proposed approach with the use of the optimal set of r_k and r_{2k}. The simulation settings were the same as in Fig. 1. It is observed that the modified Pisarenko's estimator with the practical k determined from (25) significantly outperformed that of $k = 1$, and it gave nearly uniform frequency estimation performance. Its MSFEs deviated from the CRLB only within 2 dB, and the validity of the theoretical calculations was again confirmed. We also see that there was no difference in performance when the ideal k of (24) was employed.

Fig. 4 shows the MSFEs of the six estimators versus SNR at $N = 256$ while Fig. 5 plots the mean square errors versus N at SNR = 20 dB. In both cases, the frequency was fixed at $\omega = 0.3\pi$. It is seen that the proposed approach outperformed the other three computationally simple frequency estimation methods, even with $k = 1$, although the performance with k^* was much closer to the CRLB or that of the DFS interpolator and ML estimator. We also observe from Fig. 4 that the DFS interpolator was a biased estimator at SNR $\geq$ 30 dB. In addition, the proposed algorithm outperformed the DFS method for SNR $>$ 30 dB because of the bias resulted from presence of the second negative-frequency sinusoid, which had a prominent effect at high SNR conditions, while the latter provided better estimates for SNR $\leq$ 20 dB

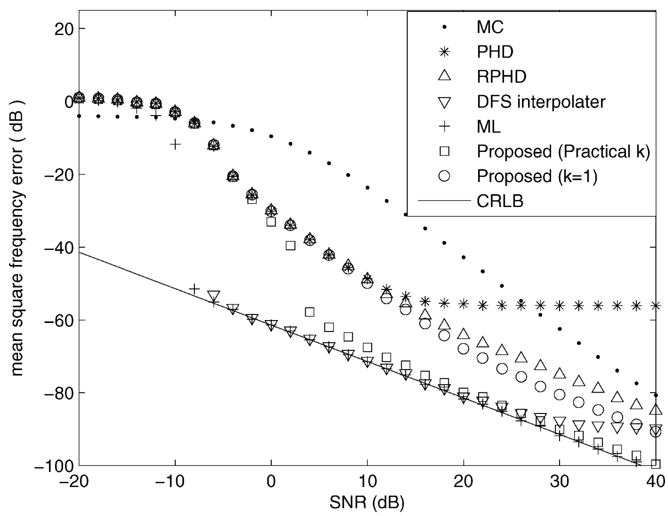

Fig. 4. Mean square frequency error versus SNR at $\omega = 0.3\pi$ and $N = 256$.

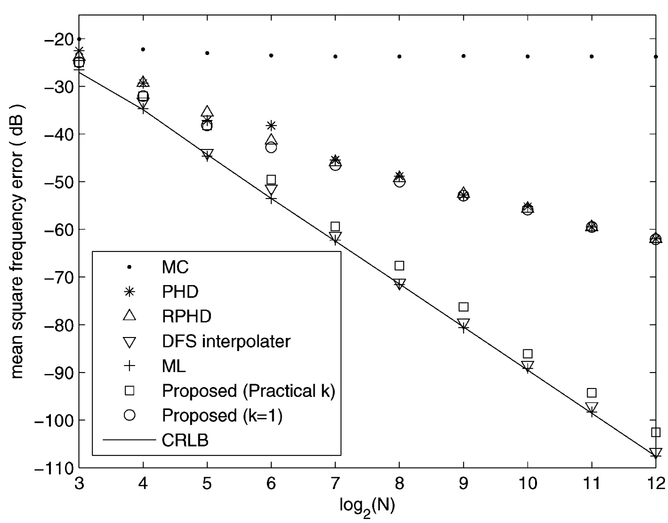

Fig. 5. Mean square frequency error versus N at $\omega = 0.3\pi$ and SNR = 20 dB.

as the bias was now negligible. That is, the DFS interpolator is superior to the proposed scheme in terms of smaller computational load and higher accuracy in low SNR conditions.

IV. Conclusion

Novel sample covariance expressions are proposed for the PHD method to achieve asymptotically unbiased frequency estimation for a single real tone in white noise. Extension to sample covariances with higher lags is investigated and the procedure of finding the best estimate among all admissible solutions from various sets of sample covariances is presented. The performance of the proposed approach is theoretically analyzed and confirmed via computer simulations. The simple version of the new method has higher estimation accuracy than that of the modified covariance, PHD, and reformed PHD algorithms although their computational requirements are comparable, while the optimum version of our proposal can provide more accurate and nearly uniform estimation performance but with higher computational complexity. As a research work, we will derive the limiting distribution [9], [18] of the proposed approach.

Appendix

In this appendix, we first prove the asymptotically unbiasedness of the frequency estimator and then the derivation of (21) will be given. Using Taylor's series to expand $f(\hat{\rho}_k)$ around $\rho_k = \cos(k\omega)$ up to the first-order term yields

$$\hat{\rho}_k \approx \cos(k\omega) - \left.\frac{f(\hat{\rho}_k)}{f'(\hat{\rho}_k)}\right|_{\hat{\rho}_k=\cos(k\omega)}. \tag{A.1}$$

The expression of $f'(\rho_k)$ is simply

$$f'(\rho_k) = 4r_k\rho_k - r_{2k} \tag{A.2}$$

From (12) and (13), r_k and r_{2k} can be expanded as

$$\begin{aligned} r_k =& (N-5k+1)\alpha^2\cos(k\omega) + \alpha^2\cos(k\omega)\sum_{n=3k}^{N-2k}\cos(2n\omega+2\phi) \\ &+ \sum_{n=4k}^{N-k} s(n-k)\left[q(n)+q(n-2k)\right] + 2\cos(k\omega)s(n-k)q(n-k) \\ &+ q(n-k)\left[q(n)+q(n-2k)\right] \end{aligned} \tag{A.3}$$

$$\begin{aligned} r_{2k} =& (N-5k+1)\alpha^2\cos(2k\omega) + \alpha^2\cos(2k\omega)\sum_{n=3k}^{N-2k}\cos(2n\omega+2\phi) \\ &+ \sum_{n=5k}^{N} s(n-2k)\left[q(n)+q(n-4k)\right] \\ &+ 2\cos(2k\omega)s(n-2k)q(n-2k) + q(n-2k)\left[q(n)+q(n-4k)\right]. \end{aligned} \tag{A.4}$$

Using (19) and (A.2)–(A.4) and considering $N \to \infty$, the second term of (A.1) is calculated as

$$\begin{aligned} &\lim_{N\to\infty} \left.\frac{f(\hat{\rho}_k)}{f'(\hat{\rho}_k)}\right|_{\hat{\rho}_k=\cos(k\omega)} \\ &\quad = \lim_{N\to\infty} \frac{\frac{1}{N}f(\rho_k)}{\frac{1}{N}f'(\rho_k)} \\ &\quad = \frac{2\alpha^2\cos(k\omega)\rho_k^2 - \alpha^2\cos(2k\omega)\rho_k - \alpha^2\cos(k\omega)}{4\alpha^2\cos(k\omega)\rho_k - \alpha^2\cos(2k\omega)} \\ &\quad = \frac{0}{2\cos^2(k\omega)+1} = 0 \end{aligned} \tag{A.5}$$

which shows that the proposed estimator is asymptotically unbiased.

We now derive (21) as follows. From (19), it is easy to show that

$$\begin{aligned} E\{f^2(\hat{\rho}_k)\}\,|_{\rho_k=\cos(k\omega)} =& \cos^2(2k\omega)E\{r_k^2\} \\ &- 2\cos(k\omega)\cos(2k\omega)E\{r_kr_{2k}\} + \cos^2(\omega)E\{r_{2k}^2\}. \end{aligned} \tag{A.6}$$

The required terms, namely, $E\{r_k^2\}$, $E\{r_kr_{2k}\}$, and $E\{r_{2k}^2\}$, are computed as

$$\begin{aligned} &E\{r_k^2\} \\ &= \alpha^4\cos^2(k\omega)[(N-5k+1) + g(\omega,\phi,N-2k,0,3k)]^2 \\ &\quad + \alpha^2\sigma^2[4N-22k+4+(4N-24k+4)\cos(2k\omega)] \\ &\quad + \alpha^2\sigma^2[(1+2\cos^2(k\omega))g(\omega,\phi,N-2k,0,3k) \\ &\quad + g(\omega,\phi,N-2k,-2k,5k) \\ &\quad + 4\cos(k\omega)g(\omega,\phi,N-2k,-k,4k)] \\ &\quad + 2(2N-10k+2)\sigma^4 \end{aligned} \tag{A.7}$$

$$\begin{aligned} &E\{r_kr_{2k}\} \\ &= \alpha^4\cos(k\omega)\cos(2k\omega)[(N-5k+1) + g(\omega,\phi,N-2k,0,3k)]^2 \\ &\quad + \alpha^2\sigma^2[(4N-24k+4)\cos(k\omega) + (4N-26k+4)\cos(3k\omega)] \\ &\quad + \alpha^2\sigma^2[(2\cos(2k\omega)+1)g(\omega,\phi,N-2k,-k,4k) \\ &\quad + 2\cos(k\omega)\cos(2k\omega)g(\omega,\phi,N-2k,0,3k) \\ &\quad + 2\cos(k\omega)g(\omega,\phi,N-2k,-2k,5k) \\ &\quad + g(\omega,\phi,N-2k,-3k,6k)] \end{aligned} \tag{A.8}$$

$$\mathrm{var}(\hat{\rho}_k) \approx \frac{2k+\mathcal{F}(\omega,\phi,N,k)+\mathcal{G}(\omega,\phi,N,k)+\mathcal{H}(\omega,\phi,N,k)}{2\mathrm{SNR}(N-5k+1+g(\omega,\phi,N-2k,0,3k))^2(\cos(2k\omega)+2)^2}+\frac{\cos^2(k\omega)(N-6k+1)+\cos^2(2k\omega)(N-5k+1)}{\mathrm{SNR}^2(N-5k+1+g(\omega,\phi,N-2k,0,3k))^2(\cos(2k\omega)+2)^2}. \tag{A.11}$$

and

$$\begin{aligned}E\{r_{2k}^2\} &= \alpha^4\cos^2(2k\omega)[(N-5k+1)+g(\omega,\phi,N-2k,0,3k)]^2\\ &\quad+\alpha^2\sigma^2[4N-24k+4+(4N-28k+4)\cos(4k\omega)]\\ &\quad+\alpha^2\sigma^2[4\cos(2k\omega)g(\omega,\phi,N-2k,-2k,5k)\\ &\quad+(2\cos^2(2k\omega)+1)g(\omega,\phi,N-2k,0,3k)\\ &\quad+g(\omega,\phi,N-2k,-4k,7k)]+2(2N-12k+2)\sigma^4 \end{aligned}\tag{A.9}$$

where

$$\begin{aligned}\mathcal{F}(\omega,\phi,N,k) &= \cos^2(2k\omega)[(1+2\cos^2(k\omega))g(\omega,\phi,N-2k,0,3k)\\ &\quad+g(\omega,\phi,N-2k,-2k,5k)\\ &\quad+4\cos(k\omega)g(\omega,\phi,N-2k,-k,4k)]\\ \mathcal{G}(\omega,\phi,N,k) &= -2\cos(k\omega)\cos(2k\omega)\\ &\quad\times[(2\cos(2k\omega)+1)g(\omega,\phi,N-2k,-k,4k)\\ &\quad+2\cos(k\omega)\cos(2k\omega)g(\omega,\phi,N-2k,0,3k)\\ &\quad+2\cos(k\omega)g(\omega,\phi,N-2k,-2k,5k)\\ &\quad+g(\omega,\phi,N-2k,-3k,6k)]\\ \mathcal{H}(\omega,\phi,N,k) &= \cos^2(k\omega)[4\cos(2k\omega)g(\omega,\phi,N-2k,-2k,5k)\\ &\quad+(2\cos^2(2k\omega)+1)g(\omega,\phi,N-2k,0,3k)\\ &\quad+g(\omega,\phi,N-2k,-4k,7k)]\end{aligned}$$

and

$$\begin{aligned}g(\omega,\phi,N,k,b) &= \sum_{n=b}^{N}\cos((2n+k)\omega+2\phi)\\ &= \frac{\sin((2N+k+1)\omega+2\phi)-\sin((2b-1)+k)\omega+2\phi)}{2\sin(\omega)}.\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}E\{f'(\hat{\rho}_k)\} &= \alpha^2(\cos(2k\omega)+2)(N-5k+1+g(\omega,\phi,N-2k,0,3k)).\end{aligned}\tag{A.10}$$

Substituting (A.6)–(A.10) into (20) with SNR $=\alpha^2/(2\sigma^2)$ and after simplifications, we obtain (A.11), shown at the top of the page. For sufficiently large N, the terms $\mathcal{F}(\omega,\phi,N,k)$, $\mathcal{G}(\omega,\phi,N,k)$, and $\mathcal{H}(\omega,\phi,N,k)$ can be ignored since they will approach to zero with a convergence rate of N^{-1}. In doing so, the asymptotic expression of (21) is obtained.

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References

[1] S. L. Marple, *Digital Spectral Analysis with Applications*. Englewood Cliffs, NJ: Prentice-Hall, 1987.

[2] S. M. Kay, *Modern Spectral Estimation : Theory and Application*. Englewood Cliffs, NJ: Prentice-Hall, 1988.

[3] P. Stoica, “List of references on spectral line analysis,” *Signal Process.*, vol. 31, no. 3, pp. 329–340, Apr. 1993.

[4] B. G. Quinn and E. J. Hannan, *The Estimation and Tracking of Frequency*. Cambridge, U.K.: Cambridge Univ. Press, 2001.

[5] P. Stoica and R. Moses, *Spectral Analysis of Signals*. Upper Saddle River, NJ: Prentice-Hall, 2005.

[6] R. J. Kenefic and A. H. Nuttall, “Maximum likelihood estimation of the parameters of tone using real discrete data,” *IEEE J. Ocean. Eng.*, vol. 12, no. 1, pp. 279–280, Jan. 1987.

[7] M. D. Macleod, “Fast nearly ML estimation of the parameters of real or complex single tones or resolved multiple tones,” *IEEE Trans. Signal Process.*, vol. 46, no. 1, pp. 141–148, Jan. 1998.

[8] K. S. Song and T. H. Li, “A statistically and computationally efficient method for frequency estimation,” *Stochastic Processes Appl.*, vol. 86, pp. 29–47, 2000.

[9] T. H. Li and K. S. Song, “Asymptotic analysis of a fast algorithm for efficient multiple frequency estimation,” *IEEE Trans. Inf. Theory*, vol. 48, no. 10, pp. 2709–2719, Oct. 2002.

[10] K. Mahata, “Subspace fitting approaches for frequency estimation using real-valued data,” *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 3099–3110, Aug. 2005.

[11] E. Aboutanios and B. Mulgrew, “Iterative frequency estimation by interpolation on Fourier coefficients,” *IEEE Trans. Signal Process.*, vol. 53, no. 4, pp. 1237–1247, Apr. 2005.

[12] B. Volcker and P. Handel, “Frequency estimation from proper sets of correlations,” *IEEE Trans. Signal Process.*, vol. 50, no. 4, pp. 791–802, Apr. 2002.

[13] S. S. Abeysekera, “Efficient frequency estimation using the pulse-pair method at various lags,” in *Proc. Int. Symp. Circuits Syst.*, Bangkok, Thailand, May 2003, vol. 4, pp. 109–112.

[14] S. S. Abeysekera, “Efficient frequency estimation using the pulse-pair method at various lags,” *IEEE Trans. Commun.*, vol. 54, no. 9, pp. 1542–1546, Sep. 2006.

[15] L. B. Fertig and J. H. McClellan, “Instantaneous frequency estimation using linear prediction with comparisons to the DESAs,” *IEEE Signal Process. Lett.*, vol. 3, no. 2, pp. 54–56, Feb. 1996.

[16] D. W. Tufts and P. D. Fiore, “Simple effective estimation of frequency based on Prony’s method,” in *Proc. Int. Conf. Acoust. Speech Signal Process.*, Atlanta, GA, May 1996, vol. 5, pp. 2801–2804.

[17] H. C. So and K. W. Chan, “Reformulation of Pisarenko harmonic decomposition method for single-tone frequency estimation,” *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 1128–1135, Apr. 2004.

[18] P. Stoica, T. Soderstrom, and F. Ti, “Asymptotic properties of the high order Yule-walker estimates of sinusoidal frequencies,” *IEEE Trans. Acoust. Speech Signal Process.*, vol. 37, no. 11, pp. 1721–1734, Nov. 1989.

[19] V. F. Pisarenko, “The retrieval of harmonics by linear prediction,” *Geophys. J. Roy. Astron. Soc.*, vol. 33, pp. 347–366, 1973.

[20] Y. Xiao and Y. Tadokoro, “On Pisarenko and constrained Yule-Walker estimator of tone frequency,” *IEICE Trans. Fundam. Electron. Commun. Comput. Sci.*, vol. E77-A, no. 8, pp. 1404–1406, Aug. 1994.

[21] M. P. Fitz, “Further results in the fast estimation of a single frequency,” *IEEE Trans. Commun.*, vol. 42, no. 2, pp. 862–864, Feb. 1994.

[22] M. L. Fowler and J. A. Johnson, “Extending the threshold and frequency range for phase-based frequency estimation,” *IEEE Trans. Signal Process.*, vol. 47, no. 10, pp. 2857–2863, Oct. 1999.

[23] K. W. Chan and H. C. So, “An exact analysis of Pisarenko’s single-tone frequency estimation algorithm,” *Signal Process.*, vol. 83, no. 3, pp. 685–690, Mar. 2003.

[24] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1993, pp. 56–57.