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Optimum Detection of Spread Spectrum Signals

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Abstract

Signal detection and processing are now mostly frequency domain operations due to the easy availability of FFT chips. The paper derives the optimal frequency domain detector which (i) computes the spectral components of the input, (ii) weights each component with the corresponding true spectral component of true signal and (iii) compares the weighted sum against a threshold. When the true signal spectrum is not available, a sub-optimum detector takes simply the peak of the input spectrum as the test statistic. However, the segment length used in the Bartlett spectral estimator for computing the input spectrum has an effect on the detection performance. The best segment length is inversely proportional to the signal bandwidth. Verification of the theoretical development is by simulation of the detection of a frequency hop, digital FM signal.

I. INTRODUCTION

IN electronic warfare systems, there is a need to detect the occurrence of an unknown spread spectrum signal in the presence of noise. Let the intercepted data be

$$y(n) = \begin{cases} s(n) + \phi(n), & \text{signal present} \\ \phi(n), & \text{no signal present} \end{cases} \tag{1}$$

where $s(n)$ is a signal and $\phi(n)$ the background noise. The problem is to decide, from $y(n)$, $n = 0, ..., N-1$, whether $s(n)$ is present.

Due to the easy availability of fast Fourier transform (FFT) chips, the detection process is now mostly a frequency domain operation. A common detector, which has the names of radiometer, energy detector or quadratic detector [1], uses the test statistic

$$\Lambda_1(m) = |Y(m)|^2 \tag{2}$$

and compares Λ_1 against a threshold for detection. The spectral components

$$Y(m) = \sum_{n=0}^{N-1} y(n) e^{-j\frac{2\pi mn}{N}} \tag{3}$$

are computed from $y(n)$ by FFT. This detector is optimum if $s(n)$ is a Gaussian, bandlimited white noise signal. Optimum detection here means that for a given probability of false alarm (P_{FA}), the detector gives the maximum probability of detection (P_D).

Another detector uses only the peak of $|Y(m)|^2$ as a test statistic. This is optimum if $s(n)$ is a sinusoid of unknown amplitude, phase and frequency [2]. In many applications,

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$s(n)$ is a spread spectrum or sonar or radar signal and therefore neither a sinusoid nor a white noise signal. It is thus necessary to find an optimum detector for a general $s(n)$.

This paper develops a new optimum detector with the test statistic

$$\Lambda = \sum_{m=0}^{N-1} |Y(m)|^2 W(m) \tag{4}$$

where $W(m)$ is a spectral weighting sequence. The selection of $W(m)$ to optimize detection is the topic of Section II.

In the remainder of the paper, Section III studies the common but sub-optimum test statistic, which is the peak of $|Y(m)|^2$. This detector appears in practice because of its simplicity. However, computation of $|Y(m)|^2$ through (3) is now no longer the best spectral estimator for detection. A beter alternative is the Bartlett spectral estimator [3] where the estimated spectrum is obtained by averaging the spectra from each of the K segments of $L = \frac{N}{K}$ points formed by partitioning an N point data sequence before taking FFT and averaging the individual spectra to give the estimated spectrum. There is a preferred segment length L which Section III shows to be inversely proportional to the bandwidth of $s(n)$. Section IV contains a simulation experiment to compare the relative performance of the two new detectors. The test signal $s(n)$ is the output of the frequency hop radio described in [4]. The conclusions are in Section V.

II. THE OPTIMAL DISCRETE DETECTOR

For mathematical tractability, this section considers a different optimal criterion, and it is a maximization of the deflection ratio (DR) [1,2]

$$\text{DR} = \frac{\Big[E\{\Lambda/1\} - E\{\Lambda/0\}\Big]^2}{VAR\{\Lambda/0\}} \tag{5}$$

In (5), $E\{.\}$ denotes the expected value, $VAR\{.\}$ the variance, and $\Lambda/1$ the value of the test statistic when a signal is present, and $\Lambda/0$ when not. The DR, which is a signal to noise parameter, is an intuitively appealing measure since its maximization will also maximize the separation between the means of $\Lambda/1$ and $\Lambda/0$, normalized by the variance of $\Lambda/0$. However, a detector that provides a maximum DR is not necessarily the one that provides maximum P_D for a fixed P_{FA} [5]. The problem is to select $W(m)$ to maximize the DR of (5).

Let

$$S(m) = \sum_{n=0}^{N-1} s(n) e^{-j\frac{2\pi mn}{N}} \tag{6}$$

and

$$\Phi(m) = \sum_{n=0}^{N-1} \phi(n) e^{-j\frac{2\pi mn}{N}} \tag{7}$$

Then from (3), when $s(n)$ is present,

$$|Y(m)|^2 = \widetilde{G}_{yy}(m) = \Big(S(m) + \Phi(m)\Big)\Big(S^*(m) + \Phi^*(m)\Big) \tag{8}$$

with * denoting the complex conjugate. Taking expected value gives

$$E\{\widetilde{G}_{yy}(m)\} = \overline{G}_{yy}(m) = \overline{G}_{ss}(m) + \overline{G}_{\phi\phi}(m) \tag{9}$$

where

$$\overline{G}_{ss}(m) = E\{|S(m)|^2\}$$
$$\overline{G}_{\phi\phi}(m) = E\{|\Phi(m)|^2\}$$

and

$$E\{S(m)\Phi(m)\} = 0 \tag{10}$$

Similarly, when there is no signal,

$$E\{\widetilde{G}_{yy}(m)\} = E\{\widetilde{G}_{\phi\phi}(m)\} = \overline{G}_{\phi\phi}(m) \tag{11}$$

where $\widetilde{G}_{\phi\phi}(m)$ is similarly defined as in (8) when $s(n) = 0$.
Hence

$$E\{\Lambda/1\} - E\{\Lambda/0\} = \sum_{m=0}^{N-1} W(m)\overline{G}_{ss}(m) \tag{12}$$

Next, using

$$VAR\{\Lambda/0\} = E\{(\Lambda/0)^2\} - E^2\{\Lambda/0\} \tag{13}$$

it is easy to verify that

$$VAR\{\Lambda/0\} = \sum_{m=0}^{N-1} W^2(m)\Big[VAR\{\widetilde{G}_{\phi\phi}(m)\}\Big] \tag{14}$$

From [3], the variance of $\widetilde{G}_{\phi\phi}(m)$ is $\frac{N^2\sigma_\phi^4}{2}$, where $\sigma_\phi^2 = E\{\phi^2(n)\}$. Note that the N^2 term appears in the variance because the Fourier transform in (7) is not divided by N. Hence the DR for the detector of (9) is, from (12) and (14)

$$\mathrm{DR} = 2\frac{\left[\sum_m W(m)\overline{G}_{ss}(m)\right]^2}{N^2\sigma_\phi^4 \sum_m W^2(m)} \tag{15}$$

In (15) and in the sequel, unless otherwise stated, the summation limits are from 0 to $N-1$. Applying the Schwartz's inequality [6] to (15) gives the $W(m)$ that maximizes the DR as

$$W(m) = \overline{G}_{ss}(m) \tag{16}$$

Substituting (16) into (4) yields, in the sense of a detector whose DR is a maximum,

$$\Lambda = \sum_m \overline{G}_{ss}(m)\widetilde{G}_{yy}(m) \tag{17}$$

with the corresponding

$$\mathrm{DR} = \frac{2\sum_m \overline{G}_{ss}^2(m)}{N^2\sigma_\phi^4} = \frac{2\sum_m W^2(m)}{N^2\sigma_\phi^4} \tag{18}$$

The optimum detector in (17) first finds the spectral estimate of the data, and weights it, at each frequency bin, with the expected value of the signal spectral estimate. This expected value is calculated based on the same number of segments and segment length used in the spectral estimate of the data. The detector then sums all the weighted spectral components and compares the sum against a threshold.

From (10) and (16)

$$W(m) = E\{S(m)S^*(m)\} \tag{19}$$

Hence

$$W(m) = \sum_u \sum_v E\{s(u)s(v)\}e^{-j\frac{2\pi m}{N}(u-v)} \tag{20}$$

Let $u - v = i$ and (20) becomes

$$W(m) = \sum_u \sum_{i=-u}^{N-1-u} R_{ss}(i)e^{-j\frac{2\pi mi}{N}} \tag{21}$$

where

$$R_{ss}(i) = E\{s(u)s(v)\} = R_{ss}(u-v) = R_{ss}(-i) \tag{22}$$

and is the autocorrelation function of $s(n)$. Using (22) in (21) simplifies it, with $\omega_m = \frac{2\pi m}{N}$, to

$$W(m) = \sum_i 2(N-i)R_{ss}(i)\cos\omega_m i - NR_{ss}(0) \tag{23}$$

Squaring (23) yields

$$W^2(m) = N^2R_{ss}^2(0) + \sum_i \sum_k 4(N-i)(N-k)R_{ss}(i)R_{ss}(k)\cos\omega_m i\, \cos\omega_m k$$
$$-2NR_{ss}(0)\sum_i 2(N-i)R_{ss}(i)\cos\omega_m i \tag{24}$$

Summing (24) over all m gives

$$\sum_m W^2(m) = \sum_m N^2R_{ss}^2(0) + \sum_i \sum_k 4(N-i)(N-k)R_{ss}(i)R_{ss}(k)\sum_m \cos\omega_m i\, \cos\omega_m k$$
$$-2NR_{ss}(0)\sum_i 2(N-i)R_{ss}(i)\sum_m \cos\omega_m i \tag{25}$$

Using the summations

$$\sum_{m=0}^{N-1} \cos\omega_m i\, \cos\omega_m k = \begin{cases} \frac{N}{2}, & \text{for } i = k \neq 0 \\ N, & \text{for } i = k = 0 \\ 0, & \text{for } i \neq k \end{cases} \tag{26}$$

and

$$\sum_{m=0}^{N-1} \cos\omega_m i = \begin{cases} N, & \text{for } i = 0 \\ 0, & \text{for } i \neq 0 \end{cases} \tag{27}$$

in (25), and after some simplifications, it results in

$$\sum_m W^2(m) = N^3 R_{ss}^2(0) + 2N \sum_{i=1}^{N-1} (N-i)^2 R_{ss}^2(i) \tag{28}$$

which is a convenient formula for the computation of DR in (18).

III. A Sub-optimum Detector

In practice, the true spectrum of $s(n)$ is often not known although some knowledge of spectral features such as bandwidth or location of the spectral peak will be available. It will not be possible to compute the optimum weights $W(m)$ when the true spectrum of $s(n)$ is not known. Here it is more appropriate to use the detector

$$\Lambda_a = \widetilde{G}_{yy}(m) \tag{29}$$

and select the m that gives the highest $\widetilde{G}_{ss}(m)$ and compare it against a threshold. Of course if the peak location of the spectrum of $s(n)$ is known *a priori* to be at m_0, the Λ_a simply equals $\widetilde{G}_{yy}(m_0)$

An alternative to (3) in computing $|Y(m)|^2 = \widetilde{G}_{yy}(m)$ is the Bartlett estimator [3]

$$\hat{G}_{yy} = \frac{1}{K} \sum_{k=0}^{K-1} |Y_k(m)|^2 \tag{30}$$

where

$$Y_k(m) = \sum_{n=kL}^{(k+1)(L-1)} y(n) e^{-j\frac{2\pi mn}{L}}, \qquad KL = N \tag{31}$$

The corresponding DR of (29) is hence dependent on L, the number of points in a Bartlett estimator. Depending on the bandwidth of $s(n)$, there is an optimum L that maximizes the DR of (29).

For Λ_a of (29), its DR is

$$\mathrm{DR}_a = \frac{\left[E\{\widetilde{G}_{ss}(m_0)\}\right]^2}{VAR\{\widetilde{G}_{\phi\phi}(m)\}} \tag{32}$$

which on using (21), since $W(m)$ in (23) equals $E\{\widetilde{G}_{ss}(m)\}$, gives, with L replacing N,

$$\mathrm{DR}_a = \frac{N\left[\sum_{i=0}^{L-1} 2(L-i)R_{ss}(i)\cos\omega_0 i - LR_{ss}(0)\right]^2}{L^3\sigma_\phi^4} \tag{33}$$

where $\omega_0 = \frac{2\pi m_0}{L}$.

To see that there is indeed an optimum segment length L that maximizing DR_a of (29), consider the example of $s(n)$ being an ideal bandpass process of bandwidth $2\pi\Delta$ and center frequency $\omega_0 = 0.5\pi$. Then

$$R_{ss}(i) = 2\Delta \frac{\sin \pi \Delta i}{\pi \Delta i} \cos \omega_0 \; i \tag{34}$$

Table I lists the DR_a with Δ as a parameter. It clearly shows that the optimum L is inversely proportional to the signal bandwidth Δ, in Hz.

TABLE I

DR_a

	Number of points L in a Bartlett segment						
Δ	4	8	16	32	64	128	256
1/32	0.50	0.97	1.75	2.40	1.63	0.91	0.48
1/16	1.95	3.53	4.82	3.27	1.81	0.95	0.49
1/8	7.23	9.84	6.60	3.63	1.91	0.98	0.495
1/4	21.43	13.68	7.39	3.85	1.96	0.99	0.498

IV. Simulation Example

The combat net radio (primary) of [4] is the example for simulation. It transmits a frequency hop signal at a data rate of 19.2 kHz. The pulse shaping filter is a 5th order Butterworth with a 3 dB cut-off at 9.6 kHz. The modulation process is digital FM (continuous phase frequency shift keying) so that the input signal to the IF amplifier in Figure 1 is

$$s(t) = A \cos\left(\omega_c t + \omega_d \int_0^t m(t) dt\right) \tag{35}$$

where $m(t)$ is the output of the pulse shaping filter. The simulation parameters are

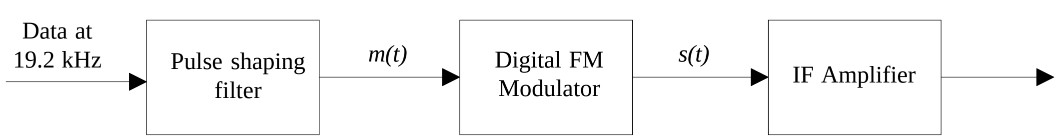

Fig. 1. Generation of the signal

$$\omega_c = 2\pi \times 10.7 \times 10^6 \text{ rad/sec.}$$

$$\omega_d = 2\pi \times 6 \times 10^3 \text{ rad/sec.}$$

$$\text{Sampling frequency} = 40 \times 10^6 \text{ Hz}$$

The spectrum of $s(n)$ is in Figure 2. The autocorrelation of the FM signal is shown in Fig. 3, whose main lobe spectrum is about 6,600 samples. The ROC curves of the detector of Section III are shown in Fig. 4. The results are based on 10,000 trials. For Λ_a of (29), various segment lengths of L indeed give different performance with $L = 6,144$, which is the main lobe width of the autocorrelation of $s(n)$, providing the best detection.

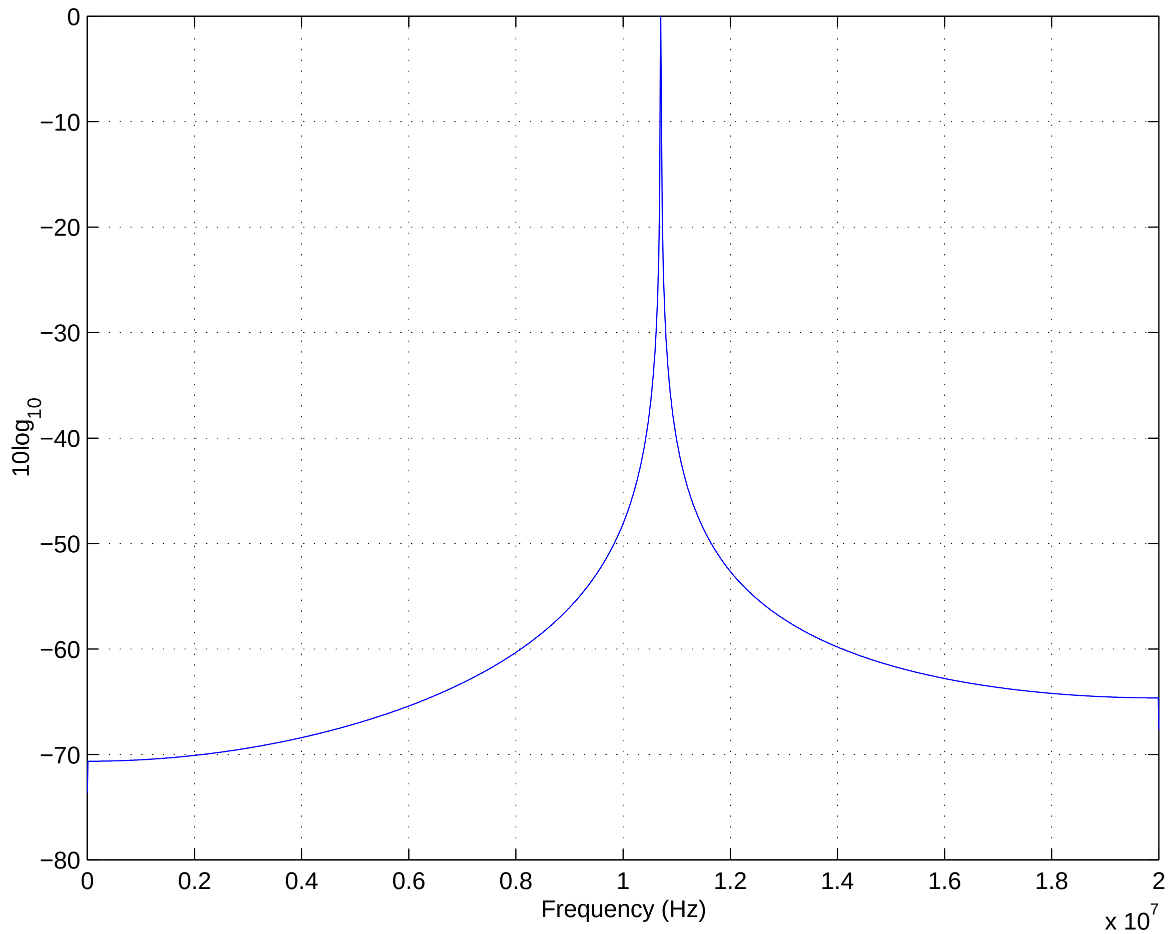

Fig. 2. The power spectrum of the FM signal

V. CONCLUSIONS

This paper has considered the optimum detection of a signal in the frequency domain. The optimum detector, in the sense of maximizing the deflection ratio, first computes the spectral components of an input, weights each component with the true spectral components of the signal, then sums up the weighted components and checks it against a threshold for detection. This operation is of course similar to the matched filter detector in the time domain. In practice, the signal true spectral components are normally not known. A sub-optimum detector uses instead the maximum of the input spectral components as a test statistic. In this case, the segment length of FFT plays a part in the performance of the detector. The optimum length is approximately inversely proportional to the bandwidth of the signal. Since in many detection cases the signal bandwidth is

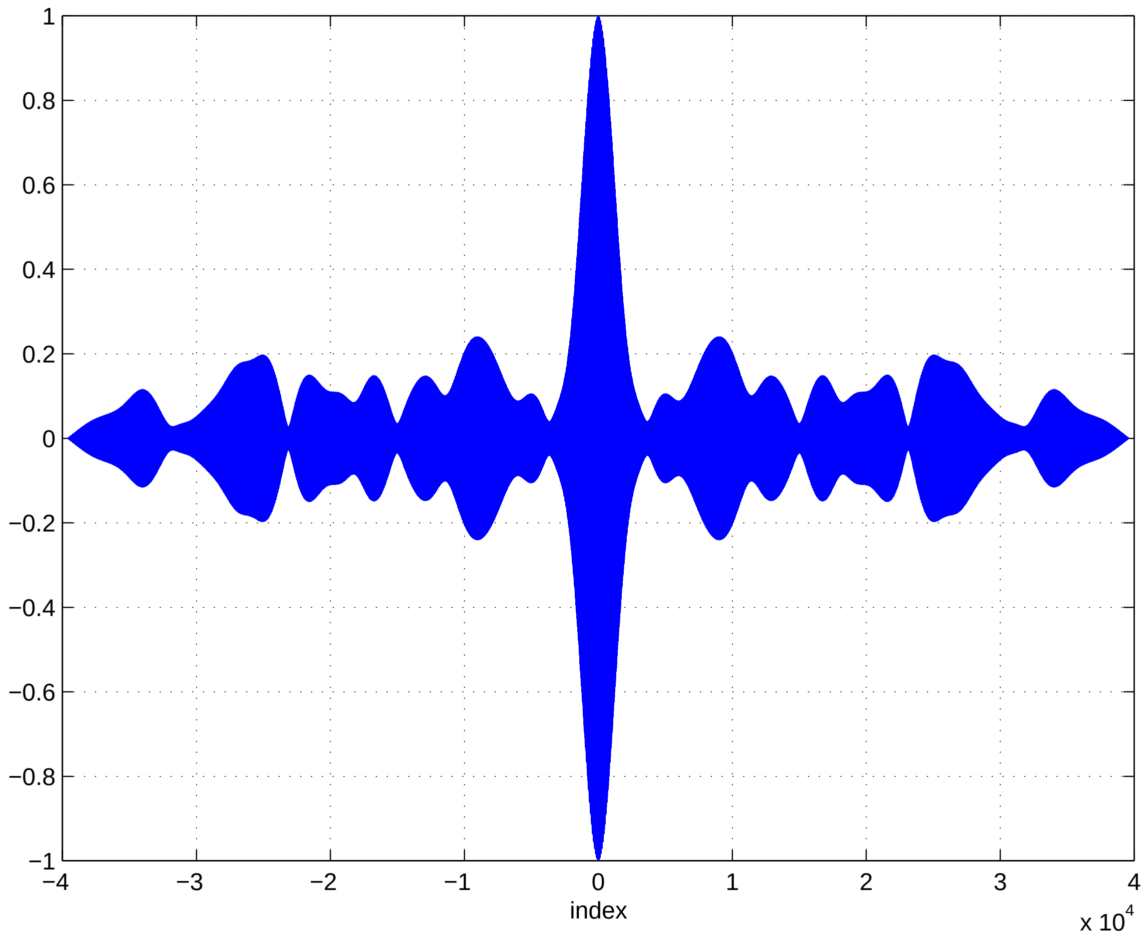

Fig. 3. The autocorrelation of the FM signal

normally known, even if approximately, this knowledge of the optimum segment to use in finding the spectrum for detection is of major importance in improving the detector performance.

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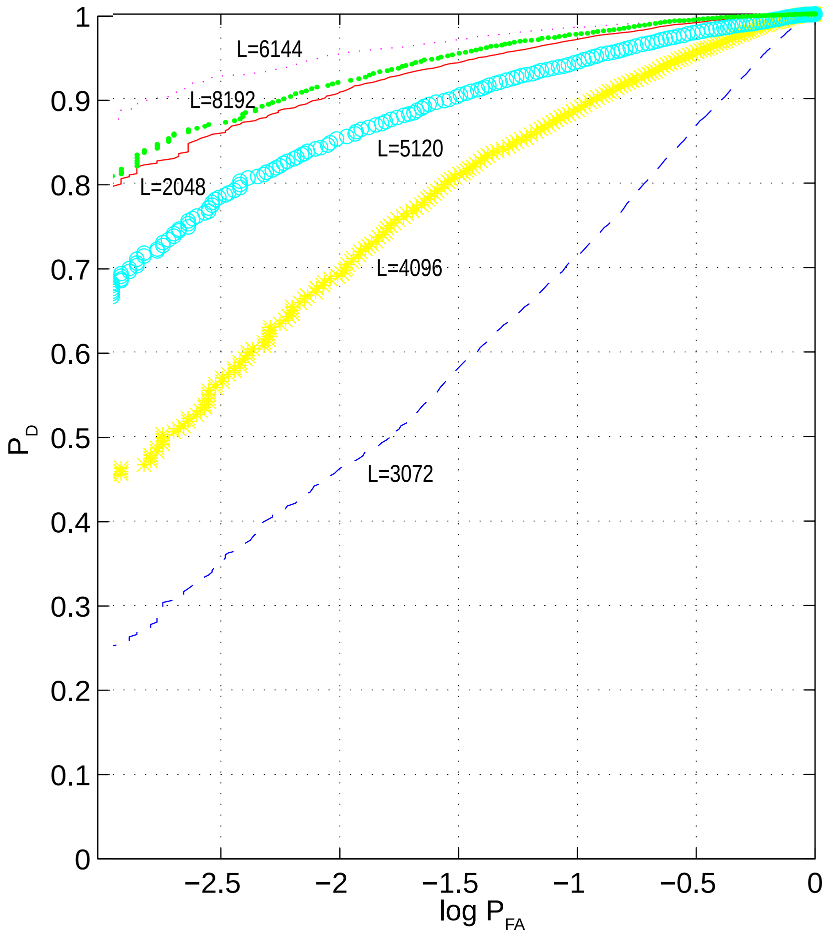

Fig. 4. The ROC curves for detection of a digital FM signal