A Simple Capacity Formula for Correlated Diversity Rician Fading Channels
Q. T. Zhang, Senior Member, IEEE, and D. P. Liu, Member, IEEE

Abstract—Appropriate exploitation of the randomness of multipath propagation in wireless communications can considerably increase the system capacity. The average capacity of correlated diversity Rician channels, however, is not available in the literature due to the nonlinear log function involved in the conditional channel capacity and to the complicated distribution of correlated Rician channels. Rather than using brute force, we tackle this issue by combining a widely adopted statistical technique with an elegant lemma by Porteous, ending up with a simple and accurate formula for average channel capacity on correlated diversity Rician channels. Numerical results are also presented for illustration.

Index Terms—Channel capacity, fading channels, wireless SIMO systems.

I. INTRODUCTION

It is well known that appropriate exploitation of the randomness of multipath propagation in wireless communications can significantly increase the system capacity. The conditional channel capacity of a general wireless multiple-input multiple-output (MIMO) system is addressed in [1]. The average channel capacity, however, depends on the characteristics of a fading environment, and relevant results are available in the literature only for some special channel conditions. A commonly used assumption for average channel-capacity analysis is that all links between any pair of transmit and receive antennas are considered to be independent and identically distributed (i.i.d.) Gaussian variables. Under this assumption, [2] and [3] have derived, respectively, the average and limiting channel capacity. The i.i.d. Gaussian assumption can be relaxed, to certain extent, if one considers the average channel capacity of diversity reception instead. Along this direction, the average channel capacity of i.i.d. diversity Rician channels is determined in [4], and its counterpart for a single-channel system with some practical boundary conditions taken into account is addressed in [5].

The purpose of this letter is to determine the average channel capacity of a diversity system, which has receive antennas operating on general correlated Rician channels. We use to denote the -by-1 vector of channel gains that link the transmitter to the receive antennas. For Rician fading channels, the gain vector follows the complex Gaussian distribution with mean vector and covariance matrix . Symbolically, we can write

\[ \mathbf{x} \sim CN_n(\mu_x, \mathbf{R}_x). \]  

where “\( \sim \)” means “distributed as”. As a convention, we use \( v(k) \) to denote the \( k \)th entry of vector \( \mathbf{v} \) and \( R(i, j) \) to denote the \( (i, j) \)th entry of matrix \( \mathbf{R} \). The Rician factor for the \( i \)th channel is thus equal to \( K_i = |\mu_x(i)|^2/R_x(i, i) \). Without loss of generality, \( \mathbf{x} \) is assumed to be normalized so that all fading paths, on average, have unit total power. Conditioned on \( \mathbf{x} \), the channel capacity is given by [2]

\[ C = \log_2 \left( 1 + \rho \mathbf{x}^\dagger \mathbf{x} \right) = \log_2 \xi \text{ bits/s/Hz} \]  

where \( \rho \) represents the signal-to-noise ratio (SNR) at the transmitter, and the superscript \( \dagger \) denotes Hermitian transposition. The symbol \( \xi \equiv 1 + \rho \mathbf{x}^\dagger \mathbf{x} \) will subsequently be used for simplicity. We need to determine \( E[C] \).

II. METHODOLOGY

The expression given in (2) involves the nonlinear log function thereby making it extremely difficult to determine \( E[C] \) by brute force. We avoid this difficulty by invoking a lemma due to Porteous [6], which can be stated as follows.

Lemma 1: If \( u \sim \chi^2(k) \), then

\[ E(\ln u) = \ln k - \frac{1}{k} - \frac{1}{3k^2} + \frac{2}{15k^4} + O(k^{-6}). \]  

The inspiration to use Porteous’ lemma arises from the resemblance between (3) and \( E[C] \) in their functional forms on one hand, and from the observation that the argument \( \xi \) in (2) is of a quadratic form in Gaussian vector on the other. Such a quadratic form, with the help of appropriate statistical techniques [8], [10], [11], can be accurately approximated by a single scaled chi-square variable. Wisely combining the \( \chi^2 \)-approximation technique with Porteous’ lemma constitutes the philosophy behind our approach.

III. AVERAGE CAPACITY

Let us begin with deriving an equivalent expression for \( \mathbf{x}^\dagger \mathbf{x} \). Eigendecompose \( \mathbf{R}_x \) such that \( \mathbf{R}_x = \mathbf{U} \Lambda \mathbf{U}^\dagger \) where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) is the diagonal matrix of the eigenvalues of \( \mathbf{R}_x \), and the columns of \( \mathbf{U} \) consist of the corresponding eigenvectors. We decorrelate vector \( \mathbf{x} \) to obtain

\[ \mathbf{e} = \Lambda^{-1/2} \mathbf{U}^\dagger \mathbf{x}. \]
It is easy to examine that the covariance matrix $R_e$ and mean vector $\mu_e$ are given by

$$R_e = I, \quad \mu_e = \Lambda^{-1/2} U^\dagger \mu_e. \tag{5}$$

It follows from (4) that $x = U^\dagger A^\dagger \mu_e$ whereby we can write

$$x^\dagger x = \mu_e^\dagger \Lambda \mu_e = \sum_{k=1}^n \lambda_k |e(k)|^2. \tag{6}$$

Since $e(k) \sim CN_1(\mu_e(k), 1)$, the use of Giri [7, Theorem 3.1] allows us to assert

$$|e(k)|^2 \sim \frac{1}{2} \chi^2(2, 2|\mu_e(k)|^2) \tag{7}$$

where the notation $\chi^2(m, \omega)$ is used to denote the noncentral chi-square variable with $m$ degrees of freedom and noncentrality parameter $\omega$. When $\omega = 0$, we simply write $\chi^2(m)$. Given the independence of $\{e(k), k = 1, 2, \ldots, n\}$, it also follows that

$$x^\dagger x \sim \sum_{k=1}^n \lambda_k \chi^2(2, 2|\mu_e(k)|^2) \tag{8}$$

which indicates that $x^\dagger x$ has the same distribution as a weighted sum of $n$ independent noncentral chi-square variables. Inserting (8) into the expression for $\xi$ yields

$$\xi = 1 + \rho_0 x^\dagger x \sim 1 + \frac{\rho_k}{2} \sum_{k=1}^n \lambda_k \chi^2(2, 2|\mu_e(k)|^2). \tag{9}$$

It is clear that $\xi$ is distributed as a linear summation of independent chi-square variables.

It is common practice in statistics to approximate a weighted sum of chi-square variables by a single one with different degrees of freedom and an adequate scaling factor. This technique has been widely adopted in statistics and engineering, see, for example, [8]–[11]. Thus we can write

$$1 + \frac{\rho_k}{2} \sum_{k=1}^n \lambda_k \chi^2(2, 2|\mu_e(k)|^2) \approx \alpha \chi^2(\ell), \tag{10}$$

The parameters $\alpha$ and $\ell$ should be chosen such that both sides have the same first two moments. Recall that if $z \sim \chi^2(m, \lambda)$, then $E[z] = m + \lambda$ and $\text{var}(z) = 2(m + 2\lambda)$ [8], [11]. Equating the first two moments of both sides of (10) leads to

$$\alpha \ell = 1 + \rho_k \sum_{k=1}^n \lambda_k (1 + |\mu_e(k)|^2)$$

$$2\alpha^2 \ell = \rho_k \sum_{k=1}^n \lambda_k^2 (1 + 2|\mu_e(k)|^2). \tag{11}$$

The right-hand sides of the above equations can be simplified if we notice that $\sum_{k=1}^n \lambda_k = \text{trace} R_x$ and $\sum_{k=1}^n \lambda_k^2 = \text{trace} R_x^2$. We can use (5) to further show that

$$\sum_{k=1}^n \lambda_k |\mu_e(k)|^2 = \mu_e^\dagger A \mu_e = \mu_e^\dagger \mu_e \tag{12}$$

and in a similar manner, we can show $\sum_{k=1}^n \lambda_k^2 |\mu_e(k)|^2 = \mu_e^\dagger \mu_e$. As such, we are able to rewrite (11) as

$$\alpha \ell = 1 + \rho_k (\text{trace} R_x + \mu_e^\dagger \mu_e)$$

$$2\alpha^2 \ell = \rho_k^2 (\text{trace} R_x^2 + 2 \mu_e^\dagger \mu_e). \tag{13}$$

Define $a$ and $b$ such that

$$a = 1 + \rho_k (\text{trace} R_x + \mu_e^\dagger \mu_e)$$

$$b = \left(\frac{\rho_k^2}{2}\right) (\text{trace} R_x^2 + 2 \mu_e^\dagger \mu_e) \tag{14}$$

and solve (13) for $\alpha$ and $\ell$ yielding

$$\alpha = \frac{b}{a}, \quad \ell = \frac{a^2}{b}. \tag{15}$$

At this point, we point out the fact that $a$ is the mean value of $\xi$; namely $a = E[\xi]$.

We are now in a position to determine the average channel capacity. To this end, insert (15) into (9)–(10), and use Lemma 1 to obtain

$$E[C] \approx \log_2 (a \ell) - c = \log_2 a - c \tag{16}$$

where $c$ is defined by

$$c = \frac{1}{\ln 2} \left(\frac{1}{\ell} + \frac{1}{3\ell^2} - \frac{2}{15\ell^4}\right). \tag{17}$$

The expression in (16) is intuitively appealing if we consider it in the framework of Jensen’s inequality ([12, p. 25]). Rewrite (16) as

$$E[C] \approx E[\log_2 \xi] = \log_2 E[\xi] - c. \tag{18}$$

Since $\ln \xi$ is a strictly concave function of random variable $\xi$, it follows from the Jensen’s inequality that $E[\ln \xi] < \ln E[\xi]$. Equation (16) reveals that if $E[\ln \xi]$ is used to approximate $E[\log_2 \xi]$, a correction term must be added. This correction term is determined by $c$.

IV. NUMERICAL RESULTS

To examine the accuracy of our formula derived above, let us consider a diversity system using a uniform linear array of $n$ antennas of spacing $d$. Suppose the correlation coefficient between two antennas separated by distance $kd$ is described by the exponential function [13], as shown by $\exp(-kd + j\theta_k)$ with $\theta_k$ denoting the phase. Clearly, $\gamma \hat{A} \exp(-d)$ represents the absolute correlation value between two adjacent antennas.

We first study the dependence of $E[C]$ on the diversity order $n$, using two methods: one based on approximation formula (16) and another based on Monte Carlo calculation by using (2). In what follows, results obtained from Monte Carlo calculation are also called simulation results for brevity. The parameters we used were: $\gamma = 0.6$, $\rho_k = 3$ dB, and $K = 0.25$ for all branches.

To see the role played by the correction term $c$, we tabulate the results in Table I where $E[C]_{\text{ana}}$ and $E[C]_{\text{sim}}$ signify the average channel capacity obtained through (16) and Monte Carlo method, respectively. Each value of $E[C]_{\text{sim}}$ is obtained by averaging over 4000 independent computer trials. Indeed, the role
Table I

<table>
<thead>
<tr>
<th>n</th>
<th>$E[C]_{\text{sim}}$</th>
<th>$E[C]_{\text{ana}}$</th>
<th>relative error</th>
<th>$\ln E[C]$</th>
<th>$-c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.0625</td>
<td>2.0194</td>
<td>0.0209</td>
<td>2.3192</td>
<td>-0.2997</td>
</tr>
<tr>
<td>3</td>
<td>2.5952</td>
<td>2.5350</td>
<td>0.0232</td>
<td>2.8044</td>
<td>-0.2694</td>
</tr>
<tr>
<td>4</td>
<td>2.9678</td>
<td>2.9218</td>
<td>0.0155</td>
<td>3.1669</td>
<td>-0.2451</td>
</tr>
<tr>
<td>5</td>
<td>3.2675</td>
<td>3.2334</td>
<td>0.0104</td>
<td>3.4563</td>
<td>-0.2229</td>
</tr>
<tr>
<td>6</td>
<td>3.5511</td>
<td>3.5132</td>
<td>0.0107</td>
<td>3.6973</td>
<td>-0.1840</td>
</tr>
<tr>
<td>7</td>
<td>3.7694</td>
<td>3.7481</td>
<td>0.0056</td>
<td>3.9037</td>
<td>-0.1556</td>
</tr>
<tr>
<td>8</td>
<td>3.9608</td>
<td>3.9483</td>
<td>0.0032</td>
<td>4.0842</td>
<td>-0.1360</td>
</tr>
<tr>
<td>9</td>
<td>4.1263</td>
<td>4.1197</td>
<td>0.0016</td>
<td>4.2447</td>
<td>-0.1250</td>
</tr>
<tr>
<td>10</td>
<td>4.2720</td>
<td>4.2719</td>
<td>0.0000</td>
<td>4.3891</td>
<td>-0.1171</td>
</tr>
<tr>
<td>11</td>
<td>4.4220</td>
<td>4.4146</td>
<td>0.0017</td>
<td>4.5203</td>
<td>-0.1057</td>
</tr>
<tr>
<td>12</td>
<td>4.5528</td>
<td>4.5456</td>
<td>0.0016</td>
<td>4.6406</td>
<td>-0.0950</td>
</tr>
</tbody>
</table>

Fig. 1. Average capacity versus diversity order on correlated Rician channels.

of the correction term is crucial especially for small $n$. In the worst case when $n = 2$, $\ln E[C]$ significantly deviates from $E[C]_{\text{sim}}$ with a relative error as large as 12.45%. With the correction term $c$, however, the deviation of $E[C]_{\text{ana}}$ from $E[C]_{\text{sim}}$ drops to 2.1%. As a general rule, the relative error drops rapidly with increased $n$, and further drops as $\gamma$ becomes smaller.

For more intuitive comparison, we depict the approximate and simulation average capacity $E[C]_{\text{ana}}$ and $E[C]_{\text{sim}}$ versus $n$ in Fig. 1. The parameter setting in this figure is the same as that used to obtain Table I except for $\gamma = 0.2$. We set $\gamma = 0.2$ and $\gamma = 0.8$ to obtain two sets of curves as shown in Fig. 1. It is clear that the approximation curve is almost identical to its simulated counterpart for $\gamma = 0.2$, and provides an accurate approximation to the latter even when $\gamma$ is as large as 0.8.

Fig. 2 shows the influence of the antenna correlation on the average capacity. The same correlation model was used. The remaining parameter was $\rho_t = 6$ dB, and different Rician factors were used for different branches such that the average $K$ is 0.32. As expected, the average channel capacity decreases as the correlation $\gamma$ increases. Again, good agreement between approximation and simulation results is observed.

V. Conclusion

In this correspondence, we have derived a simple average-capacity formula for correlated diversity Rician channels. The lemma by Porteous is employed to simplify the expected-value evaluation of the nonlinear log function in the formula of the conditional channel capacity. The resulting formula takes a simple form of $E[\ln \xi] = E[\ln \xi] - c$, indicating that in determining the average channel capacity, one can exchange the order of expectation and logarithm as long as an appropriate correction term is added. The accuracy of the formula is confirmed through computer simulations.

References