Abstract— Postdetection combining is a popular means to improve the bit error performance of DPSK and noncoherent FSK (NFSK) systems over fading channels. Nevertheless, the error performance of such systems in an arbitrarily correlated Nakagami environment is not available in the literature. The difficulty arises from inherent nonlinearity in noncoherent detection and from attempts to determine explicitly the probability density function of the total signal-to-noise ratio at the combiner output. In this paper, we directly determine the error probability from the characteristic function of decision variables, resulting in closed-form solutions involving matrix differentiation. The performance calculation is further simplified by developing a recursive technique. The theory is illustrated by analyzing two feasible antenna arrays used in base stations for diversity reception, ending up with some findings of interest to system design.

Index Terms—DPSK, diversity reception, Nakagami channels, noncoherent FSK.

I. INTRODUCTION

DIVERSITY reception is an effective means to combat multipath fading. The fading environment of multichannels is statistically described by their joint distribution. Nakagami distribution is one such model in common use. This is largely attributable to its flexibility in that it can be used to account for both fast and slow fading, and includes the classical Rayleigh fading as a special case. Furthermore, a recent study of propagation in mobile channels further justifies the use of such a model from the physical point of view [8].

Various combining schemes in a Nakagami environment have been studied [1]–[7]. Most of these studies focus on various predetection combining schemes operating in mutually independent branch environments [6]–[7]. The independent assumption greatly simplifies the analysis. However, the results so far obtained may fail to reflect the situations in the real world. For example, typical correlation between two antennas used in a base station is 0.7 or even higher as a result of space limitation [15]. Correlation among branch antennas has been taken in account in some recent publications [1]–[3]. The difficult issue is to determine the probability density function (PDF) of the instantaneous signal-to-noise ratio (SNR) at the combiner’s output, and the state of the art is that only a couple of relevant PDF’s are available in the mathematical literature. As a consequence, the aforementioned studies either concentrate on dual-branch combiners [1], [2] or multibranch combiners with special correlation matrix [3]. The statistical structure assumed in [3] is constant or exponential correlation among the antennas. In reality, correlation among the antennas can take on an arbitrary symmetrical structure, since it depends not only on the antenna array configuration, but also on the incident angle of the incoming wavefront. Analytical results are still not available for this general case.

The situation for postdetection combining is even worse, and its performance over correlated Nakagami channels remains unknown. The difficulty in the past arose from the attempt to express the conditional error probability in terms of a total signal-to-noise ratio at the combiner’s output, and then to find an explicit expression for probability density function of the total SNR. Both issues proved to be difficult due to the nonlinearity involved in postdetection combining and to the current status in multivariate analysis. In this paper, we derive general solutions to postdetection combiners for noncoherent detection of DPSK and FSK signals over arbitrarily correlated Nakagami channels by taking a completely different approach. We directly work on the domain of characteristic functions (CF), thereby eliminating the need to explicitly determine the PDF of the SNR at the combiner’s output. The remainder of the paper is organized as follows. We briefly describe the necessary background for correlated Nakagami distribution in Section II, followed by an exposition of the CF method for determining error probability in Section III. Error performance of FSK and DPSK systems is derived in Sections IV and V, respectively. Section VI addresses the issue of efficient calculation of the error performance. The analytical results are then applied in Section VI to study two possible antenna array configurations for base stations. Section VII finishes the paper with concluding remarks.

II. CORRELATED NAKAGAMI CHANNELS

We are interested in diversity reception systems which combine $L$ branches with joint correlated Nakagami distribution. For many combining schemes in use, the bit error performance depends on the SNR’s at different diversity branches only.
through their summation

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_L$$  \hspace{1cm} (1)

where $\gamma_l$ represents the SNR at the $l$th branch antenna. Nakagami amplitude distribution implies that the variables $\gamma_1, \cdots, \gamma_L$ are jointly gamma distributed. Correlated gamma variables are usually defined by their joint characteristic function [9]. [10]. Let $R_T$ denote the covariance matrix of the SNR vector $\mathbf{r} = [\gamma_1, \gamma_2, \cdots, \gamma_L]$ and define $\mathbf{T}$ as $\mathbf{T} = \text{diag} \{ t_1, \cdots, t_L \}$. Then, the joint CF of the instantaneous SNR is given by [13]

$$\phi_{\mathbf{r}}(jt_1, \cdots, jt_L) = \exp \left( j \sum_{k=1}^{L} \gamma_k t_k \right)$$

$$= \det (I - jT)^{-\gamma}$$  \hspace{1cm} (2)

where $m$ is the fading parameter and $\Gamma$ is the positive definite matrix determined by the branch covariance matrix $R_T$, as shown below. Denote $s = jt$ and $s = (s_1, \cdots, s_L)$. Using the rules (10.16) and (10.19–10.21) for derivatives involving matrices [14], we can show that

$$E[\gamma] = \frac{\partial \phi_{\mathbf{r}}}{\partial s_k} \bigg|_{s=0} = m\Gamma(k, k)$$

$$E[\gamma \gamma^*] = \frac{\partial^2 \phi_{\mathbf{r}}}{\partial s_k \partial s_l} \bigg|_{s=0} = m^2 \Gamma(k, l) \Gamma(l, k) + m \Gamma(k, \ell) \Gamma(l, k)$$  \hspace{1cm} (3)

III. THE CF METHOD

Various methods can be used to determine the bit error performance of a signaling scheme. But not all of them can lead to a solution in a simple form, depending on the decision variable and the fading environment. In this paper, we will use the method of characteristic function (CF), which is particularly powerful for signals suffering from Nakagami fading. The idea of the CF method is described below. Suppose $D$ is the decision variable with probability density function (PDF) $f(D)$. The CF of $D$ is defined as $\phi_D(jt) = \mathbb{E}[e^{jtD}]$. Without loss of generality, we assume that an erroneous decision corresponds to $D < 0$. The error probability can be directly determined from the CF, as shown by

$$P_e = \int_{-\infty}^{0} f(D) \, dD$$

$$= \int_{-\infty}^{0} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_D(jt) \exp(-jtD) \, dt \right] \, dD$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_D(jt) \, dt \left[ \int_{-\infty}^{0} e^{-jtD} \, dD \right].$$

Note that the integral inside the squared brackets on the last line can be obtained from the Fourier transform of a unit-step function; the result is $\{\pi\delta(t) - (1/jt)\}$. It follows that

$$P_e = \frac{1}{2} - \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\phi_D(jt)}{t} \, dt.$$  \hspace{1cm} (9)

Here we have used the fact that $\phi_D(0)$ represents the total probability and is therefore equal to unity.

The solution to the integral in (9) depends on the channel conditions. We identify two cases. Let $F(s)$ denote the ratio $\{\phi_D(s)\}/s$. In the first case, the denominator of $F(s)$ is at least two orders higher than its numerator and, furthermore, all poles of $F(s)$ on the right-half complex plane or on the left-half plane are of integer order. In this case, we can relate the path integral in (9) to a contour integral and determine its value using the residue theorem. It turns out that

$$P_e = 1 + \text{Res} \left\{ \frac{\phi_D(s)}{s}, \text{poles in Re}(s) > 0 \right\}$$

$$= -\text{Res} \left\{ \frac{\phi_D(s)}{s}, \text{poles in Re}(s) < 0 \right\}.$$  \hspace{1cm} (10)

The first line should be used if all poles in $\text{Re}(s) > 0$ are of integer order, and the second line should be used if all poles in $\text{Re}(s) < 0$ are of integer order. In any event, we have a closed-form solution. If, as encountered in some situations, all poles in the complex plane are of integer order, then the half-plane having a fewer number of poles should be chosen so as to simplify the calculation.

In the second situation, not all poles in $\text{Re}(s) < 0$ or in $\text{Re}(s) > 0$ are of integer order. Note that the real part of $\phi_D(jt)$ is an even function of $t$ since it is the Fourier transform of a real valued PDF. Thus, we can simplify (9) as

$$P_e = \frac{1}{2} - \frac{1}{2\pi j} \int_{-\infty}^{\infty} \text{Re} \left( \frac{\phi_D(jt)}{t} \right) \, dt$$

whose calculation relies on numerical methods.
IV. NFSK SYSTEMS

Consider an order-$L$ diversity system for noncoherent detection of FSK (NFSK) signals, as illustrated in Fig. 1. The signal components received at different antennas are jointly Nakagami distributed with an arbitrary covariance matrix, whereas the noise components at different branches are Gaussian and spatially independent. We will first obtain the decision variable and then derive its error performance.

A. Decision Variable

The received signal at the $k$th antenna in the symbol duration $0 < t < T$ can be written as

$$r_k(t) = \alpha_k \cos(\omega^*_k t + \psi_k) + n_k(t)$$  \hspace{1cm} (12)

where $i = 1, 2$ and $k = 1, 2, \ldots, L$, and $n_k(t)$ is zero-mean white Gaussian noise process with one-sided spectral density $\sigma_n^2$. The carrier frequency depends on the message with $\omega^*_1$ corresponding to symbol 1 and $\omega^*_2$ to symbol 0. The influence of the $k$th fading channel is characterized by the channel gain $\alpha_k$ and phase delay $\psi_k$, which remain unchanged over the symbol interval $T$. In a complex form, the influence of the fading channel can be compactly expressed as

$$g_k = \alpha_k \exp(j\psi_k).$$  \hspace{1cm} (13)

Without loss of generality, suppose symbol 1 is transmitted, and let us consider the branch $k$. The upper bandpass filters (BPF) is centered at the carrier frequency $\omega^*_k$, thereby allowing the desired signal to pass through; while the lower BPF is centered at $\omega^*_2$, hence blocking the signal component. The noise output components from the upper and lower BPF’s can also be represented in a complex form, denoted here by $n_{k1}$ and $n_{k2}$, respectively. They are mutually independent and each follows zero-mean complex Gaussian distribution with variance equal to twice that for $n_k(t)$. The BPF outputs are further applied to the square-law detectors. The diversity system combines the outputs from all upper branches and accumulates information from all lower branches to enhance symbol detection. These two combined outputs, when expressed in terms of the complex channel gains and noise components, are given by

$$u_1 = \sum_{k=1}^{L} |Ag_k + N_{k1}|^2, \quad u_2 = \sum_{k=1}^{L} |N_{k2}|^2$$  \hspace{1cm} (14)

respectively. The decision variable is then taken as their difference

$$D = u_1 - u_2.$$  \hspace{1cm} (15)

Since symbol 1 is transmitted, an erroneous decision is made when $D < 0$, and a correct event otherwise. The situation is reversed when symbol 0 is sent.

B. Bit Error Performance

To determine the error probability, we begin with the conditional CF of $D$. To simplify notation, we use $\chi^2(\nu, \lambda)$ to denote a chi-square variable with $\nu$ degrees of freedom and noncentrality parameter $\lambda$. For a given set of channel gains $g = \{g_k, k = 1, \ldots, L\}$, the variables $|Ag_k + N_{k1}|^2$ and $|N_{k2}|^2$ have the following distributions [11]:

$$|Ag_k + N_{k1}|^2 \sim \frac{\sigma_n^2}{2}\chi^2(2, 2\gamma_k), \quad |N_{k2}|^2 \sim \frac{\sigma_n^2}{2}\chi^2(2, 0).$$  \hspace{1cm} (16)

Since the noise components $\{N_{k2}\}$ are independent for all $k$ and $t$, it follows [12, p. 134] that the conditional CF of $D$ equals

$$\phi_D(jt|g) = (1 + jt\sigma_n^2)^{-L}(1 - jt\sigma_n^2)^{-L}\exp(a(t)\gamma)$$  \hspace{1cm} (17)

where $a(t)$ is defined by

$$a(t) = \frac{jt\sigma_n^2}{1 - jt\sigma_n^2}$$  \hspace{1cm} (18)

and $\gamma$ denotes the SNR at the combiner’s output, given by

$$\gamma = \sum_{k=1}^{L} \frac{A^2\alpha_k^2}{\sigma_n^2} = \sum_{k=1}^{L} \gamma_k$$  \hspace{1cm} (19)
where $\gamma_k = A^2 \sigma_k^2 / \sigma_n^2$ is the instantaneous SNR at branch $k$. It is clear that the conditional CF depends on the channel gains only through the SNR $\gamma$. We next average the conditional CF over the PDF of $\gamma, f_r(x)$, yielding

$$
\phi_D(jt) = \int_{-\infty}^{\infty} \phi_D(jt|\gamma)f_r(x)\,dx
= (1 + j\sigma_n^2)^{-L} (1 - j\sigma_n^2)^{-L} \cdot \int_{-\infty}^{\infty} \exp[a(t)x]f_r(x)\,dx
$$

(20)

We recognize that the last integral is exactly the CF of $\gamma$, except that $jt$ is replaced by $a(t)$. Thus, we can write

$$
\phi_D(jt) = (1 + j\sigma_n^2)^{-L} (1 - j\sigma_n^2)^{-L} \phi_r(a)
$$

(21)

which, when combined with (2), produces

$$
\phi_D(jt) = (1 + j\sigma_n^2)^{-L} (1 - j\sigma_n^2)^{-L} \cdot \det \left( I - \frac{j\sigma_n^2}{1 - j\sigma_n^2} \Gamma \right)^{-m}.
$$

(22)

To use residue to determine the error probability, we need to know the location of poles introduced by the determinant. If we denote the eigenvalues of $\Gamma$ by $\lambda_k$, it is easy to show

$$
\det \left( I - \frac{\sigma_n^2}{1 - \sigma_n^2} \Gamma \right)^{-m} = \prod_{k=1}^{L} \left[ 1 - \frac{1 - \sigma_n^2}{1 + \lambda_k} \right]^{-m}.
$$

(23)

Clearly, all the poles introduced by the determinant are located on $\Re \{s\} > 0$ and, moreover, these poles can be of nonintegral order, depending on the value of the fading parameter. On the other hand, $\phi_D(s)$ has only one pole of integral order $L$ on the left-half plane $\Re (s) < 0$. Consequently, we choose to calculate the error probability using the pole on $\Re (s) < 0$. The use of the second line of (10) yields

$$
P_e = -\text{Res} \left\{ \frac{\phi_D(s)}{s}, s = -\frac{1}{\sigma_n^2} \right\}.
$$

(24)

The residue represents the closed-loop integration of $\phi_D(s)/s$ over variable $s$. Change of variable $z = \sigma_n^2 s$ allows us to write

$$
P_e = -\text{Res} \left\{ \frac{\phi_D(z)}{z}, z = -1 \right\}
= -\frac{1}{(L-1)!} \int_{-1}^{1} \left[ \frac{1}{z(1-z)} \det (R)^{-m} \right] dz
$$

(25)

with $R = I - \frac{z}{1-z} \Gamma$.

(26)

This is the basic formula for diversity reception of NFSK over Nakagami fading channels with an arbitrary branch covariance matrix. We only need to calculate the $(L-1)$th derivative. It is apparent that in noncoherent BFSK, the noise component introduces poles of integral order, making it possible to obtain a closed-form solution. If we put $L = 1$ and $m = 1$, then (25) reduces to

$$
P_e = \frac{0.5}{1+0.5\gamma_1}
$$

(27)

which is the result for noncoherent FSK system on Rayleigh channel [16, p. 200].

V. DSPK

The diversity system we will study consists of $L$ branches, as illustrated in Fig. 2. The received signal at the $\theta$th branch, over the symbol interval $(n-1)T < t < nT$, is given by

$$r(t) = \alpha_e(t)\{A \cos (\omega_c t + \theta_n + \phi(t))\} + n(t)
$$

(28)

where $\alpha_e$ and $\phi$ signify the gain and the phase delay of the $\theta$th channel; $A$ is the magnitude of the transmit signal, and $\omega_c$ the carrier frequency; $n(t)$ is a zero-mean white Gaussian process with one-sided spectral density $\sigma_n^2$. The $n$th message symbol is encoded into the phase difference $\theta_n - \theta_{n-1}$. Consider the $\theta$th branch. Let

$$g_n = \alpha_e(n) \exp \{ j\phi(n) \}
$$

(29)

denote its complex channel gain, and assume that the channel is slowly fading so that $g_n$ remains unchanged over two successive symbol intervals. Let $v_n(n)$ denote the complex output from the two samplers at the $\theta$th branch. Then, the complex outputs over two successive symbols are

$$v_n(n) = Ag_n \exp \{ j\phi_n \} + N_n(\omega)
$$

(30)

$$v_n(n-1) = Ag_n \exp \{ j\phi_{n-1} \} + N_n(n-1)
$$

where $N_n(\omega)$ is a white complex Gaussian sequence with mean zero and variance $\sigma_n^2$. With these notations, it follows from Fig. 2 that the final output of the $\theta$th branch equals

$$u_n(n) = \Re \{v_n(n)v^*_n(n-1)\}
$$

(31)

from which the decision variable follows:

$$D = \sum_{n=1}^{L} D_n = 2 \sum_{n=1}^{L} \Re \left\{ \sum_{n=1}^{L} v_n(n)v^*_n(n-1) \right\}
$$

(32)

where $D_n \triangleq 2u_n(n)$. The use of the scaling factor of 2 is for theoretical convenience and has no effect on the error performance.

To determine the bit error performance, we assume, without loss of generality, that the symbol is such that no phase change happens over the $(n-1)$th and $n$th symbol intervals. Then, an error event occurs when $D < 0$. We take two steps to analyze the bit error performance of DPSK. First, we determine the conditional CF of $D$ by assuming that all channel gains $g_n$ are given. Then, we average the conditional over the joint Nakagami distribution of the channel gains to obtain the final result. Let us begin with the conditional CF’s of $D_n$. To simplify notation, define

$$z_n = \begin{pmatrix} v_n(n) \\ v_n(n-1) \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

(33)

Then, $D_n$ can be written in a quadratic form, as shown by

$$D_n = z_n^T \Omega z_n,
$$

(34)
Given the channel gains $g_k, k = 1, \ldots, L$, $D_k$ are mutually independent. Under this condition, $z_k$ has the complex Gaussian distribution

$$z_k \sim CN(A(\alpha_2(\tau))e^{-j\alpha_2(\tau)}e, \sigma_n^2I)$$

from which the conditional CF of the quadratic form $D_k$ follows:

$$\phi_{D_k}(jt|g) = \det(I - jt\sigma_n^2\Omega)^{-1} \exp\{jtA^2\alpha_2^2(\tau) \cdot e^T(\Omega^{-1} - jt\sigma_n^2I)^{-1}e\}. \quad (36)$$

Note that due to the special structure of the matrix $\Omega$, the exponent in (36) can be greatly simplified. Denote

$$b(t) = jte^T(\Omega^{-1} - jt\sigma_n^2I)^{-1}e.$$ \quad (37)

The properties of $\Omega$ to be used are $\Omega = \Omega^{-1}$ and its special eigenstructure. Specifically, $\Omega$ has eigenvalues $-1$ and $1$ with the corresponding eigenvectors given by

$$v_1 = \frac{1}{\sqrt{2}}(1, -1), \quad v_2 = \frac{1}{\sqrt{2}}(1, 1). \quad (38)$$

Using this eigenstructure, we can easily simplify (37) to obtain

$$b(t) = \frac{2jt}{1 - jt\sigma_n^2} = \frac{2\alpha(t)}{\sigma_n^2}.$$ \quad (39)

We can also show that

$$\det(I - jt\sigma_n^2\Omega)^{-1} = (1 + jt\sigma_n^2)^{-1}(1 - jt\sigma_n^2)^{-1}. \quad (40)$$

The results obtained above, together with the mutual independence of $D_k$ for a given set of channel gains, allow us to write

$$\phi_D(jt|g) = \prod_{k=1}^L \phi_{D_k}(jt|g) = (1 + jt\sigma_n^2)^{-L}(1 - jt\sigma_n^2)^{-L} \exp\{2\alpha(t)\gamma\}. \quad (41)$$

It is clear that the conditional CF for DPSK has exactly the same form as that for noncoherent FSK, except that $a(t)$ is replaced by $2\alpha(t)$. The same is true of the unconditional CF; thus, we have

$$\phi_D(jt) = (1 + jt\sigma_n^2)^{-L}(1 - jt\sigma_n^2)^{-L} \cdot \det(I - \frac{2j\sigma_n^2}{1 - jt\sigma_n^2} \Gamma)^{-m}. \quad (42)$$

Comparing to (22), we see that the CF for DPSK has exactly the same as that for noncoherent FSK, except that $\Gamma$ in (22) is replaced by $2\Gamma$. Consequently, the formula (25) remains applicable and the only change is that

$$R = I - \frac{2\gamma}{1 - \gamma} \Gamma. \quad (43)$$

For $L = 1$ and $m = 1$, the above formula reduces to

$$P_e = \frac{0.5}{1 + \gamma_1} \quad (44)$$

which is identical to the well-known result for nondiversity reception of DPSK signals suffering from Rayleigh fading [16].

From the above analysis, we see that it is the special eigenstructure of $\Omega$ that introduces a pole of order $m$ on $\Re \{s\} < 0$, making it possible to use the residue theorem. In the performance analysis of DPSK systems subject to small Doppler frequency shift, it is common practice to assume that the channel gain remains unchanged over two successive symbol intervals. This assumption makes it possible to use a joint gamma distribution to obtain a closed-form solution for DPSK in a Nakagami environment. Without this assumption, the entries of vector $e^T(\Omega^{-1} - jt\sigma_n^2I)^{-1}e$ instead. This latter vector consists of both amplitude and phase variation, thereby requiring a joint complex Nakagami distribution for its characterization. Unfortunately, such a distribution is not available in the literature. We therefore remark that only when Doppler shift is small, can the problem of DPSK in a Nakagami environment be resolved leading to a closed-form solution in form similar to that for NFSK.
VI. Efficient Evaluation

The closed-form solution provided in (25) involves a derivative of order \((L - 1)\). Determination of such a derivative can be quite complicated or even prohibitive when \(L\) is large, since the function to be differentiated is the product of three factors. Furthermore, the function includes a matrix as its argument, thereby further complicating the calculation. It is therefore necessary to develop an efficient technique to resolve the problem. To this end, we denote

\[
\varphi(z) = \frac{1}{z(1-z)^L} \det(R)^{-m}. \tag{45}
\]

To avoid differentiating a product, we consider the use of the log function to convert \(\varphi(z)\) as a summation. Thus, we start from the relation

\[
\frac{d\varphi(z)}{dz} = \varphi(z) \frac{d\ln \varphi(z)}{dz} \tag{46}
\]

and invoke the formula (14) of [14, p. 230]

\[
\frac{d^n \varphi(z)}{d z^n} = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{d^k \varphi(z)}{d z^k} \frac{d^{n-k-1} \ln \varphi(z)}{d z^{n-k-1}} \tag{47}
\]

to rewrite the error probability as

\[
P_e = -\frac{1}{(L-1)!} \sum_{k=0}^{L-2} \binom{L-2}{k} \frac{d^k \varphi(z)}{d z^k} \cdot \frac{d^{L-k-1} \ln \varphi(z)}{d z^{L-k-1}} \bigg|_{z=1}. \tag{48}
\]

As a convention, the derivative with order zero simply represents the original function. We note that in (47), the derivative of order \(n\) depends only on the derivatives of lower order up to \((n - 1)\). Thus, we can calculate the derivative of order \((L - 1)\) recursively by starting from the original function \(\varphi(z)\) and increasing the order \(n\) by one in every iteration until \(n = L - 1\). We need to determine the derivatives of \(\ln \varphi(z)\), which is explicitly expressible as

\[
\frac{d^n \ln \varphi(z)}{d z^n} = (-1)^n(n-1)! z^{n-1} + \frac{(n-1)! L}{(1-z)^n} \tag{49}
\]

It remains to determine the last term on the right. By using the formulas (10.19–10.21) for matrix differentiation in [14, p. 45] and (47) above, we are able to obtain

\[
\frac{d^n}{d z^n} \ln(\det R) = \begin{cases} \ \left(\frac{1}{1-z}\right)^2 \text{tr}[R^{-1}], & n = 1 \\ -\sum_{k=0}^{n-1} \binom{n-1}{k}(n-2)\cdots(n-k)(k+1) \left(\frac{1}{1-z}\right)^{k+2} \text{tr}[\nabla V_{n-k-1}], & n > 1. \end{cases} \tag{50}
\]

Here, \(V_i\) is defined as \(V_i = \frac{d^i}{d z^i} R^{-1}\). When \(i = 1, V_i\) is a product of three matrices. For \(i > 1\), we need to continue the computation and the result is the summation of matrix products. The coefficients in the sum have the same form as those in a multinomial expansion. The explicit expression is

\[
V_j = \begin{cases} R^{-1}, & j = 0 \\ -R^{-1} \Gamma R^{-1} / (1-z)^2, & j = 1 \\ \sum_{j_1+j_2+j_3=j} \binom{j-1}{j_1,j_2,j_3} \cdot V_{j_1} \left(\frac{j_2+1}{1-z}\right)^{j_2+2} \Gamma R^{-1}, \ V_{j_3}, & j > 1. \end{cases} \tag{51}
\]

In this expression,

\[
\binom{j-1}{j_1,j_2,j_3} \equiv \frac{(j-1)!}{j_1!j_2!j_3!}.
\]

Equations (48)–(51) constitute efficient formulas for recursive evaluation of the bit error performance.

VII. Illustrating Examples

As an illustration of the theory, we study some feasible diversity systems for a base station. Two kinds of antenna configurations will be considered here, due to their relative ease of installation. They are uniform linear and triangular arrays, as sketched in Fig. 3. The correlation matrix of an antenna array depends not only on the array configuration but also on the incident angle of the incoming signal. Different parameters affect the correlation differently. Increase in the antenna separation \(d\) between two antennas tends to reduce their correlation. The antenna height behaves in a similar manner; the higher the antennas, the lower the correlation. The correlation matrix of a given antenna array varies with
the signal incident angle. Correlation between two antennas attains its maximum value when the signal arrives from the broadside, and reduces gradually as the signal direction moves toward the endfire direction.

Suppose now that an antenna array is used to receive signals from a mobile unit operating at 850 MHz. The signal suffers from Nakagami fading. As a typical value, we assume that the antenna is 100 ft in height. A signal at 850 MHz implies that its wavelength is \( \lambda = 0.353 \) m or 1.16 ft. For both linear and triangular arrays, any two adjacent antennas are assumed to have equal separation, denoted here by \( d \). To be representative of practical operational conditions with a linear array, we assume that in all cases treated below, signals impinge upon the array at 45° relative to the broadside of the array. For a triangular array, we assume that signals come from the broadside of the line linking antennas \( A \) and \( B \); refer to Fig. 3. Then, the incident angles to \( AC \) and to \( BC \) are both equal to 45°. We therefore expect that the correlation between antennas \( A \) and \( C \) (or between \( B \) and \( C \)) is higher than that between \( A \) and \( B \), regardless of their equal separation.

Figs. 4–6 are shown for DPSK; the discussion of NFSK is deferred to Fig. 7. We first study the influence of the fading parameter on the error performance using linear and triangular arrays, both having three antennas and with antenna spacing \( d = 8\lambda = 9.26 \) ft. This separation is close to the practical number used in the real world [15]. With this parameter setting, we can determine the correlation matrix of an antenna array in accordance with empirical results of Lee [15, p. 203]. The correlation matrix for the triangular array is given by

\[
\mathbf{R}_t = \begin{pmatrix}
1 & 0.727 & 0.913 \\
0.727 & 1 & 0.913 \\
0.913 & 0.913 & 1
\end{pmatrix}
\]  

(52)

and for the linear array, it is given by

\[
\mathbf{R}_l = \begin{pmatrix}
1 & 0.795 & 0.605 \\
0.795 & 1 & 0.795 \\
0.605 & 0.795 & 1
\end{pmatrix}
\]  

(53)

Here, for simplicity, we have assumed that all branches have the same average SNR. We then determine \( \Gamma \) from \( \mathbf{R}_t \) and scaling it by a scaling factor \( \frac{\text{SNR}}{m} \) so that we can write

\[
\Gamma = \frac{\text{SNR}}{m} \cdot \sqrt{\mathbf{R}_t}
\]  

(54)

where \( \sqrt{\mathbf{R}_t} \) denotes the matrix with its entries equal to the square root of the corresponding entries of \( \mathbf{R}_t \). Note the correlation matrix of the linear array has a Toeplitz structure, whereas that of the triangular array does not.

The results for linear and triangular arrays are graphed in Fig. 4(a) and (b), respectively. As expected, the error performance improves with increased fading parameter since small \( m \) corresponds to fast fading. In particular, \( m = 1 \) corresponds to the situation of Rayleigh fading. Since \( P_e \) is not a linear function of \( m \), we do not expect a uniform change of \( P_e \) with \( m \). This becomes evident in the curves. We also see that the linear array has a better performance than the triangular array, at the cost of a larger array dimension.
We next study the way the diversity order affects the performance. The correlation matrix for a five-antenna linear array with \( d = 8 \lambda = 9.26 \text{ ft} \) is

\[
R_r = \begin{pmatrix}
1 & 0.795 & 0.605 & 0.375 & 0.283 \\
0.795 & 1 & 0.795 & 0.605 & 0.375 \\
0.605 & 0.795 & 1 & 0.795 & 0.605 \\
0.375 & 0.605 & 0.795 & 1 & 0.795 \\
0.283 & 0.375 & 0.605 & 0.795 & 1
\end{pmatrix}.
\] (55)

The correlation matrices for \( L = 2 \), 3, and 4 are the submatrix of \( R_r \). The fading parameter we used was \( m = 1.5 \) and the results are shown in Fig. 5. The curves show that increased diversity order significantly improves the system performance over fading channels.

The use of a large diversity order to improve the system immunity to multipath fading, however, is often restricted by the system complexity. It is therefore interesting to see how the error performance is affected by the antenna separation. The variation of system error performance with antenna separation, \( d \), is shown in Fig. 6 for a triangular array. The fading parameter was \( m = 1.5 \). The correlation matrix still takes the form

\[
R_r = \begin{pmatrix}
1 & x & y \\
x & 1 & y \\
y & y & 1
\end{pmatrix}.
\] (56)

The values for \( x \) and \( y \) vary with \( d \). For \( d = 6, 8, 10, 12, 16, \) and \( 20 \), \((x, y)\) equals \((0.815, 0.994), (0.727, 0.913), (0.620, 0.802), (0.543, 0.723), (0.450, 0.672), \) and \((0.360, 0.527)\), respectively. The error performance improves with increased antenna separation at a varying degree, depending on the value of \( d \). The results suggest that an appropriate antenna separation should be such that the correlation falls into the range from 0.6–0.65.

The results shown above are all for DPSK systems. A comparison between DPSK and noncoherent FSK are illustrated in Fig. 7 for a triangular array with spacing \( d = 8 \lambda \). The cases of \( m = 1.6 \) and \( m = 4 \) are considered. It is observed that the DPSK scheme is about 3 dB better than NFSK.

VIII. CONCLUSION

In this paper, we have examined the error performance of postdetection combiners for noncoherent detection of DPSK and FSK signals over Nakagami channels with an arbitrary covariance matrix. We tackled the problem in the characteristic function domain and ended up with closed form solutions in terms of differentiation. We also developed an efficient technique to simplify the error performance calculation. It was revealed that if channel gains remain unchanged over two successive symbol intervals, the error probability of DPSK has the same expression as NFSK except that the \( \Gamma \)-matrix used for NFSK is replaced by \( 2\Gamma \) for DPSK. Without this condition, the error probability of DPSK would take a different form.

The formulas were then used to study two types of diversity schemes that can be used in a base station, one with linear antenna arrays and the other with triangular arrays. Computer results show that increase of the diversity order can significantly improve the error performance at the cost of increased antenna site. Properly increasing antenna separation can also help improve the performance. However, when the correlation reaches 0.6, further increasing the separation will basically trade array dimension for little improvement. As an overall consideration, the triangular array with adjacent separation ranging from 7–10 wavelengths is a good candidate for practical application.

REFERENCES


Q. T. Zhang (S’84–M’85–SM’95) received the B.Eng. degree from Tsinghua University, Beijing, and the M.Eng. degree from South China University of Technology, Guangzhou, China, both in radio communications, and the Ph.D. degree in electrical engineering from McMaster University, Hamilton, Ontario, Canada.

After graduation from McMaster in 1986, he spent a few years at the Communications Research Laboratory of the same institution, as a Senior Research Engineer and Adjunct Professor conducting research on communications and signal processing. In January 1992, he joined the Spar Aerospace Ltd., Satellite and Communication Systems Division, Montreal, as a Senior Member of Technical Staff. At Spar Aerospace, he was first responsible for the image performance analysis of the Radar Satellite and was subsequently engaged in the development of the advanced satellite communication systems for the next generation. Since September 1993, he has been a tenured Associate Professor with the Department of Electrical Engineering, Ryerson Polytechnic University, Toronto. His research is primarily in the areas of wireless communications and ATM networks with current emphasis on frequency and interference management, handoffs, reliable reception over various fading channels, intelligent beamforming, ATM switches, video traffic modeling, and source compression.