Probability of Error for Equal-Gain Combiners over Rayleigh Channels: Some Closed-Form Solutions

Q. T. Zhang

Abstract—The probability of error of a general equal-gain combiner (EG) in closed form is not available in the literature. In this letter, by formulating the problem in the framework of signal detection and evaluating the probability of error directly from the characteristic function (CF) of the signal at the combiner output, we obtain closed-form solutions for the probability of error of the EG combiner with two or three branches operating over a Rayleigh-fading channel.

Index Terms—Bit-error performance, diversity reception, equal-gain combiner, FSK, PSK, Rayleigh fading.

I. INTRODUCTION

REGARDLESS OF ITS practical and theoretical importance, the error performance of the equal-gain combiner (EGC) in closed form is not available in the literature. The crucial issue is to determine the probability density function (pdf) of the signal component at the combiner output. For a maximum ratio combiner operating in a Rayleigh or Nakagami environment, this corresponds to seeking the pdf of a sum of Gamma-distributed variables. It is a relatively easy task since the Gamma distribution has finite poles in its characteristic function. For an EG combiner, however, one has to determine the pdf of a sum of Rayleigh or Nakagami amplitudes, which does not share the desirable properties of the Gamma distribution. Consequently, the residue theorem becomes completely useless in the performance evaluation. Indeed, determining the pdf of the sum of multiple Rayleigh amplitudes alone has been a dilemma for many years, dating back to Rayleigh [1]–[3].

In this correspondence, we derive a closed-form solution for the error rate to the problem of order-three diversity with coherent detection by formulating it in the framework of statistical decision theory and evaluating the probability of error directly from the characteristic function (CF) of the combiner output.

II. SYSTEM DESCRIPTION

Consider the widely used predetection diversity system using EG combining [3], [4], as illustrated in Fig. 1. It is assumed that the signals received at the L antennas are independent and jointly Gaussian distributed. The signals are corrupted by zero-mean, white Gaussian noise, which is uncorrelated between different branches. The input at the kth antenna over 0 ≤ t < T can be written as

\[ r_k(t) = \text{Re}\{s_k(t) + \eta_k(t)\} = x_k(t) \left\{ \sqrt{2 \over T} \cos[\omega_c t + \theta_m(t) + \alpha_k(t)] \right\} + \eta_k(t). \] (1)

In this expression, \( \eta_k(t) \) is zero-mean, white Gaussian noise at the kth branch with one-sided power spectral density \( N_k \); \( x_k(t) \) is its complex representation, having a zero-mean complex Gaussian distribution with variance \( N_k \); \( \omega_c \) is the carrier angular frequency and \( \theta_m(t) \) is the information signal depending on the transmitted symbol \( m \). \( T \) and \( \alpha_k(t) \) represent, respectively, the symbol duration and the random phase uniformly distributed over \([0, 2\pi]\). The cosine function has been normalized so that the square of the magnitude \( x_k \geq 0 \) represents the symbol energy. \( s_k(t) \) is the signal component received at the kth branch; in a complex form, it can be expressed as

\[ s_k(t) = x_k(t)e^{j\omega z_k(t)} \sqrt{2 / T} \cos[\omega_c t + \theta_m(t)]. \] (2)

The amplitude \( x_k(t) \) remains constant within a symbol duration but changes from symbol to symbol, following the Rayleigh distribution

\[ f_{x_k}(x) = {2x \over \Omega_k} \exp\left( -{x^2 \over \Omega_k} \right) \] (3)

with \( \Omega_k = E[x_k^2] \) denoting the average signal energy at branch k. It follows from Fig. 1 that the combiner output is given by

\[ y(t) = \sum_{k=1}^{L} x_k(t) \sqrt{2 \over T} \cos[\omega_c t + \theta_m(t)] + \sum_{k=1}^{L} \eta_k(t). \] (4)

Here, we continue using the same symbols for the noise components since the process of co-phasing does not change their statistical characteristics.
III. COHERENT DETECTION

For coherent detection, the signal received at each branch is co-phased, summed, and coherently demodulated. For coherent binary phase shift keying (BPSK), the signal space is one-dimensional (1-D) and the resultant decision variable is

$$\gamma = \pm (x_1 + x_2 + \cdots + x_L) + \sum_{k=1}^{L} \eta_k$$

(5)

where \(\eta_k\) is the noise projection of \(\eta_k(t)\) onto the base and has the same statistical properties as the latter. The first term assumes the positive sign if the transmitted symbol \(m = 1\) and assumes the negative sign if \(m = 0\). The decision rule is

$$\gamma \begin{cases} > 0, & m = 1 \\ < 0, & m = 0 \end{cases}$$

(6)

For orthogonal BFSK, the decision variable and the decision rule are the same except that the signal space is two-dimensional (2-D) and, hence, the noise variance is twice that of BPSK.

To determine the error performance, we begin with the CF of the decision variable. By definition, the CF of \(\gamma\) is given by

$$\phi_\gamma(t) = E[\exp(j\gamma t)]$$

(7)

with \(E[a]\) denoting the expectation operator. Applying the assumption of independence between the signal and noise components to (5), we find that

$$\phi_\gamma(t) = \prod_{k=1}^{L} \phi_{x_k}(t)\phi_{\eta_k}(t)$$

(8)

where the CF of the Rayleigh-signal component, \(\phi_{x_k}(t)\), can be expressed by virtue of the sine and cosine transforms

$$\phi_{x_k}(t) = \int_{0}^{\infty} \frac{2x}{\Omega_k} \exp \left\{ -\frac{x^2}{\Omega_k} \right\} \cos(xt)dx$$

$$+ j \int_{0}^{\infty} \frac{2x}{\Omega_k} \exp \left\{ -\frac{x^2}{\Omega_k} \right\} \sin(xt)dx$$

$$= I_1 \left(1; \frac{1}{2}; -\frac{\Omega_k t^2}{4} \right) + j \sqrt{\frac{\pi \Omega_k}{4}} t \exp \left\{ -\frac{\Omega_k t^2}{4} \right\}.$$}

(9)

On the last line, we have used formulas (14) and (19) of [6, pp. 15 and 73]. Here, \(I_1(a; b; z)\) denotes the confluent hypergeometric function. The CF of the Gaussian noise \(\eta_k\) is well known, given by

$$\phi_{\eta_k}(t) = \exp \left\{ -\frac{\Omega_k t^2}{4} \right\}.$$}

(10)

Let \(\eta_L\) be the power spectral density function of the total noise such that

$$\eta_L = N_1 + N_2 + \cdots + N_L$$

(11)

and let \(\rho_k\) be the average bit energy to average noise density ratio (signal-to-noise ratio, SNR) at the \(k\)th branch such that

$$\rho_{Lk} = \frac{\Omega_k}{\eta_L/L}, \quad k = 1, 2, \cdots, L.$$}

(12)

We substitute (9) and (10) into (8) to obtain

$$\phi_\gamma(t) = e^{\frac{yx^2}{2}} \prod_{k=1}^{L} \left\{ I_1 \left(1; \frac{1}{2}; -\frac{\Omega_k t^2}{4} \right) + j \sqrt{\frac{\pi \Omega_k}{4}} t \exp \left\{ -\frac{\Omega_k t^2}{4} \right\} \right\}.$$}

(13)

Having obtained the CF of the decision variable, we may, in principle, determine its pdf through the use of inverse Fourier transform, from which the probability of error can be calculated. We note, however, that the calculation of such an inverse Fourier integral is not a simple matter. It is, therefore, preferable to determine the probability of error directly from the CF of the decision variable. To this end, we invoke the following lemma originated by Gil-Palaez (see [5]).

**Lemma 1 (Gil-Palaez):** Let \(F(x)\) be a 1-D cumulative distribution and \(\phi(\xi)\), the corresponding characteristic function with real variable \(\xi\). Then

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \frac{\Im\{\phi(\xi)e^{-j\xi x}\}}{\xi} d\xi$$

(14)

where \(\Im(z)\) denotes the imaginary part of \(z\).

It follows from (13) and Lemma 1 that the probability of error equals

$$P_e = \Pr \{\gamma < 0\} = F(0) = \frac{1}{2} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im\{\phi_\gamma(t)\}}{t} dt.$$}

(15)

This formula provides a general basis for the performance evaluation of an EGC with coherent detection. It applies to, though not addressed here, arbitrary fading channels as long as their characteristic functions exist. The solution takes the form of a one-fold integral, the calculation of which relies on numerical methods. For the special cases considered in this correspondence, however, we can obtain a closed-form solution.

Now let us consider the case \(L = 3\). In this case, the CF in (13) becomes

$$\phi_\gamma(t) = \exp \left\{ -\frac{(\Omega_1 + \Omega_2 + \Omega_3 + \eta_3)t^2}{4} \right\} \prod_{k=1}^{3} \left\{ I_1 \left(1; \frac{1}{2}; -\frac{\Omega_k t^2}{4} \right) + j \sqrt{\frac{\pi \Omega_k}{4}} t \exp \left\{ -\frac{\Omega_k t^2}{4} \right\} \right\}.$$}

(16)

Define \(\psi(t; a, b, c)\) such that

$$\psi(t; a, b, c) = e^{-\frac{t^2}{2} + \frac{at^3}{3} + \frac{bt^4}{4} + \frac{ct^5}{5}} \times \sqrt{\frac{\pi a}{2}} \frac{1}{t} F_1 \left(1; \frac{1}{2}; \frac{bt^2}{4} \right) F_1 \left(1; \frac{1}{2}; \frac{ct^2}{4} \right)$$}

(17)

and \(\varphi(t)\) such that

$$\varphi(t) = \frac{1}{8} \sqrt{\pi^3 \Omega_1 \Omega_2 \Omega_3} \exp \left\{ -\frac{(\Omega_1 + \Omega_2 + \Omega_3 + \eta_3)t^2}{4} \right\} \prod_{k=1}^{3} \left\{ I_1 \left(1; \frac{1}{2}; -\frac{\Omega_k t^2}{4} \right) + j \sqrt{\frac{\pi \Omega_k}{4}} t \exp \left\{ -\frac{\Omega_k t^2}{4} \right\} \right\}.$$}

(18)

and, hence, we write

$$\frac{\Im\{\phi_\gamma(t)\}}{t} = \psi(t; \Omega_1, \Omega_2, \Omega_3) + \psi(t; \Omega_2, \Omega_3, \Omega_1)$$

$$+ \psi(t; \Omega_3, \Omega_1, \Omega_2) - \varphi(t).$$}

(19)
Inserting (19) into (15) and using of [7, Formula 7.622(1)], we obtain the following.

**Theorem 1:** Consider the problem of coherent detection of binary signals using a three-branch EGC. Then, the probability of error can be expressed as

\[
P_e = \frac{1}{2} \mathcal{F}(\Omega_1, \Omega_2, \Omega_3, \kappa \rho_3) - \mathcal{F}(\Omega_2, \Omega_3, \Omega_1, \kappa \rho_3)
\]

\[
= \mathcal{F}(\Omega_3, \Omega_1, \Omega_2, \kappa \rho_3) + \pi \sqrt{\frac{\Omega_1 \Omega_2 \Omega_3}{(\Omega_1 + \Omega_2 + \Omega_3 + \kappa \rho_3)^3}}
\]

(20)

where the constant \(\kappa\) equals 1 for BPSK and equals 2 for BFSK. We can alternatively express \(P_e\) in terms of SNR’s at individual branches

\[
P_e = \frac{1}{2} \mathcal{F}(\rho_{31}, \rho_{32}, \rho_{33}, 3\kappa) - \mathcal{F}(\rho_{32}, \rho_{33}, \rho_{31}, 3\kappa)
\]

\[
= \mathcal{F}(\rho_{33}, \rho_{31}, \rho_{32}, 3\kappa) + \pi \sqrt{\frac{\rho_{31} \rho_{32} \rho_{33}}{(\rho_{31} + \rho_{32} + \rho_{33} + 3\kappa)^3}}
\]

(21)

The function \(\mathcal{F}\) is defined as

\[
\mathcal{F}(x, y, z, c) = \frac{1}{2} \sqrt{x(x + y + c)(x + z + c)}
\]

\[
\times \, _2F_1\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; \frac{yz}{(x + y + c)(x + z + c)}\right)
\]

(22)

with \(\, _2F_1\) denoting the Gauss hypergeometric function.

We observe that \(P_e\) depends on the signal energy and noise power density only through the SNR’s of the individual channels.

We obtain \(P_e\) for the case \(L = 2\) by letting \(\Omega_3 = 0\) and \(N_3 = 0\) in (20)

\[
P_e = \frac{1}{2} \left\{ 1 - \sqrt{\frac{\rho_{31}(\rho_{21} + 2\kappa)}{\rho_{21} + \rho_{22} + 2\kappa}} + \sqrt{\frac{\rho_{22}(\rho_{22} + 2\kappa)}{\rho_{21} + \rho_{22} + 2\kappa}} \right\}
\]

(23)

In this expression, if we further let \(\Omega_2 = N_2 = 0\), the result for a nondiversity receiver follows:

\[
P_e = \frac{1}{2} \left[ 1 - \frac{1}{\sqrt{\frac{\rho_{11}}{\rho_{11} + \kappa}}} \right]
\]

(24)

which is identical to the well known result in the literature [9, p. 717]. The Gauss hypergeometric functions in (22) can be evaluated using the series expansion [8]

\[
\, _2F_1\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; z\right)
\]

\[
= 1 + \frac{z}{2} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \frac{1}{2^n} \prod_{k=2}^{n} \left(1 - \frac{3}{2k}\right) z^n
\]

(25)

which converges very fast in our case since \(0 < z < 1\).

Fig. 2 shows the error performance of a three-branch receiver using BPSK with equal and unequal SNR’s. Comparing BPSK and coherent BFSK for different \(L\) is shown in Fig. 3. As expected, the error performance improves with increased order of diversity.

**IV. NONCOHERENT DETECTION**

For noncoherent detection, the combined signal is applied to a noncoherent detector. Consider the case \(L = 2\). Let \(\xi = x_1 + x_2\) and let \(f_\xi(x)\) denote its pdf. Given \(\xi\), the probability of error equals \(P_e(\xi) = 0.5 \exp(-\xi^2/g)\), where \(g = a\sigma_2^2\), and \(a = 1\) for DPSK and \(a = 2\) for noncoherent BFSK.

The average probability of error is the statistical average of \(P_e(\xi)\) over \(f_\xi(x)\). We express \(f_\xi(x)\) in terms of \(\phi_\xi(t)\) and reorganize, obtaining

\[
P_e = \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_\xi(t) dt \int_{-\infty}^{\infty} e^{-z^2/2} e^{-jxt} dx
\]

(26)

where \(\phi_\xi(t)\) can be found from (9). We next write the second integral as sine and cosine transforms and determine them.
using formulas 3.896(3-4) of [7, p. 480]. It ends up with the expression of \( P_e \) as the sum of four integrals. These integrals can further be simplified by using formula 7.622(1) of [7] and the recursive relation (13.4.6) of [8]. The result is summarized as follows.

**Theorem 2:** The error performance of a two-branch EG combiner, when used for noncoherent detection of binary DPSK and FSK signals, is given by

\[
P_e = \frac{1}{4} + \frac{1}{2} F(2\kappa, \rho_{21}, \rho_{22}, 0) - \frac{1}{2} F(\rho_{21}, \rho_{22}, 2\kappa, 0) - \frac{1}{2} F(\rho_{22}, 2\kappa, \rho_{21}, 0) - \frac{\pi}{8} \sqrt{\frac{2\kappa \rho_{21} \rho_{22}}{(\rho_{21} + \rho_{22} + 2\kappa)^3}}.
\] (27)

The error performance for nondiversity reception [9, p. 718] can be obtained from (27) by letting \( \Omega_2 = \eta_2 = 0 \).

V. CONCLUSION

In this letter, we have examined the probability of error \( (P_e) \) of EGC’s with two or three branches operating in a Rayleigh-fading environment. We tackled the problem by directly determining \( P_e \) from the relevant characteristic functions, ending up with some closed-form solutions which are expressed in terms of Gauss hypergeometric functions.

The new formulas include the classical results for nondiversity reception in a Rayleigh fading environment as a special case. Furthermore, the assumption of equal SNR at each branch is not necessary for the use of the new formulas.

REFERENCES