The Use of Fuzzy Spaces in Signal Detection

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Abstract: The Fuzzy CFAR (Constant False Alarm Rate) detector, which is based on the M-out-of-N binary detector, is characterized and compared with the optimal Neyman-Pearson detector. It replaces the crisp M-out-of-N binary threshold with a soft, continuous threshold, implemented as a membership function. This function is chosen so that the output is equal to the false alarm rate of the binary detector, and therefore maps the observation set to a False Alarm Space corresponding to the false alarm rate, P_{FA}. An analogous membership function is also developed mapping observations to a Detection Space which corresponds to the detection rate, P_D. These two spaces allow different detectors to be compared directly with respect to the two important detection performance indices, P_{FA} and P_D. Comparison of the False Alarm Space and Detection Space indicates that the Fuzzy CFAR detector and Neyman-Pearson detector detect signals in a different manner and have different detection properties. Nevertheless, performance results illustrate that the Fuzzy CFAR detector achieves detection performance comparable to the optimal Neyman-Pearson detector.

Keywords: Decision Making; Signal Detection

1. Introduction

Constant False Alarm Rate (CFAR) detectors [4, 8-10] and other threshold-based detectors [7, 12] form the basis of many decision making systems such as in radar detection and digital communication. Detection algorithms are generally optimized with respect to a particular set of cost functions chosen for the specific application. In this paper we present the Fuzzy

CFAR detector [5-6] and compare it to the Neyman-Pearson detector [1, 12-13], which provides optimal detection in many situations in which cost is too vague for meaningful assignment of cost functions [13]. We describe the underlying principles and detection criteria for each detector, and also show how the Fuzzy CFAR detector is adapted from the classic *M*-out-of-*N* binary detector [10]. We go on to discuss the use of Fuzzy Spaces [6], explaining how they allow analysis and direct comparison of the detection properties of detectors.

2. Detection Criteria

In signal detection, we are often interested in determining the presence of a weak signal, corrupted by noise. CFAR detectors generally implement decision rules based on a crisp threshold producing binary output, either 0 or 1 [10]. Often, such discontinuous decision rules give rise to a significant loss of information, producing far from optimal detection performance. The Fuzzy CFAR detector replaces the binary threshold with a soft, continuous threshold producing smooth output in order to reduce information loss. The following sections compare the detection criteria of this detector with the optimal Neyman-Pearson detector, and the classic *M*-out-of-*N* detector which it adapts.

Noise can be modeled as a random process, reducing the detection problem to a statistical hypothesis test. An appropriate hypothesis test for detection of a signal in a noisy channel is

$$\begin{array}{lll} H_0: & y = n & \text{"}No\,Target\text{"} \\ H_1: & y = M + n & \text{"}Target\text{"} \end{array} , \tag{2.1}$$

where y represents an observation of the channel, n the observed noise, and M the signal which may be present in the channel. In this study, we consider only the basic case of a constant signal in zero-mean white Gaussian noise, with observations assumed independent and, therefore, uncorrelated [1]. Although simple, this model allows us to introduce the concepts required to characterize the Fuzzy CFAR detector, and to compare it to the Neyman-Pearson and M-out-of-N detectors. We now present a brief review of each detector, including the derivation of the Fuzzy CFAR detector from the M-out-of-N detector.

2.1 Neyman-Pearson Criterion

The Neyman-Pearson criterion specifies the optimal decision rule for a hypothesis test with constrained rate of false alarm and is often used when the cost functions associated with hypotheses are unknown. The Neyman-Pearson detector is implemented as shown in Figure 1. For each observation set, N independent measurements of the detection channel are made. The mean value is then thresholded to provide the target decision.

For our chosen noise/signal model with N independent observations, the Neyman-Pearson threshold, TH_X, is given implicitly by [1]

$$P^{FA} := \Pr\left(\overline{X} \geq TH_X \,\middle|\, H_0\right), \quad \overline{X} := \frac{1}{N}\sum_{i=1}^{N} Y_i \quad , \tag{2.2}$$

where P^{FA} is the chosen false alarm rate; hypothesis "*Target*" is accepted when $X \geq TH_X$. This produces the Neyman-Pearson decision rule

$$\overline{X} \underset{NoTarget}{\overset{Target}{\gtrless}} TH_X := s\big/\sqrt{N}\; \Phi^{-1}\left(1 - P^{FA}\right), \tag{2.3}$$

where Φ^{-1} is the inverse cumulative distribution function of the standard Gaussian distribution, and s^2 is the noise mean power. The detection rate is given by [1]

$$P^D := \Pr\left(X \geq TH_X \,\middle|\, H_1\right), \tag{2.4}$$

which for Gaussian noise is

$$P^D = 1 - \Phi\left[\Phi^{-1}\left(1 - P^{FA}\right) - \sqrt{N}\, M_1\big/s\right], \tag{2.5}$$

where Φ is the cumulative distribution function of the standard Gaussian distribution.

2.2 *M*- out - of - *N* Criterion

The *M*-out-of-*N* or Binary detector found widespread application in the early years of radar when operators were still using head-phones to listen to the detection channel [10]. Although now little used as a detector, it forms the basis for the new class of fuzzy detectors introduced by the authors in [5-6]. The *M*-out-of-*N* detector is shown in Figure 2 as used for radar detection. At each range bin, N observations are made, as for the Neyman-Pearson detector. However, each observation, y_i, is thresholded independently, producing binary output, $m(y_i)$,

$$m : y_i \;\mapsto\; \begin{cases} 0 : \; y_i < TH_Y{}^N \\ 1 : \; y_i \geq TH_Y{}^N \end{cases} \;. \tag{2.6}$$

These binary outputs are then summed and a "*Target*" is declared when the sum is greater than or equal to M, *i.e*.

$$\sum_{i=1}^{N} m(y_i) \underset{NoTarget}{\overset{Target}{\gtrless}} M \;. \tag{2.7}$$

The threshold, $TH_Y{}^N$, is defined implicitly by

$$P^{FA} := \prod_{i=j_1}^{j_M} \left\{ \Pr\left(Y_i \geq TH_Y{}^N \,|\, H_0\right)\right\}, \tag{2.8}$$

or equivalently

$$\sqrt[M]{P^{FA}} \;:=\; \Pr\left(Y_i \geq TH_Y{}^N \,|\, H_0\right) \;\text{ for all }\; Y_i \;=\; Y_{j_1}, \quad , Y_{j_M}, \tag{2.9}$$

reflecting the fact that at least M of the N observations should exceed the chosen threshold with constrained false alarm rate.

The detection rate is given by

$$P^D \;=\; \prod_{i\,=\,j_1}^{j_M} \left\{ \Pr\left(Y_i \geq TH_Y{}^N \,|\, H_1\right)\right\}, \tag{2.10}$$

which for Gaussian noise is

$$P^D = \left\{1 - \Phi\left[\Phi^{-1}\left(1 - \sqrt[N]{P^{FA}}\right) - M_1 / s\right]\right\}^N . \tag{2.11}$$

2.3 Fuzzy Criterion

The Fuzzy CFAR detector adapts the M-out-of-N detector by replacing the binary threshold of Equation 2.6 with a soft threshold, shown in Figure 3. The reason for adapting the M-out-of-N detector in this way is to deal with the uncertainty in selecting a fixed value of the binary threshold, TH_Y^N; this new fuzzy threshold provides soft decisions with a smooth transition from certain acceptance to certain rejection of a hypothesis. In this way the detector retains more information than the binary detector.

The soft threshold is implemented as a fuzzy membership function [3, 14], $\Theta\ (y_i)$, assigning membership to hypothesis H_0, "*No Target*". Furthermore, it is defined so that membership values are distributed uniformly on [0,1] under H_0,

$$\Theta : y_i \mapsto \Pr\left(z \geq y_i \,\middle|\, z \sim H_0\right), \tag{2.12}$$

so that $\Theta(y_i)$ is the probability that an observation exceeds threshold y_i under the "*No Target*" hypothesis. The reason for this particular definition is two-fold:

- The membership function is monotone decreasing, guaranteeing that stronger observations are assigned smaller membership to the "*No Target*" hypothesis, *i.e.* $y_i \geq y_j \iff \Theta(y_i) \leq \Theta(y_j)$; (2.13)
- The M-out-of-N criterion, as interpreted in equation (2.9), requires that the false alarm rate associated with at least M **individual** observations be smaller than some threshold for a "*Target*" to be declared. With our choice of membership function, the M-out-of-N decision rule can be written as

$$\Theta(y_i) \underset{NoTarget}{\overset{Target}{\underset{\geq}{<}}} \sqrt[M]{P^{FA}} \quad \forall\, y_i = y_{j_1}, \ldots, y_{j_M} , \tag{2.14}$$

hence presenting the threshold explicitly in terms of false alarm rate.

The Fuzzy CFAR criterion adapts this decision rule by declaring "*Target*" when the false alarm rate associated with the **entire** observation set is small enough. Since observations are independent, the Fuzzy CFAR decision rule for N observations is given by

$$\prod_{i=1}^{N} \Theta(y_i) \underset{NoTarget}{\overset{Target}{\substack{< \\ \geq}}} P^{FA}, \tag{2.15}$$

where threshold, P^{FA}, is just the false alarm rate of the associated M-out-of-N detector, allowing the False Alarm Space concept to be developed in Section 3. The block diagram of the Fuzzy CFAR detector is shown in Figure 4. With this detector, if even a single observation suggests "*Target*" with sufficient certainty ($\Theta(y_i) << 1$), the detector outputs "*Target*." This will be discussed further in Section 3.2.

In our simple example, the membership function is

$$\Theta : y \mapsto 1 - \Phi\left({}^{y}\!/_{s} \right), \tag{2.16}$$

the complement of the cumulative distribution function under H_0, and has the form shown in Figure 3. The false alarm rate of the N scan detector is

$$P_N^{\ FA} = a \sum_{i=1}^{N-1} (-1)^i \frac{(\ln a)^i}{i!}, \tag{2.17}$$

while the detection rate is given by the integral

$$P_N^{\ D} = \int_{a \prod_{i=1}^{N} \Theta(y_i)} dy_1 \ldots dy_N \, . \tag{2.18}$$

3. Fuzzy Spaces

A standard method for comparing the performance of detectors is to consider the detection rate achieved at chosen false alarm rates. It may often be instructive to consider which observations give rise to the differences in the quoted detection and false alarm rates. In this section, we look into this by defining two new spaces, the False Alarm Space and the Detection Space, which make the comparison of such differences trivial.

3.1 False Alarm Space

The "*No Target*" membership function, Θ, defined in equation (2.12) maps the observation space, **S** := $Â^{N}$, into a fuzzy hyper-cube [4], $Á^{N}$:= $[0,1]^{N}$. The threshold of each detector forms an N – 1 dimensional subspace of $Á^{N}$, partitioning it into a False Alarm Region, $Á_1$, and a Rejection Region, $Á_0$. Under H_0, observations which map to a point in $Á_1$ result in an incorrect acceptance of H_1, *i.e.* a false alarm. Observations which map to a point in $Á_0$ result in a correct rejection of H_1. Due to the choice of membership function, Θ, and the independence of observations, the false alarm rate of a detector is simply the hyper-volume of its associated False Alarm Region, $Á_1$ [6]. We therefore call $Á^{N}$ the False Alarm Space.

Figure 5 shows the threshold and False Alarm Region for each detector projected into the False Alarm Space. In order to clearly illustrate the form of each threshold, we show the False Alarm Space at 10% false alarm rate in two dimensions only. However, the thresholds have the same general form at higher dimensions. The False Alarm Regions differ considerably, highlighting how different observations generate false alarms for each detector:

- The Neyman-Pearson detector favors selecting the "*Target*" hypothesis, H_1 when $\Theta(y_i)$ for **both** observations is sufficiently small – the False Alarm Region has non-linear bound;
- The M-out-of-N binary detector with 2-out-of-2 decision rule selects H_1, whenever $\Theta(y_i)$ for **both** observations is smaller than the threshold, and with 1-out-of-2 decision rule selects H_1 whenever $\Theta(y_i)$ for **either** observation is smaller than the threshold $\sqrt[1]{P^{FA}} = P^{FA}$ – the False Alarm Region is bounded by a square;

- The Fuzzy CFAR detector favors selecting H_1 when $\Theta(y_i)$ for **either** observation is sufficiently small – the False Alarm Region is bounded by a hyperbola.

It is these differences in the False Alarm Regions of each detector that give rise to differences in detection performance, discussed in the following section.

3.2 Detection Space

A second membership function, Ψ, is defined to measure the degree to which an observation is consistent with hypothesis H_1, "*Target*". This membership function is analogous to the false alarm membership function described in Section 2.3 and is defined so that membership values, $\Psi(y_i)$, are distributed uniformly on [0,1] under hypothesis H_1. Ψ maps the observation space, $\mathbf{S} = \hat{A}^{\boldsymbol{N}}$, into a fuzzy hyper-cube, $D^{\boldsymbol{N}} := [0,1]^N$.

The detection threshold partitions $D^{\boldsymbol{N}}$ into a Detection Region, D_1, and a Miss Region, D_0. Under H_1, observations which map to a point in D_1 result in a correct acceptance of H_1, *i.e.* a target is detected, while observations which map to a point in D_0 result in an incorrect rejection of H_1, *i.e.* a missed target. The detection rate of the detector is thus the hyper-volume of the Detection Region, D_1 [6] hence we call $D^{\boldsymbol{N}}$ the Detection Space.

For Gaussian noise, the "*Target*" membership function is

$$\Psi : y \mapsto 1 - \Phi\left({y - M_1}\big/{s} \right), \tag{3.1}$$

the complement of the cumulative distribution function under H_1.

A plot of the two-dimensional Detection Space for each detector is shown in Figure 6.

- The Neyman-Pearson detector favors detection when $\Psi(y_i)$ for **both** observations is sufficiently large;
- The 2-out-of-2 binary detector has square Detection Region, while the 1-out-of-2 detector has square Miss Region;

- The Fuzzy CFAR detector favors detection when $\Psi(y_i)$ for **either** observation is sufficiently large.

Without calculating the actual detection rates of the detectors, it is apparent that the Fuzzy CFAR detector generates fewer detections than the Neyman-Pearson detector when observations are approximately equal (region D_1^-), but more detections when observations are significantly different (region D_1^+). For a general N scan, the Neyman-Pearson detector selects "*Target*" when all observations are sufficiently large, whereas the Fuzzy CFAR detectors also allows the "*Target*" hypothesis to be selected when even a single observation is sufficiently large.

These differences in the Detection Regions of the Neyman-Pearson and Fuzzy Detectors may be significant in certain applications, such as in radar detection of Swerling 2 & 4 fluctuating targets [11] when signal observations may vary more rapidly than expected.

4. Performance

The performance characteristics of each detector are summarized in Figures 7 & 8. Figure 7 plots the detection rate for each detector at several practical false alarm rates, while Figure 8 shows the signal-to-noise ratio (SNR) required for each detector to achieve the specified detection performance.

The M-out-of-N detector provides poorer detection performance than either the Neyman-Pearson detector or the Fuzzy CFAR detector, due mainly to its use of a binary threshold. It is also evident that for a two scan scheme, the performance of the Fuzzy CFAR detector differs by no more than 0.2 dB from that of the optimal Neyman-Pearson detector. As discussed in Section 3.2, the Fuzzy CFAR detector may also be less susceptible to degradation of detection performance when, due to uncertain knowledge of target fluctuations, signal observations fluctuate more rapidly than expected.

5. Conclusion

This paper has reviewed the principles of signal detection using both binary and fuzzy thresholds to implement decision rules. The Fuzzy CFAR detector has been shown to out-perform the classic M-out-of-N binary detector, which it extends, and to have performance comparable to that of the optimal Neyman-Pearson detector. By particular choice of the threshold membership function, the threshold of the Fuzzy CFAR detector is related directly to the false alarm rate.

A method for analyzing the performance of a detector by projecting its threshold into two fuzzy spaces, the False Alarm Space and the Detection Space, has been demonstrated. The false alarm rate and detection rate of the detector can be calculated from the hyper-volume of the False Alarm Region and Detection Region, respectively. Moreover, detectors can be characterized and compared in terms of these spaces by examining which observations give rise to false alarms and valid detections.

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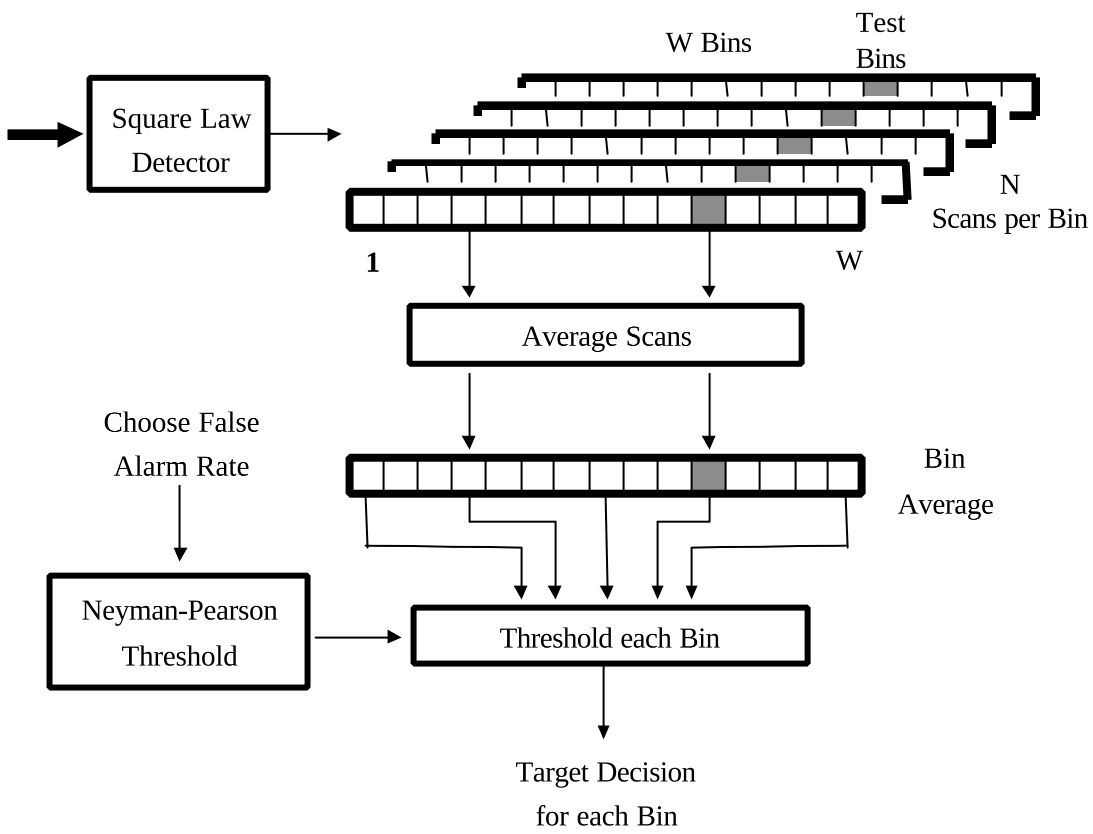
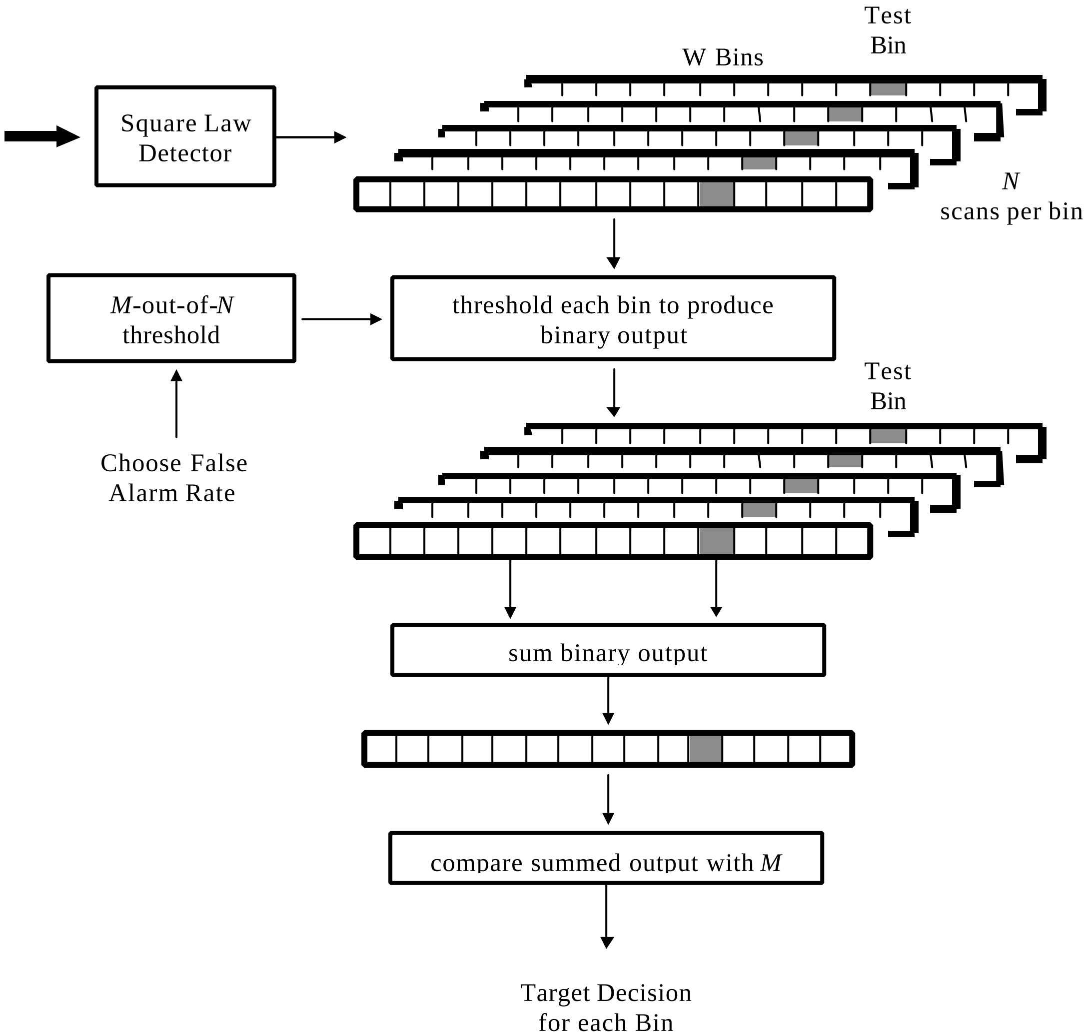
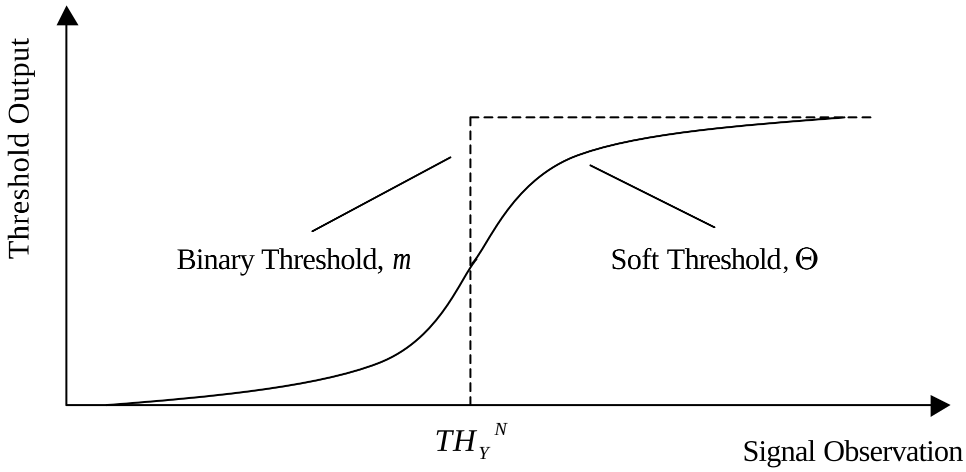
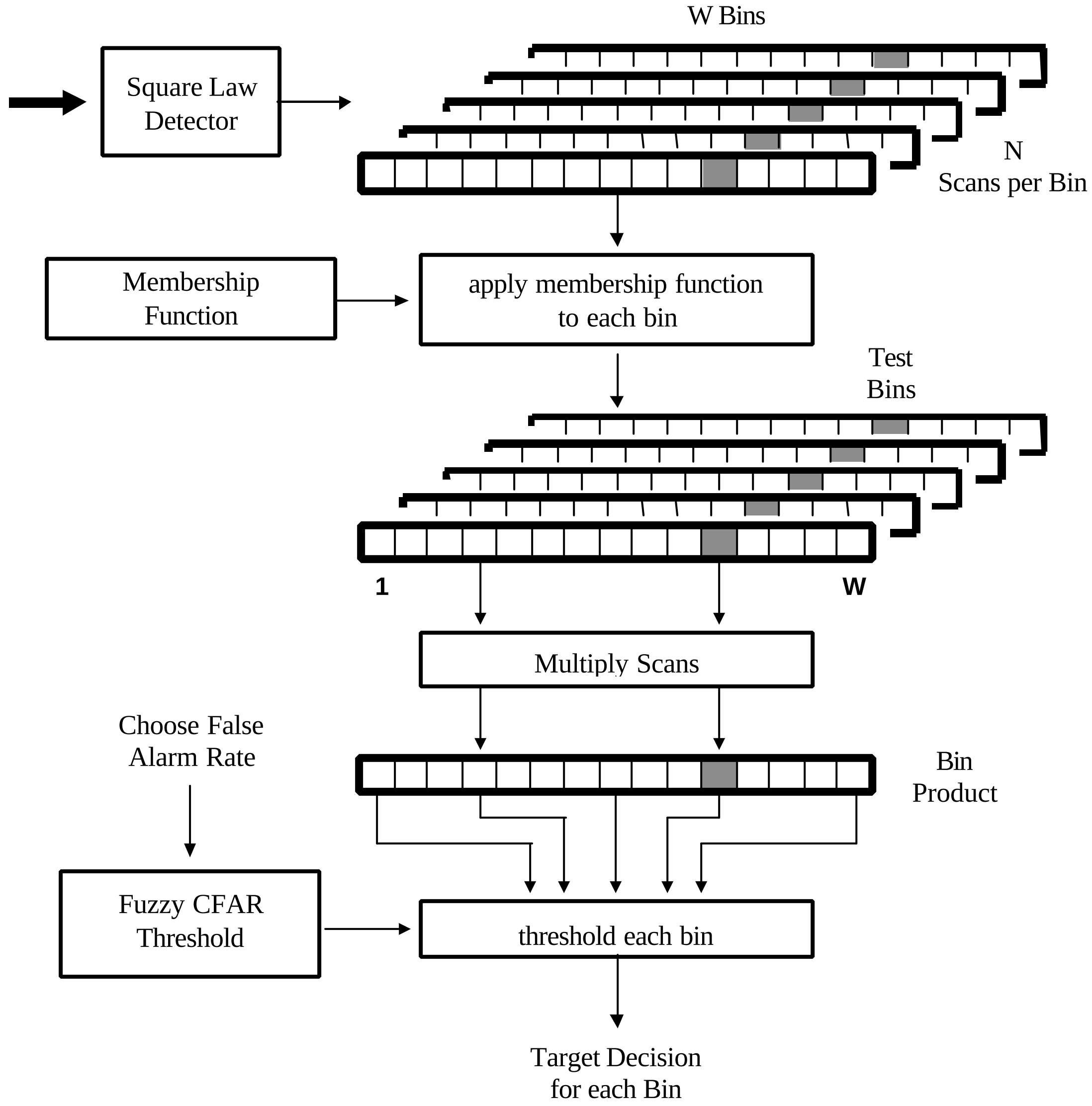
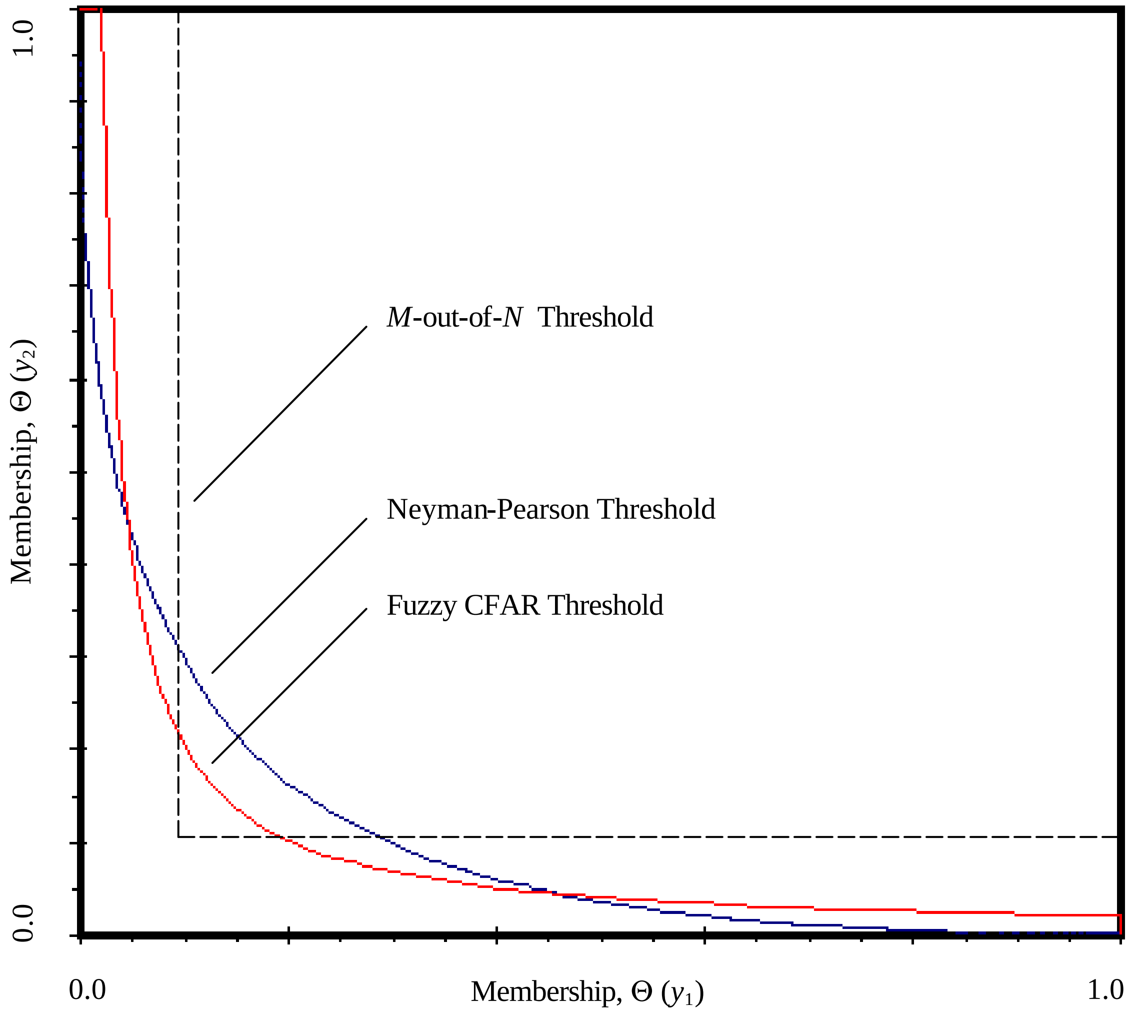

W Bins
Test
Bins
Square Law
Detector
N
Scans per Bin
1
W
Average Scans
Choose False
Alarm Rate
Bin
Average
Neyman-Pearson
Threshold
Threshold each Bin
Target Decision
for each Bin

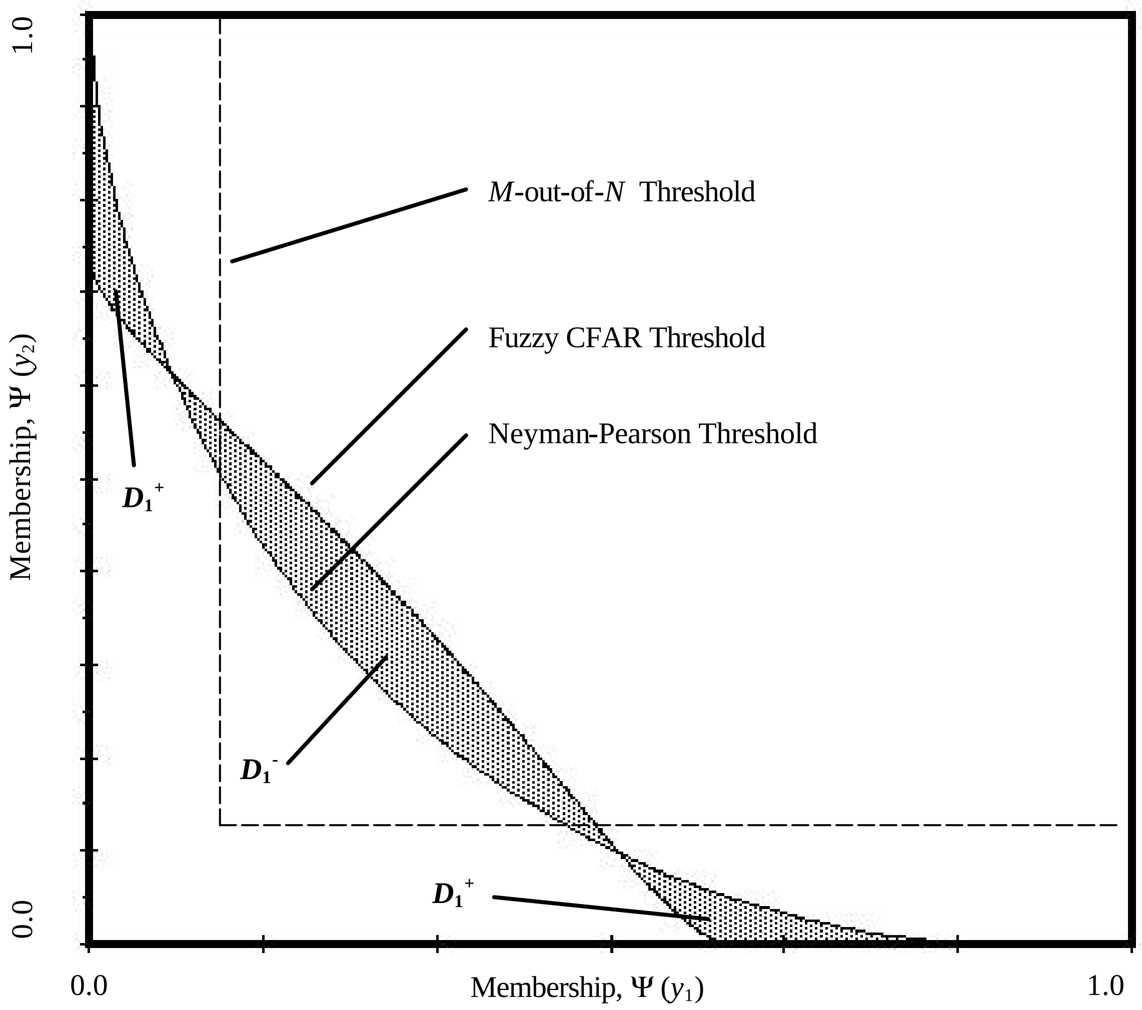

Test
Bin
W Bins
Square Law
Detector
N
scans per bin
M-out-of-N
threshold
threshold each bin to produce
binary output
Test
Bin
Choose False
Alarm Rate
sum binary output
compare summed output with M
Target Decision
for each Bin

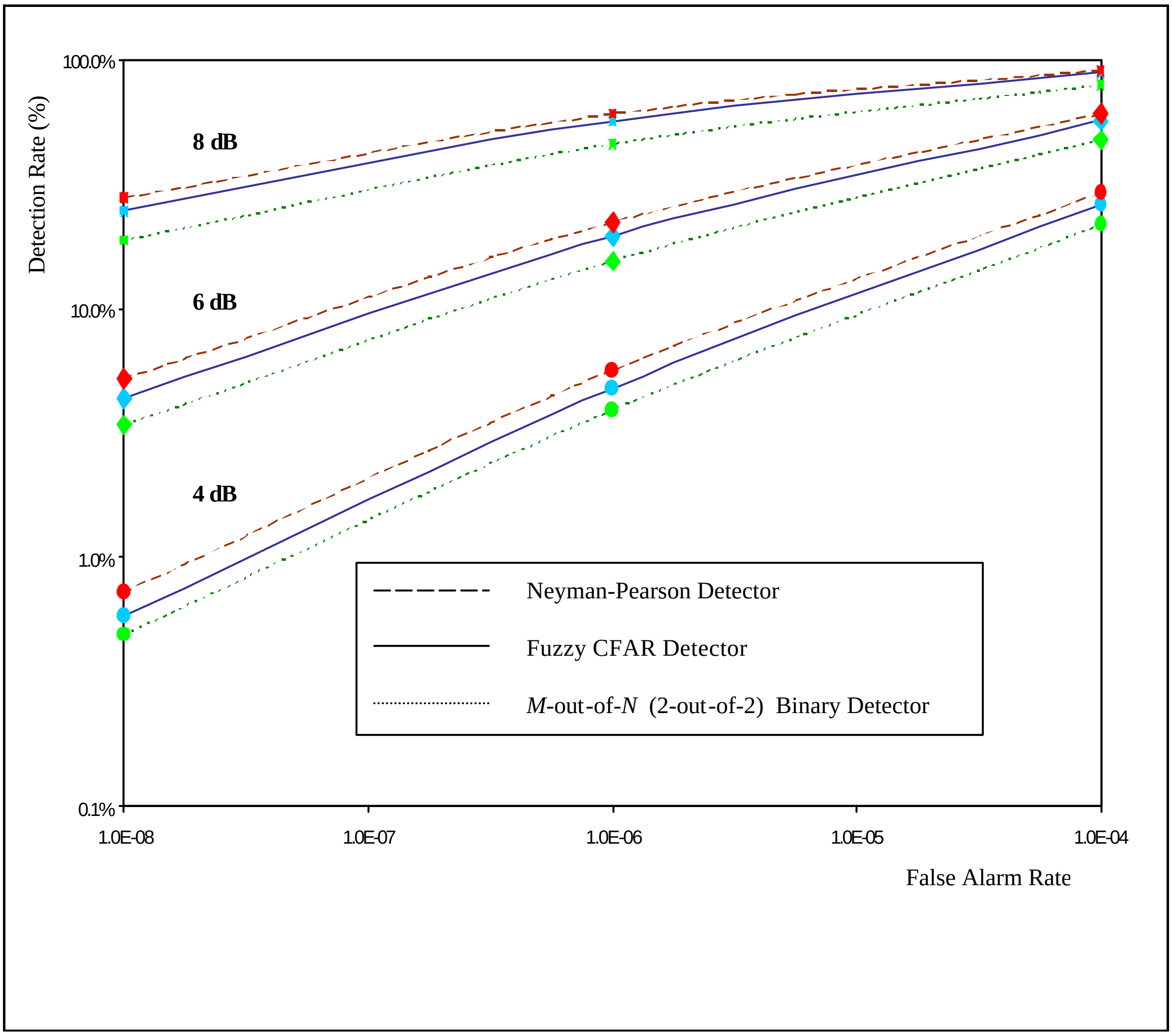


Threshold Output
Binary Threshold, m
Soft Threshold, Θ
TH_Y^N
Signal Observation


W Bins
Square Law Detector
N
Scans per Bin
Membership Function
apply membership function to each bin
Test Bins
1
W
Multiply Scans
Choose False Alarm Rate
Bin Product
Fuzzy CFAR Threshold
threshold each bin
Target Decision for each Bin


1.0
0.0
Membership, Θ (y_2)
M-out-of-*N* Threshold
Neyman-Pearson Threshold
Fuzzy CFAR Threshold
0.0
Membership, Θ (y_1)
1.0


1.0
0.0
Membership, Ψ (y_2)
M-out-of-N Threshold
Fuzzy CFAR Threshold
Neyman-Pearson Threshold
D_1^+
D_1^-
D_1^+
0.0
Membership, Ψ (y_1)
1.0



Detection Rate (%)
100.0%
10.0%
1.0%
0.1%
8 dB
6 dB
4 dB
Neyman-Pearson Detector
Fuzzy CFAR Detector
M-out-of-*N* (2-out-of-2) Binary Detector
1.0E-08
1.0E-07
1.0E-06
1.0E-05
1.0E-04
False Alarm Rate

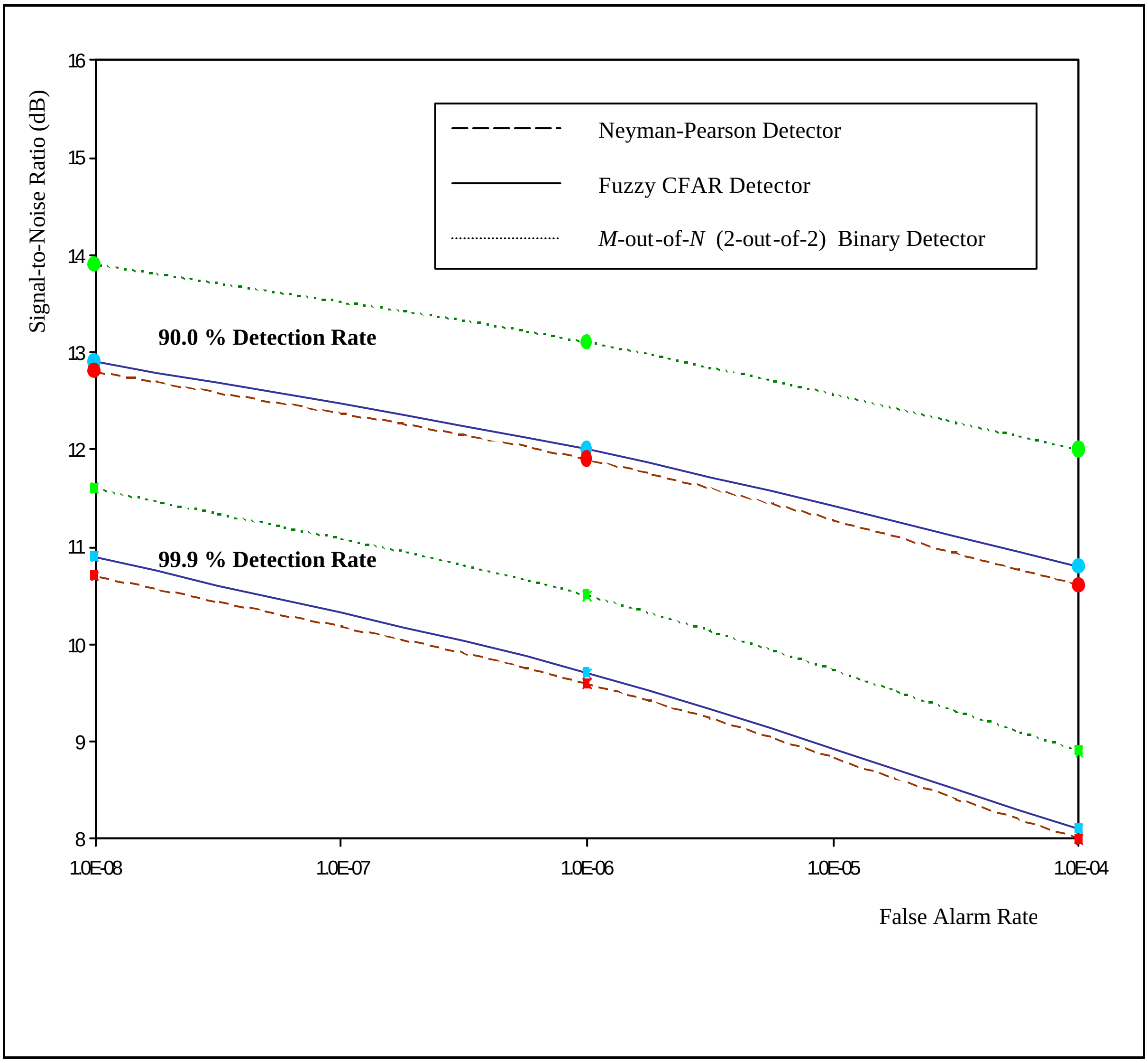

Neyman-Pearson Detector
Fuzzy CFAR Detector
M-out-of-N (2-out-of-2) Binary Detector
90.0 % Detection Rate
99.9 % Detection Rate
Signal-to-Noise Ratio (dB)
16
15
14
13
12
11
10
9
8
1.0E-08
1.0E-07
1.0E-06
1.0E-05
1.0E-04
False Alarm Rate

Figure Captions

Figure 1. Block Diagram of the Neyman-Pearson Detector

Figure 2. Block Diagram of the *M*-out-of-*N* Binary Detector

Figure3. Comparison of Binary and Soft Threshold

Figure 4. Block Diagram of the Fuzzy CFAR Detector

Figure 5. Threshold and False Alarm Region of the Fuzzy CFAR, Neyman-Pearson and *M*-out-of-*N* Detector.

(Signal-to-Noise Ratio = 0 dB, False Alarm Rate = 10%)

Figure 6. Threshold and Detection Region of the Fuzzy CFAR, Neyman-Pearson and *M*-out-of-*N* Detector.

(Signal-to-Noise Ratio = 0 dB, False Alarm Rate = 10%)

Figure 7. Plot of Detection Rate against False Alarm Rate

(Signal-to-Noise Ratio 4 dB, 6 dB, and 8 dB)

Figure 8. Signal-to-Noise Ratio against False Alarm Rate

(Detection Rate 90.0 % and 99.9 %)