Lecture 8. Digital Communications
Part III. Digital Demodulation

• Binary Detection
• M-ary Detection
Digital Communications

Analog Signal

Bit sequence 0001101110.....

Source → A-D Conversion → Digital Baseband Modulation → Digital Bandpass Modulation

User → D-A Conversion → Digital Baseband Demodulation → Digital Bandpass Demodulation
Digital Demodulation

What are the sources of signal corruption?

How to detect the signal (to obtain the bit sequence \( \{ \hat{b}_i \} \))?

How to evaluate the fidelity performance?
Sources of Signal Corruption

- Suppose that channel bandwidth is properly chosen such that most frequency components of the transmitted signal can pass through the channel.

- Thermal Noise: caused by the random motion of electrons within electronic devices.
Modeling of Thermal Noise

- The thermal noise is modeled as a WSS process \( n(t) \).
  - The thermal noise is superimposed (added) to the signal: \( y(t) = s(t) + n(t) \)
  - At each time slot \( t_0 \), \( n(t_0) \sim \mathcal{N}(0, \sigma^2_0) \) (i.e., zero-mean Gaussian random variable with variance \( \sigma^2_0 \)):
    \[
    f_n(x) = \frac{1}{\sqrt{2\pi\sigma_0}} \exp\left(-\frac{x^2}{2\sigma_0^2}\right)
    \]
    \[
    \Pr\{X < -a\} = \int_{-\infty}^{-a} f_n(x) \, dx
    \]
    \[
    = \int_{a}^{\infty} f_n(-y) \, dy = \Pr\{X > a\}
    \]
    \[
    \Pr\{X > a\} = \int_{a}^{\infty} f_n(x) \, dx = Q\left(\frac{a}{\sigma_0}\right)
    \]
Modeling of Thermal Noise

- The thermal noise has a power spectrum that is constant from dc to approximately $10^{12}$ Hz: $n(t)$ can be approximately regarded as a white process.

The thermal noise is also referred to as additive white Gaussian Noise (AWGN), because it is modeled as a white Gaussian WSS process which is added to the signal.
Detection

Transmitted signal

\[ s(t) = \begin{array}{cccccccccccccc}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array} \]

Received signal

\[ y(t) = s(t) + n(t) \]

Step 1: Filtering

Step 2: Sampling

Step 3: Threshold Comparison

Sample > 0 \(\rightarrow\) 1
Sample < 0 \(\rightarrow\) 0

Lin Dai (City University of Hong Kong) EE3008 Principles of Communications Lecture 8
Bit Error Rate (BER)

- **Bit Error:** \( \{ \hat{b}_i \neq b_i \} = \{ \hat{b}_i = 1 \text{ but } b_i = 0 \} \cup \{ \hat{b}_i = 0 \text{ but } b_i = 1 \} \)

- **Probability of Bit Error (or Bit Error Rate, BER):**

\[
P_b = \Pr\{\hat{b}_i = 1, b_i = 0\} + \Pr\{\hat{b}_i = 0, b_i = 1\}
\]
Digital Demodulation

- How to design the filter, sampler and threshold to minimize the BER?
Binary Detection

- Optimal Receiver Design
- BER of Binary Signaling
Binary Detection

- Bit sequence \( \{b_i\} \)
- Binary Modulation
- Transmitted signal \( s(t) = \begin{cases} s_1(t - i\tau) & \text{if } b_i = 1 \\ s_2(t - i\tau) & \text{if } b_i = 0 \end{cases} \)
- Baseband/Bandpass Channel
- Received signal \( y(t) = s(t) + n(t) \)

\[
\hat{b}_i = \begin{cases} 1 & \text{if } z(t_0) > \lambda \\ 0 & \text{if } z(t_0) < \lambda \end{cases}
\]

Threshold Comparison

Sample at \( t_0 \)

Filter \( h(t) \)

- BER:
  \[
P_b = \Pr\{\hat{b}_i = 1, b_i = 0\} + \Pr\{\hat{b}_i = 0, b_i = 1\}
\]

How to choose the threshold \( \lambda \), sampling point \( t_0 \) and the filter to minimize BER?
Receiver Structure

- Transmitted signal: \( s(t) = \begin{cases} s_1(t) & \text{if } b_1 = 1 \\ s_2(t) & \text{if } b_1 = 0 \end{cases} \quad 0 \leq t \leq \tau 

- Received signal: \( y(t) = s(t) + n(t) = \begin{cases} s_1(t) + n(t) & \text{if } b_1 = 1 \\ s_2(t) + n(t) & \text{if } b_1 = 0 \end{cases} \)

- Filter output: \( z(t) = s_o(t) + n_o(t) = \begin{cases} s_{o,1}(t) + n_o(t) & \text{if } b_1 = 1 \\ s_{o,2}(t) + n_o(t) & \text{if } b_1 = 0 \end{cases} \)

where \( n_o(t) = \int_0^t n(x)h(t-x)dx \), \( s_{o,i}(t) = \int_0^t s_i(x)h(t-x)dx \), \( i = 1, 2 \).

\( n(t) \) is a white process with two-sided power spectral density \( N_0/2 \).

Is \( n_o(t) \) a white process? No!
**Receiver Structure**

- **Transmitted signal:**
  \[
  s(t) = \begin{cases} 
  s_1(t) & \text{if } b_1 = 1 \\
  s_2(t) & \text{if } b_1 = 0 
  \end{cases} 
  \quad 0 \leq t \leq \tau
  \]

- **Received signal:**
  \[
  y(t) = s(t) + n(t)
  \]

- **Filter output:**
  \[
  z(t) = s_o(t) + n_o(t) = \begin{cases} 
  s_{o,1}(t) + n_o(t) & \text{if } b_1 = 1 \\
  s_{o,2}(t) + n_o(t) & \text{if } b_1 = 0 
  \end{cases}
  \]

- **Sampler output:**
  \[
  z(t_0) = s_o(t_0) + n_o(t_0) = \begin{cases} 
  s_{o,1}(t_0) + n_o(t_0) & \text{if } b_1 = 1 \\
  s_{o,2}(t_0) + n_o(t_0) & \text{if } b_1 = 0 
  \end{cases}
  \]

  \[
  n_o(t_0) \sim \mathcal{N}(0, \sigma_o^2) \quad \Rightarrow \quad 
  \begin{align*}
  z(t_0) | b_1 = 1 & \sim \mathcal{N}(s_{o,1}(t_0), \sigma_o^2) \\
  z(t_0) | b_1 = 0 & \sim \mathcal{N}(s_{o,2}(t_0), \sigma_o^2)
  \end{align*}
  \]
Received signal $y(t) = s(t) + n(t)$

Filter $h(t)$

$z(t)$

Sample at $t_0$

$z(t_0)$

Threshold Comparison

$\hat{b}_i = 1$ if $z(t_0) > \lambda$

$\hat{b}_i = 0$ if $z(t_0) < \lambda$

• BER:

$P_b = \Pr\{\hat{b}_1 = 1, b_1 = 0\} + \Pr\{\hat{b}_1 = 0, b_1 = 1\} = \Pr\{z(t_0) > \lambda, b_1 = 0\} + \Pr\{z(t_0) < \lambda, b_1 = 1\}$

$= \Pr\{z(t_0) > \lambda \mid b_1 = 0\} \Pr\{b_1 = 0\} + \Pr\{z(t_0) < \lambda \mid b_1 = 1\} \Pr\{b_1 = 1\}$

$= \frac{1}{2} \left[ \Pr\{z(t_0) > \lambda \mid b_1 = 0\} + \Pr\{z(t_0) < \lambda \mid b_1 = 1\} \right]$ (Pr \{b_1 = 0\} = Pr \{b_1 = 1\} = \frac{1}{2})

Recall that $z(t_0) \mid b_1 = 0 \sim \mathcal{N}(s_{o,2}(t_0), \sigma_0^2)$ and $z(t_0) \mid b_1 = 1 \sim \mathcal{N}(s_{o,1}(t_0), \sigma_0^2)$

How to obtain $\Pr\{z(t_0) > \lambda \mid b_1 = 0\}$ and $\Pr\{z(t_0) < \lambda \mid b_1 = 1\}?$
BER

Received signal $y(t) = s(t) + n(t)$

Filter $h(t)$

$z(t)$

Sample at $t_0$

$z(t_0)$

Threshold Comparison

$\hat{b}_i = 1$ if $z(t_0) > \lambda$

$\hat{b}_i = 0$ if $z(t_0) < \lambda$

$z(t_0) | b_1 = 0 \sim \mathcal{N}(s_{o,2}(t_0), \sigma_0^2)$

$z(t_0) | b_1 = 1 \sim \mathcal{N}(s_{o,1}(t_0), \sigma_0^2)$

$f_Z(z(t_0) | b_1 = 0)$

$f_Z(z(t_0) | b_1 = 1)$

$P_b$ is the area of the yellow zone!

$P_b$ is determined by the threshold $\lambda$!

$\hat{b}_i = b_i$

$\hat{b}_i = \hat{b}_i$

$\hat{b}_i = \hat{b}_i$

BER: $P_b = \frac{1}{2} \left[ \Pr \{ z(t_0) < \lambda | b_1 = 1 \} + \Pr \{ z(t_0) > \lambda | b_1 = 0 \} \right]$

$= \frac{1}{2} \left[ Q \left( \frac{\lambda - s_{o,2}(t_0)}{\sigma_0} \right) + Q \left( \frac{s_{o,1}(t_0) - \lambda}{\sigma_0} \right) \right]$
Optimal Threshold to Minimize BER

Received signal \( y(t) = s(t) + n(t) \)

Filter \( h(t) \)

\( z(t) \)

Sample at \( t_0 \)

\( z(t_0) \)

Threshold Comparison

\( \hat{b}_i = 1 \) if \( z(t_0) > \lambda \)

\( \hat{b}_i = 0 \) if \( z(t_0) < \lambda \)

Optimal threshold to minimize BER:

Choose \( \lambda^* \) to minimize the yellow area!

\[ \lambda^* = \frac{s_{o,1}(t_0) + s_{o,2}(t_0)}{2} \]
BER with Optimal Threshold

- BER with the optimal threshold $\lambda^* = \frac{1}{2}(s_{o,1}(t_0) + s_{o,2}(t_0))$ is

$$P_b(\lambda^*) = \frac{1}{2} \left( Q \left( \frac{\lambda^* - s_{o,2}(t_0)}{\sigma_0} \right) + Q \left( \frac{s_{o,1}(t_0) - \lambda^*}{\sigma_0} \right) \right) = Q \left( \frac{s_{o,1}(t_0) - s_{o,2}(t_0)}{2\sigma_0} \right)$$

$$= Q \left( \frac{1}{2} \sqrt{\frac{\left( \int_0^{t_0} (s_1(x) - s_2(x))h(t_0 - x)dx \right)^2}{\frac{N_0}{2} \int_0^{t_0} h^2(t_0 - x)dx}} \right)$$

where $s_{o,1}(t_0) - s_{o,2}(t_0) = \int_0^{t_0} (s_1(x) - s_2(x))h(t_0 - x)dx$, and $\sigma_0^2 = \frac{N_0}{2} \int_0^{t_0} h^2(t_0 - x)dx$.

$P_b(\lambda^*)$ is determined by the filter $h(t)$ and the sampling point $t_0$!

Why?
Optimal Filter to Minimize BER

Received signal \( y(t) = s(t) + n(t) \) → Filter \( h(t) \) → Sample at \( t_0 \) → Threshold

\[
\hat{b}_i = \begin{cases} 
1 & \text{if } z(t_0) > \lambda^* \\
0 & \text{if } z(t_0) < \lambda^* 
\end{cases}
\]

- Optimal filter to minimize \( P_b(\lambda^*) \): 
  \[
  P_b(\lambda^*); \min_{h(t), t_0} P_b(\lambda^*) = \max_{h(t), t_0} \left[ \frac{\int_{0}^{t_0} (s_1(x) - s_2(x)) h(t_0 - x) dx}{\int_{0}^{t_0} h^2(t_0 - x) dx} \right]^2
  \]

\[
h(t) = k(s_1(\tau - t) - s_2(\tau - t)), \quad 0 \leq t \leq \tau \text{ and } t_0 = \tau
\]

\[
\frac{\left[ \int_{0}^{t_0} (s_1(x) - s_2(x)) h(t_0 - x) dx \right]^2}{\int_{t_0}^{t_0} \frac{N_0}{2} h^2(t_0 - x) dx} \leq \frac{\int_{0}^{t_0} (s_1(x) - s_2(x))^2 dx \int_{0}^{t_0} h^2(t_0 - x) dx}{\int_{0}^{t_0} h^2(t_0 - x) dx}
\]

\[
= \frac{\int_{0}^{t_0} (s_1(x) - s_2(x))^2 dx}{N_0/2} \leq \frac{\int_{0}^{\tau} (s_1(x) - s_2(x))^2 dx}{N_0/2}
\]

“=” holds when \( h(t) = k(s_1(t_0 - t) - s_2(t_0 - t)) \)

“=” holds when \( t_0 = \tau \)
Matched Filter

- Optimal filter: \( h(t) = k(s_1(\tau - t) - s_2(\tau - t)) \) \( (0 \leq t \leq \tau) \)

\[
H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt = k \int_0^\tau (s_1(\tau - t) - s_2(\tau - t)) e^{-j2\pi ft} dt = k(S_1^*(f) - S_2^*(f)) e^{-j2\pi f \tau}
\]

The optimal filter is called matched filter, as it has a shape matched to the shape of the input signal.

- Output of Matched Filter:

\[
z(t) = \int_0^\tau y(t) h(\tau - t) dt
\]

\[
z(\tau) = \int_0^\tau y(t)(s_1(t) - s_2(t)) dt
\]

- Correlation realization of Matched Filter:

\[
z(\tau) = \int_0^\tau y(t)(s_1(t) - s_2(t)) dt
\]
Optimal Binary Detector

Received signal
\[ y(t) = s(t) + n(t) \]

Matched Filter
\[ h(t) = s_1(\tau - t) - s_2(\tau - t) \]
\[ (0 \leq t \leq \tau) \]

Sample at \( \tau \)
\[ z(t) = \int_0^\tau y(t)(s_1(t) - s_2(t))dt \]

Or equivalently:

Received signal
\[ y(t) = s(t) + n(t) \]

Integrator
\[ z(\tau) = \int_0^\tau y(t)(s_1(t) - s_2(t))dt \]

Threshold Comparison
\[ \lambda^* = \frac{1}{2}(E_1 - E_2) \]

\[ \hat{b}_i = 1 \text{ if } z(\tau) > \lambda^* \]
\[ \hat{b}_i = 0 \text{ if } z(\tau) < \lambda^* \]

\[ \lambda^* = \frac{1}{2}(s_{o,1}(\tau) + s_{o,2}(\tau)) = \frac{1}{2} \int_0^\tau (s_1(x) + s_2(x))h(\tau - x)dx = \frac{1}{2}\left(\int_0^\tau s_1^2(t)dt - \int_0^\tau s_2^2(t)dt\right) \]

Energy of \( s_i(t) \):
\[ E_1 \quad \quad \quad \quad E_2 \]
BER of Optimal Binary Detector

- BER with the optimal threshold: 
  \[ P_b (\lambda^*) = Q \left( \frac{1}{2} \sqrt{\frac{\left( \int_0^I (s_1(x) - s_2(x))h(t_0 - x)dx \right)^2}{\frac{N_0}{2} \int_0^I h^2(t_0 - x)dx}} \right) \]

- Impulse response of matched filter: 
  \[ h(t) = s_1(\tau - t) - s_2(\tau - t) \quad (0 \leq t \leq \tau) \]

- Optimal sampling point: 
  \[ t_0 = \tau \]

\[ \downarrow \]

- BER of the Optimal Binary Detector:

  \[ P_b^* = Q \left( \frac{1}{2} \sqrt{\frac{\left( \int_0^\tau (s_1(x) - s_2(x))(s_1(x) - s_2(x))dx \right)^2}{\frac{N_0}{2} \int_0^\tau (s_1(x) - s_2(x))^2 dx}} \right) = Q \left( \sqrt{\frac{\int_0^\tau (s_1(t) - s_2(t))^2 dt}{2N_0}} \right) \]
Energy per Bit $E_b$ and Energy Difference per Bit $E_d$

- **Energy per Bit:** 
  \[ E_b = \frac{1}{2} (E_1 + E_2) = \frac{1}{2} \int_0^\tau (s_1^2(t) + s_2^2(t)) dt \]

- **Energy difference per Bit:** 
  \[ E_d = \int_0^\tau (s_1(t) - s_2(t))^2 dt \]

  - $E_d$ can be further written as

  \[
  E_d = \int_0^\tau s_1^2(t) dt + \int_0^\tau s_2^2(t) dt - 2 \int_0^\tau s_1(t)s_2(t) dt = 2(1 - \rho)E_b
  \]

  \[
  \rho = \frac{1}{2E_b} \int_0^\tau s_1(t)s_2(t) dt
  \]

**Cross-correlation coefficient** $-1 \leq \rho \leq 1$ is a measure of similarity between two signals $s_1(t)$ and $s_2(t)$.
BER of Optimal Binary Detector

- BER of the Optimal Binary Detector:

\[
P_b^* = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b(1 - \rho)}{N_0}}\right)
\]

- The BER performance is determined by 1) \(E_b/N_0\) and 2) Cross-correlation coefficient \(\rho\).

- \(P_b^*\) decreases as \(E_b/N_0\) increases.

- \(P_b^*\) is minimized when cross-correlation coefficient \(\rho=-1\).
BER of Binary Signaling

- Binary PAM, Binary OOK
- Binary ASK, Binary PSK, Binary FSK
BER of Binary PAM

- Energy of $s_i(t)$: $E_1 = E_2 = \int_0^\tau A^2 dt = A^2 \tau$

- Energy per bit: $E_{b,BPAM} = \frac{1}{2} (E_1 + E_2) = A^2 \tau$

- Cross-correlation coefficient:
  
  $$
  \rho_{BPAM} = \frac{1}{E_b} \int_0^\tau s_1(t)s_2(t)dt = -\frac{1}{E_b} \int_0^\tau A^2 dt = -1
  $$

- Power: $P_{BPAM} = A^2$

- Optimal BER:
  
  $$
  P_{b,BPAM}^* = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2A^2\tau}{N_0}}\right) = Q\left(\sqrt{\frac{2P_{BPAM}}{R_{b,BPAM}N_0}}\right)
  $$
BER of Binary OOK

- **Energy of** \( s_i(t) \): \( E_1 = A^2\tau, \ E_2 = 0 \).

- **Energy per bit:** \( E_{b,BOOK} = \frac{1}{2}(E_1 + E_2) = \frac{1}{2} A^2\tau \)

- **Cross-correlation coefficient:**

\[
\rho_{BOOK} = \frac{1}{E_b} \int_{0}^{\tau} s_1(t)s_2(t)dt = 0
\]

- **Power:** \( P_{BOOK} = A^2 / 2 \)

- **Optimal BER:** \( P_{b,BOOK}^* = Q\left( \sqrt{\frac{E_b(1 - \rho)}{N_0}} \right) = Q\left( \sqrt{\frac{E_b}{N_0}} \right) = Q\left( \sqrt{\frac{A^2\tau}{2N_0}} \right) = Q\left( \sqrt{\frac{P_{BOOK}}{R_{b,BOOK}N_0}} \right) \)
Optimal Receivers of Binary PAM and OOK

**Binary PAM:**

- Received signal: $y(t)$
- Integrator: $\int_{0}^{\tau} z(\tau) d\tau$
- Threshold Comparison:
  - $\hat{b}_i = 1$ if $z(\tau) > \lambda^*$
  - $\hat{b}_i = 0$ if $z(\tau) < \lambda^*$
- $\lambda^* = \frac{1}{2} (E_1 - E_2) = 0$

**Binary OOK:**

- Received signal: $y(t)$
- Integrator: $\int_{0}^{\tau} z(\tau) d\tau$
- Threshold Comparison:
  - $\hat{b}_i = 1$ if $z(\tau) > \lambda^*$
  - $\hat{b}_i = 0$ if $z(\tau) < \lambda^*$
- $\lambda^* = \frac{1}{2} (E_1 - E_2) = \frac{1}{2} A^2 \tau$
BER of Binary ASK

- Energy of $s_i(t)$: $E_1 = \frac{1}{2} A^2 \tau$, $E_2 = 0$.

- Energy per bit: $E_{b, BASK} = \frac{1}{2} (E_1 + E_2) = \frac{1}{4} A^2 \tau$

- Cross-correlation coefficient:
  $$\rho_{BASK} = \frac{1}{E_b} \left. \int_{0}^{\tau} s_1(t) s_2(t) dt \right| = 0$$

- Power: $P_{BASK} = \frac{A^2}{4}$

- Optimal BER: $P_{b, BASK}^* = Q \left( \sqrt{ \frac{E_b (1 - \rho)}{N_0} } \right) = Q \left( \sqrt{ \frac{E_b}{N_0} } \right) = Q \left( \sqrt{ \frac{A^2 \tau}{4N_0} } \right) = Q \left( \sqrt{ \frac{P_{BASK}}{R_{b, BASK} N_0} } \right)$

\( \tau \) is an integer number of $1/f_c$
BER of Binary PSK

- **Energy of** $s_i(t)$: $E_1 = E_2 = \frac{1}{2} A^2 \tau$

- **Energy per bit**: $E_{b,\text{BPSK}} = \frac{1}{2} (E_1 + E_2) = \frac{1}{2} A^2 \tau$

- **Cross-correlation coefficient**:
  
  $$\rho_{\text{BPSK}} = \frac{1}{E_b} \int_0^{\tau} s_1(t) s_2(t) dt = -\frac{1}{E_b} \int_0^{\tau} s_1^2(t) dt = -1$$

- **Power**: $P_{\text{BPSK}} = A^2 / 2$

- **Optimal BER**: $P^*_{b,\text{BPSK}} = Q \left( \sqrt{\frac{E_b (1 - \rho)}{N_0}} \right) = Q \left( \sqrt{\frac{2E_b}{N_0}} \right) = Q \left( \sqrt{\frac{A^2 \tau}{N_0}} \right) = Q \left( \sqrt{\frac{2P_{\text{BPSK}}}{R_{b,\text{BPSK}} N_0}} \right)$
Coherent Receivers of BASK and BPSK

**BASK:**

Received signal $y(t)$

Threshold Comparison

\[ \hat{b}_1 = 1 \text{ if } z(\tau) > \lambda^* \]
\[ \hat{b}_1 = 0 \text{ if } z(\tau) < \lambda^* \]

\[ \lambda^* = \frac{1}{2}(E_1 - E_2) = \frac{1}{4} A^2 \tau \]

**BPSK:**

Received signal $y(t)$

Threshold Comparison

\[ \hat{b}_1 = 1 \text{ if } z(\tau) > \lambda^* \]
\[ \hat{b}_1 = 0 \text{ if } z(\tau) < \lambda^* \]

\[ \lambda^* = \frac{1}{2}(E_1 - E_2) = 0 \]

The optimal receiver is also called “coherent receiver” because it must be capable of internally producing a reference signal which is in exact phase and frequency synchronization with the carrier signal $\cos(2\pi f_c t)$.
BER of Binary FSK

- **Energy of** $s_i(t)$: $E_1 = E_2 = \frac{1}{2} A^2 \tau$

- **Energy per bit:**

  $$E_{b,BFSK} = \frac{1}{2} (E_1 + E_2) = \frac{1}{2} A^2 \tau$$

- **Cross-correlation coefficient:**

  $$\rho_{BFSK} = \frac{1}{E_b} \int_0^\tau s_1(t)s_2(t)dt = \frac{2}{\tau} \int_0^\tau \cos(2\pi(f_c + \Delta f)t)\cos(2\pi(f_c - \Delta f)t)dt$$

  $$= \frac{1}{\tau} \left( \int_0^\tau \cos(4\pi\Delta ft)dt + \int_0^\tau \cos(4\pi f_c t)dt \right) = \frac{1}{\tau} \int_0^\tau \cos(4\pi\Delta ft)dt = \frac{1}{4\pi\Delta f} \sin(4\pi\Delta f \tau)$$

- What is the minimum $\Delta f$ to achieve $\rho_{BFSK} = 0$?

  $$\min \Delta f = \frac{1}{4\tau} = \frac{R_{b,BFSK}}{4}$$
Coherent BFSK Receiver

- **Power:** \( P_{\text{BFSK}} = \frac{A^2}{2} \)

- **Optimal BER:** 
  \[
  P_{b,\text{BFSK}}^* = Q\left(\sqrt{\frac{E_b (1 - \rho)}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2 \tau}{2N_0}}\right) = Q\left(\sqrt{\frac{P_{\text{BFSK}}}{R_{b,\text{BFSK}} N_0}}\right)
  \]
Bandwidth Efficiency of Coherent BFSK

With $\Delta f = \frac{R_{b,BFSK}}{4}$:

- The required channel bandwidth for 90% in-band power:

$$B_{h\_90\%} = 2\Delta f + 2R_{b,BFSK} = 2.5R_{b,BFSK}$$

- Bandwidth efficiency of coherent BFSK:

$$\gamma_{BFSK} = \frac{R_{b,BFSK}}{B_{h\_90\%}} = 0.4$$
## Summary I: Binary Modulation and Demodulation

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<th>Modulation Type</th>
<th>Bandwidth Efficiency</th>
<th>BER</th>
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<tr>
<td>Binary PAM</td>
<td>1 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{2E_{b,BPAM}}{N_0}}\right)$</td>
</tr>
<tr>
<td>Binary OOK</td>
<td>1 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{E_{b,OOK}}{N_0}}\right)$</td>
</tr>
<tr>
<td>Coherent Binary ASK</td>
<td>0.5 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{E_{b,BASK}}{N_0}}\right)$</td>
</tr>
<tr>
<td>Coherent Binary PSK</td>
<td>0.5 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{2E_{b,BPSK}}{N_0}}\right)$</td>
</tr>
<tr>
<td>Coherent Binary FSK</td>
<td>0.4 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{E_{b,BFSK}}{N_0}}\right)$</td>
</tr>
</tbody>
</table>
M-ary Detection

- M-ary PAM
- M-ary PSK
Detection of 4-ary PAM

Bit sequence \( \{b_i\} \) → 4-ary PAM → Transmitted signal \( s(t) = \begin{cases} s_1(t-i\tau) & b_{2i-1}b_{2i} = 11 \\ s_2(t-i\tau) & b_{2i-1}b_{2i} = 10 \\ s_3(t-i\tau) & b_{2i-1}b_{2i} = 01 \\ s_4(t-i\tau) & b_{2i-1}b_{2i} = 00 \end{cases} \)

Baseband Channel

\( s_1(t) = A, \ 0 \leq t \leq \tau \)
\( s_2(t) = \frac{A}{3}, \ 0 \leq t \leq \tau \)
\( s_3(t) = -\frac{A}{3}, \ 0 \leq t \leq \tau \)
\( s_4(t) = -A, \ 0 \leq t \leq \tau \)

Threshold Comparison

\( \hat{b}_{2i-1}\hat{b}_{2i} = 11 \) if \( z(\tau) > \lambda_1 \)
\( \hat{b}_{2i-1}\hat{b}_{2i} = 10 \) if \( \lambda_2 < z(\tau) < \lambda_1 \)
\( \hat{b}_{2i-1}\hat{b}_{2i} = 01 \) if \( \lambda_3 < z(\tau) < \lambda_2 \)
\( \hat{b}_{2i-1}\hat{b}_{2i} = 00 \) if \( z(\tau) < \lambda_3 \)

Sample at \( \tau \)

Matched Filter

Received signal \( y(t) = s(t) + n(t) \)

\[ \text{Symbol Error:} \quad \{\hat{b}_{2i-1}\hat{b}_{2i} \neq b_{2i-1}b_{2i}\} \]
\[ = \{\hat{b}_{2i-1}\hat{b}_{2i} \neq 11 \text{ but } b_{2i-1}b_{2i} = 11\} \cup \{\hat{b}_{2i-1}\hat{b}_{2i} \neq 10 \text{ but } b_{2i-1}b_{2i} = 10\} \]
\[ \cup \{\hat{b}_{2i-1}\hat{b}_{2i} \neq 01 \text{ but } b_{2i-1}b_{2i} = 01\} \cup \{\hat{b}_{2i-1}\hat{b}_{2i} \neq 00 \text{ but } b_{2i-1}b_{2i} = 00\} \]
• Probability of Symbol Error (or Symbol Error Rate, SER):

\[ P_s = \Pr\{\hat{b}_{2i-1} \neq b_{2i}, b_{2i-1}b_{2i} = 11\} + \Pr\{\hat{b}_{2i-1} \neq 10, b_{2i-1}b_{2i} = 10\} \]
\[ + \Pr\{\hat{b}_{2i-1} \neq 01, b_{2i-1}b_{2i} = 01\} + \Pr\{\hat{b}_{2i-1} \neq 00, b_{2i-1}b_{2i} = 00\} \]

• SER vs. BER:

\[ P_s = \Pr\{b_{2i-1} \text{ is received in error or } b_{2i} \text{ is received in error}\} \]
\[ = 1 - \Pr\{b_{2i-1} \text{ is received correctly and } b_{2i} \text{ is received correctly}\} \]
\[ = 1 - \Pr\{b_{2i-1} \text{ is received correctly}\} \cdot \Pr\{b_{2i} \text{ is received correctly}\} \]
\[ = 1 - (1 - P_b)^2 = 2P_b - P_b^2 \approx 2P_b \text{ for small } P_b \]

What is the minimum SER of 4-ary PAM and how to achieve it?
SER of 4-ary PAM Receiver

Received signal $y(t)$

$z(\tau) = \begin{cases} 
\int_0^\tau s_1(t)(s_1(t) - s_2(t))dt + n_o(\tau) & \text{if } b_{2i-1}b_{2i}=11 \\
\int_0^\tau s_2(t)(s_1(t) - s_2(t))dt + n_o(\tau) & \text{if } b_{2i-1}b_{2i}=10 \\
\int_0^\tau s_3(t)(s_1(t) - s_2(t))dt + n_o(\tau) & \text{if } b_{2i-1}b_{2i}=01 \\
\int_0^\tau s_4(t)(s_1(t) - s_2(t))dt + n_o(\tau) & \text{if } b_{2i-1}b_{2i}=00 
\end{cases}$

$P_s = \Pr\{z(\tau) < \lambda_1, b_{2i-1}b_{2i} = 11\}$

Threshold Comparison

• SER:

$\hat{b}_{2i-1}\hat{b}_{2i} = 11$ if $z(\tau) > \lambda_1$

$\hat{b}_{2i-1}\hat{b}_{2i} = 10$ if $\lambda_2 < z(\tau) < \lambda_1$

$\hat{b}_{2i-1}\hat{b}_{2i} = 01$ if $\lambda_3 < z(\tau) < \lambda_2$

$\hat{b}_{2i-1}\hat{b}_{2i} = 00$ if $z(\tau) < \lambda_3$
Optimal Thresholds

\[ f_z(z(\tau) \mid b_{2i-1}b_{2i} = 00) \quad f_z(z(\tau) \mid b_{2i-1}b_{2i} = 01) \quad f_z(z(\tau) \mid b_{2i-1}b_{2i} = 10) \quad f_z(z(\tau) \mid b_{2i-1}b_{2i} = 11) \]

- SER of the optimal 4-ary PAM receiver:

\[ P_s^* = \Pr\{z(\tau) < \lambda_1^*, b_{2i-1}b_{2i} = 11\} + \Pr\{z(\tau) > \lambda_3^*, b_{2i-1}b_{2i} = 00\} \]
\[ + \Pr\{z(\tau) < \lambda_2^*, b_{2i-1}b_{2i} = 10\} + \Pr\{z(\tau) > \lambda_1^*, b_{2i-1}b_{2i} = 10\} \]
\[ + \Pr\{z(\tau) < \lambda_3^*, b_{2i-1}b_{2i} = 01\} + \Pr\{z(\tau) > \lambda_2^*, b_{2i-1}b_{2i} = 01\} \]
\[ = \Pr\{z(\tau) \mid b_{2i-1}b_{2i} = 00 > \frac{1}{2} (a_3 + a_4)\} \cdot \Pr\{b_{2i-1}b_{2i} = 00\} \]
\[ + \left( \Pr\{z(\tau) \mid b_{2i-1}b_{2i} = 01 < \frac{1}{2} (a_3 + a_4)\} + \Pr\{z(\tau) \mid b_{2i-1}b_{2i} = 01 > \frac{1}{2} (a_3 + a_4)\} \right) \cdot \Pr\{b_{2i-1}b_{2i} = 01\} \]
\[ + \left( \Pr\{z(\tau) \mid b_{2i-1}b_{2i} = 10 < \frac{1}{2} (a_2 + a_3)\} + \Pr\{z(\tau) \mid b_{2i-1}b_{2i} = 10 > \frac{1}{2} (a_2 + a_3)\} \right) \cdot \Pr\{b_{2i-1}b_{2i} = 10\} \]
\[ + \Pr\{z(\tau) \mid b_{2i-1}b_{2i} = 11 < \frac{1}{2} (a_1 + a_2)\} \cdot \Pr\{b_{2i-1}b_{2i} = 11\} \]
SER of the optimal 4-ary PAM Receiver

\[ f_Z(z(\tau) \mid b_{2i-1}b_{2i} = 00) \quad f_Z(z(\tau) \mid b_{2i-1}b_{2i} = 01) \quad f_Z(z(\tau) \mid b_{2i-1}b_{2i} = 10) \quad f_Z(z(\tau) \mid b_{2i-1}b_{2i} = 11) \]

- SER of the optimal 4-ary PAM receiver:

\[
P_s^* = \frac{1}{4} \left( \Pr \left\{ z(\tau) \mid b_{2i-1}b_{2i} = 00 > \frac{1}{2} (a_3 + a_4) \right\} + \Pr \left\{ z(\tau) \mid b_{2i-1}b_{2i} = 01 < \frac{1}{2} (a_3 + a_4) \right\} + \Pr \left\{ z(\tau) \mid b_{2i-1}b_{2i} = 10 > \frac{1}{2} (a_2 + a_3) \right\} \right.

+ \Pr \left\{ z(\tau) \mid b_{2i-1}b_{2i} = 10 < \frac{1}{2} (a_2 + a_3) \right\} + \Pr \left\{ z(\tau) \mid b_{2i-1}b_{2i} = 11 > \frac{1}{2} (a_1 + a_2) \right\} + \Pr \left\{ z(\tau) \mid b_{2i-1}b_{2i} = 11 < \frac{1}{2} (a_1 + a_2) \right\} \right)

= \frac{1}{4} \left( 2Q \left( \frac{a_3 - a_4}{2\sigma_0} \right) + 2Q \left( \frac{a_2 - a_3}{2\sigma_0} \right) + 2Q \left( \frac{a_1 - a_2}{2\sigma_0} \right) \right) = \frac{6}{4} Q \left( \frac{a_1 - a_2}{2\sigma_0} \right)

\[ a_i = \int_0^\tau s_i(t) (s_1(t) - s_2(t)) \, dt \]

\[ \sigma_0^2 = \frac{N_0}{2} \int_0^\tau (s_1(t) - s_2(t))^2 \, dt \]

\[ P_s^* = \frac{3}{2} Q \left( \sqrt{\frac{E_d}{2N_0}} \right) \]
SER and BER of Optimal 4-ary PAM Receiver

\[ P_{s,APAM}^* = \frac{3}{2} Q\left( \sqrt{\frac{E_{d,APAM}}{2N_0}} \right) = \frac{3}{2} Q\left( \sqrt{\frac{0.4E_{s,APAM}}{N_0}} \right) = \frac{3}{2} Q\left( \sqrt{\frac{0.8E_{b,APAM}}{N_0}} \right) \]

\[ P_{b,APAM}^* \approx \frac{1}{2} P_{s,APAM}^* = \frac{3}{4} Q\left( \sqrt{\frac{E_{d,APAM}}{2N_0}} \right) = \frac{3}{4} Q\left( \sqrt{\frac{0.8E_{b,APAM}}{N_0}} \right) \]

Energy difference \( E_d \):

\[ E_{d,APAM} = \int_0^\tau (s_1(t) - s_2(t))^2 dt = \int_0^\tau (s_2(t) - s_3(t))^2 dt = \int_0^\tau (s_3(t) - s_4(t))^2 dt = \frac{4}{9} A^2 \tau = 0.8E_S \]

Energy per symbol \( E_s \):

\[ E_{s,APAM} = \frac{1}{4} \int_0^\tau s_1^2(t)dt + \frac{1}{4} \int_0^\tau s_2^2(t)dt + \frac{1}{4} \int_0^\tau s_3^2(t)dt + \frac{1}{4} \int_0^\tau s_4^2(t)dt = \frac{5}{9} A^2 \tau \]

Energy per bit \( E_b \):

\[ E_{b,APAM} = \frac{1}{2} E_{s,APAM} \]
### Performance Comparison of Binary PAM and 4-ary PAM

<table>
<thead>
<tr>
<th></th>
<th>BER (optimal receiver)</th>
<th>Bandwidth Efficiency (90% in-band power)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Binary PAM</strong></td>
<td>$Q\left(\sqrt{\frac{2E_{b,BPAM}}{N_0}}\right)$</td>
<td>1</td>
</tr>
<tr>
<td><strong>4-ary PAM</strong></td>
<td>$\frac{3}{4}Q\left(\sqrt{\frac{0.8E_{b,4PAM}}{N_0}}\right)$</td>
<td>2</td>
</tr>
</tbody>
</table>

- 4-ary PAM is more bandwidth-efficient, but more susceptible to noise.
BER Comparison of Binary PAM and 4-ary PAM

- Suppose $E_{b, BPAM} = E_{b, 4PAM} = E_b$
Constellation Representation of M-ary PAM

Binary PAM:

\[ -A \quad X \quad A \]

4-ary PAM:

\[ -A \quad -A/3 \quad A/3 \quad A \]

M-ary PAM:

\[ -A \quad \ldots \quad A \]

\[ d = \frac{A - (-A)}{M - 1} \]

\[ s_i(t) = A - (i - 1) \cdot d \]

\[ i = 1, \ldots, M, \quad 0 \leq t \leq \tau \]

- Energy per symbol:

\[ E_s = \frac{1}{M} \sum_{i=1}^{M} \int_{0}^{\tau} s_i^2(t) \, dt = \frac{\tau}{3(M-1)} A^2 \tau \]

- Energy difference:

\[ E_d = \int_{0}^{\tau} (s_1(t) - s_2(t))^2 \, dt = \tau \cdot d^2 = \frac{4A^2 \tau}{(M-1)^2} = \frac{12E_s}{(M+1)(M-1)} \]

Given \( E_s, E_d \) decreases as \( M \) increases!
SER of M-ary PAM

• SER of M-ary PAM:

\[
P^*_s = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{E_d}{2N_0}}\right)
\]

\[
E_d = \frac{12}{M^2 - 1} E_s
\]

\[
E_s = E_b \log_2 M
\]

\[
P^*_s = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6E_b \log_2 M}{N_0 (M^2 - 1)}}\right)
\]

\[
E_d = \frac{12 \log_2 M}{M^2 - 1} E_b
\]

• Given \(E_b\),

\checkmark \ E_d \text{ decreases as } M \text{ increases;}

\checkmark \ P^*_s \text{ increases as } M \text{ increases.}

A larger \(M\) leads to a smaller energy difference ---- a higher SER
(As two symbols become closer in amplitude, distinguishing them becomes harder.)
Performance of M-ary PAM

- Fidelity performance of M-ary PAM:
  
  \[ P_{s}^{*} = \frac{2(M - 1)}{M} Q \left( \sqrt{\frac{6E_{b} \log_{2} M}{N_{0}(M^2 - 1)}} \right) \]

- Bandwidth Efficiency of M-ary PAM:

  \[ \gamma_{MPAM} = k = \log_{2} M \]

  (with 90% in-band power)

With an increase of \( M \), M-ary PAM becomes:
1) more bandwidth-efficient;
2) more susceptible to noise.
M-ary PSK
Coherent Demodulator of QPSK

- QPSK

\[ s_{\text{QPSK}}(t) = d_I \frac{A}{\sqrt{2}} \cos(2\pi f_c t) + d_Q \frac{A}{\sqrt{2}} \sin(2\pi f_c t) \]

\[ d_I = \begin{cases} 
1 & \text{if } b_{2i-1} = 1 \\
-1 & \text{if } b_{2i-1} = 0 
\end{cases} \]

\[ d_Q = \begin{cases} 
1 & \text{if } b_{2i} = 1 \\
-1 & \text{if } b_{2i} = 0 
\end{cases} \]

- Coherent Demodulator of QPSK

A combination of two orthogonal BPSK signals

\[ y(t) = s(t) + n(t) \]

\[ y(t) = s(t) + n(t) \]

Threshold Comparison

\[ \hat{b}_{2i-1} = \begin{cases} 
1 & \text{if } z_1(\tau) > \lambda_1^* = 0 \\
0 & \text{otherwise} 
\end{cases} \]

\[ \hat{b}_{2i} = \begin{cases} 
1 & \text{if } z_2(\tau) > \lambda_2^* = 0 \\
0 & \text{otherwise} 
\end{cases} \]
BER of Coherent QPSK

- BER of Coherent QPSK:

\[ P_{b,QPSK}^* = Q\left(\sqrt{\frac{2E_{b,QPSK}}{N_0}}\right) \]

QPSK has the same BER performance as BPSK if \( E_{b,QPSK} = E_{b,BPSK} \), but is more bandwidth-efficient!

Energy per bit: \( E_{b,QPSK} = \frac{1}{2} E_{s,QPSK} = \frac{A^2\tau}{4} = \frac{A^2}{2R_{b,QPSK}} \)

Energy per symbol: \( E_{s,QPSK} = \frac{A^2\tau}{2} = \frac{A^2}{2R_{s,QPSK}} = \frac{A^2}{R_{b,QPSK}} \)
### Performance Comparison of BPSK and QPSK

<table>
<thead>
<tr>
<th></th>
<th>BER (coherent demodulation)</th>
<th>Bandwidth Efficiency (90% in-band power)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>$Q\left(\sqrt{\frac{2E_{b,BPSK}}{N_0}}\right)$</td>
<td>0.5</td>
</tr>
<tr>
<td>QPSK</td>
<td>$Q\left(\sqrt{\frac{2E_{b,QPSK}}{N_0}}\right)$</td>
<td>1</td>
</tr>
</tbody>
</table>

- What if the two signals, QPSK and BPSK, are transmitted with the same amplitude $A$ and the same bit rate? **Equally accurate!** (BPSK requires a larger bandwidth)
- What if the two signals, QPSK and BPSK, are transmitted with the same amplitude $A$ and over the same channel bandwidth? **BPSK is more accurate!** (but lower bit rate)
M-ary PSK

- M-ary PAM: transmitting pulses with $M$ possible different amplitudes, and allowing each pulse to represent $\log_2 M$ bits.

- M-ary PSK: transmitting pulses with $M$ possible different carrier phases, and allowing each pulse to represent $\log_2 M$ bits.

$$s_i(t) = A \cos(2\pi f_c t + \theta_i + i \cdot \theta)$$

$i = 0, ..., M-1, 0 \leq t \leq \tau.
\theta = \frac{2\pi}{M}$

$\Rightarrow$

$$s_i(t) = \frac{A}{\sqrt{2}} e^{j(\theta_i + i \cdot \theta)}$$

$i = 0, ..., M-1, 0 \leq t \leq \tau.$
SER of M-ary PSK

- What is the minimum phase difference between symbols?
  \[ d = \sqrt{2} A \sin \frac{\pi}{M} \]
  \[ 2\pi / M \]

- What is the energy difference between two adjacent symbols?
  \[ E_d = \tau \cdot d^2 = 2 A^2 \tau \sin^2 \frac{\pi}{M} = 4E_s \sin^2 \frac{\pi}{M} \]

- What is the SER with optimal receiver?
  \[ P_s^* \approx 2Q \left( \sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{M} \right) \text{ with a large } M \]
• A larger $M$ leads to a smaller energy difference ---- a higher SER
(As two symbols become closer in phase, distinguishing them becomes harder.)
## Summary II: M-ary Modulation and Demodulation

<table>
<thead>
<tr>
<th>Modulation Type</th>
<th>Bandwidth Efficiency</th>
<th>BER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binary PAM</td>
<td>1 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{2E_{b, BPA M}}{N_0}}\right)$</td>
</tr>
<tr>
<td>4-ary PAM</td>
<td>2 (90% in-band power)</td>
<td>$\frac{3}{4}Q\left(\sqrt{\frac{0.8E_{b, APA M}}{N_0}}\right)$</td>
</tr>
<tr>
<td>M-ary PAM ($M&gt;4$)</td>
<td>$\log_2 M$ (90% in-band power)</td>
<td>$\frac{2}{\log_2 M}Q\left(\sqrt{\frac{6\log_2 M}{M^2 - 1} \cdot \frac{E_{b, MPA M}}{N_0}}\right)$</td>
</tr>
<tr>
<td>Binary PSK</td>
<td>0.5 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{2E_{b, BPS K}}{N_0}}\right)$</td>
</tr>
<tr>
<td>QPSK</td>
<td>1 (90% in-band power)</td>
<td>$Q\left(\sqrt{\frac{2E_{b, QPS K}}{N_0}}\right)$</td>
</tr>
<tr>
<td>M-ary PSK ($M&gt;4$)</td>
<td>$\frac{1}{2} \log_2 M$ (90% in-band power)</td>
<td>$\frac{2}{\log_2 M}Q\left(\sqrt{\frac{2\sin^2 \frac{\pi}{M} \log_2 M \cdot \frac{E_{b, MPS K}}{N_0}}}{N_0}}\right)$</td>
</tr>
</tbody>
</table>
Performance Comparison of Digital Modulation Schemes

\[ \gamma = \frac{R_b}{B_h} \]

The highest achievable bandwidth efficiency \( \gamma^* \):

\[ \frac{2^{\gamma^*} - 1}{\gamma^*} = \frac{E_b}{N_0} \]

Required \( E_b/N_0 \) when \( P_b^* = 10^{-5} \):

- BPSK: 9.6dB
- QPSK: 9.6dB
- 8PSK: 12.9dB
- 16PSK: 17.4dB
Digital Communication Systems

\[ \{b_i\} \rightarrow \text{Digital Modulation} \rightarrow \text{Transmitted Digital waveform} \rightarrow \text{Channel} \rightarrow \text{Corrupted Digital waveform} \rightarrow \text{Digital Demodulation} \rightarrow \{\hat{b}_i\} \]

- **Bandwidth Efficiency**
  \[ \gamma \triangleq \frac{\text{Information Bit Rate } R_b}{\text{Required Channel Bandwidth } B_h} \]

- **BER (Fidelity Performance)**
  \[ \text{Binary: } P_b = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) \]

- What is the highest bandwidth efficiency for given \( E_b/N_0 \)?
  Information theory -- AWGN channel capacity

- How to achieve the highest bandwidth efficiency?
  Channel coding theory

- What if the channel is not an LTI system? Wireless communication theory