Optimized Mapping Rules for Coded OFDM Systems with Peak-Power Limitation and Iterative Soft Compensation

Jun Tong, Student Member, IEEE, and Li Ping, Senior Member, IEEE

Abstract

This paper aims to study the clipping method used in orthogonal frequency-division multiplexing (OFDM) systems to reduce peak-to-average power ratio (PAPR). An iterative compensation method is applied to alleviate the clipping-induced distortion. The impact of signal mapping rules on the residual clipping effect is investigated. It is shown that the superposition coded modulation (SCM) provides a solution for minimizing the residual clipping noise power. An enhanced detection strategy is proposed to improve performance and several implementation issues are discussed. System performance is analyzed by the extrinsic information transfer (EXIT) charts technique. Numerical results show that SCM can significantly outperform other conventional coded modulation methods when the iterative clipping noise compensation is involved.

Index Terms

Clipping, extrinsic information transfer (EXIT) chart, mapping rule, orthogonal frequency-division multiplexing (OFDM), peak-to-average power ratio (PAPR), superposition coded modulation (SCM).

I. INTRODUCTION

Orthogonal frequency-division multiplexing (OFDM) is a multi-carrier transmission technique for broadband channels. It has attracted much attention due to its distinct features such as its high spectral efficiency and low receiver complexity. However, the OFDM signal exhibits a high peak-to-average power

This work was fully supported by a grant from the Research Grant Council of the Hong Kong Special Administrative Region, China, under project CityU 117305.

The authors are with the Department of Electronic Engineering, City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Kowloon, Hong Kong SAR, China. Telephone: (852) 2194 2055, (852) 2788 9574. E-mail: jun.tong@student.cityu.edu.hk, eeliping@cityu.edu.hk.
ratio (PAPR). This makes the OFDM systems more prone to the non-linear effect of transmitter devices than the conventional single-carrier schemes.

Various PAPR reduction techniques have been investigated (see [1], [2] and the references therein). They can be broadly classified into two categories. The first category includes but is not restricted to coding, partial transmit sequence, and selective mapping techniques that incur redundancy and spectral efficiency loss. The second category employs pre-distortion methods that do not introduce redundancy. In particular, deliberate clipping [3]-[15] is a straightforward and efficient approach. Clipping is generally more effective than other alternatives for PAPR suppression.

The problem of clipping is bit error rate (BER) degradation [3]-[15] due to clipping distortion. Interestingly, such performance loss should not be serious in theory. Fig. 1 shows that for Gaussian signaling in an additive white Gaussian noise (AWGN) channel, the mutual information [15] is only marginally affected by clipping with a quite deep clipping ratio of 2 dB with the resultant PAPR $\approx 3$ dB. (For the definition of clipping ratio, see (2) below.) Thus, the problem boils down to how to realize the promise of information theoretical analysis. Compensation methods have been suggested to tackle this problem. Among them, the iterative compensation [11]-[15] appears a promising direction following the success of turbo-type iterative processing in recent years.

However, as reported in [12], the effectiveness of iterative compensation is far from satisfactory. High-rate systems based on high-order modulations are most sensitive to clipping noise. For some popular high-order modulation methods [16], [17], [19], iterative compensation can provide only limited improvement. Some preliminary explanations for this disappointing observation are provided in [12] using extrinsic information transfer (EXIT) charts. So far and to the best of our knowledge, there has been no rigorous
analysis on this observation and no satisfactory solution to the problem.

This paper presents a comprehensive study on the iterative compensation technique in clipped OFDM systems. The impact of the mapping rules involved in coded modulation is analyzed. The result shows that superposition coded modulation (SCM) [20]-[22] can be employed to minimize the residual clipping noise power during iterative compensation, which can lead to significantly enhanced performance. An overall iterative decoding strategy is developed, which incorporates soft clipping-noise compensation, Gaussian approximation-based demapping, and a posteriori probability (APP) decoding. The convergence behavior of this strategy is analyzed using the EXIT chart technique. Numerical results show that SCM can significantly outperform other conventional coded modulation methods when iterative clipping noise compensation is involved.

We believe that this paper offers a practical solution to the design of OFDM systems with low PAPR and high power and spectral efficiencies.

The rest of this paper is organized as follows: Section II establishes the system model. Section III derives the optimality of SCM mapping with respect to minimizing the residual clipping noise power. Section IV develops a linearized detection model to remove the correlation between the wanted signal and the clipping noise. Several implementation details are discussed in Sections V and VI. Performance analysis and numerical results are provided in Section VII, followed by the conclusions in Section VIII.

II. SYSTEM MODEL

We now outline the basic principles of the system under consideration using Fig. 2.

A. Transmitter Principles

Consider an OFDM system employing coded modulation based on binary component codes. The transmitter structure is illustrated in the upper part of Fig. 2. The information bits are first encoded by a binary encoder (ENC). The resultant coded bits are then randomly interleaved and packed into groups $b[n] = (b_1[n], b_2[n], \ldots, b_K[n])$ of $K$ bits. The signal mapper maps each group $b[n]$ to a symbol $X[n]$ to be carried by the $n$th sub-carrier. (The related mapping rule will be discussed in detail in the next section.) This transmitter scheme basically follows the principles of bit-interleaved coded modulation with iterative decoding (BICM-ID) [16], [17]. Our discussions below will reveal the advantages of a special family of mapping rules related to SCM [20]-[22].

Denote by $N$ the number of OFDM sub-carriers and define a column vector $\mathbf{X} = [X[0], X[1], \ldots, X[N-1]]^T$ of length $N$. The time-domain signal $\mathbf{x}$ is produced by applying inverse discrete Fourier transform (IDFT) to $\mathbf{X}$,

$$\mathbf{x} = \mathbf{F}^H \mathbf{X}$$  \hspace{1cm} (1)

where $\mathbf{F}$ is the unitary DFT matrix of size $N \times N$, and $(\cdot)^H$ denotes the conjugate transpose. We will call $\mathbf{X}$ a frequency-domain signal and $\mathbf{x}$ a time-domain signal. We will use capital letters for the signals related to $\mathbf{X}$ in the frequency domain and lower case letters for those related to $\mathbf{x}$ in the time domain.
To reduce the PAPR, each entry $x[m]$ of $x$, is clipped using the following clipping function,

$$
\text{clp}(x[m]) = \begin{cases} 
  x[m], & \text{if } |x[m]| \leq A \\
  A|x[m]|/|x[m]|, & \text{if } |x[m]| > A
\end{cases}
$$

where $A$ is a clipping threshold and $| \cdot |$ represents the absolute value. The clipping ratio in decibel is denoted by $\text{CR} = 10 \log_{10}(A^2/E[|x[m]|^2])$ where $E[\cdot]$ represents mathematical expectation. The clipped signal is then transmitted. With abuse of notation, we write the transmitted signal vector as $\text{clp}(x)$.

For simplicity, we ignore the cyclic prefix that is generally adopted in the OFDM to prevent inter-block-interference. We also assume that clipping is the only source of non-linear distortion, which implies that ideal power amplifiers are used.

### B. The ESE Principle

The receiver is shown in the lower part of Fig. 2. It consists of an elementary signal estimator (ESE) and a decoder (DEC), which are connected by an interleaver ($\Pi$) and a de-interleaver ($\Pi^{-1}$). Consider an equivalent discrete domain inter-symbol interference (ISI) channel with channel coefficient vector $h$.

In the time domain, the received signal $y$ is given in a convolutional form as

$$
y \equiv h * \text{clp}(x) + w
$$

where $*$ denotes convolution and $w$ is an AWGN vector. We assume that $h$ is known to the receiver.

Let us first ignore the SC (for “soft compensation”) module in Fig. 2. In the feed-forward part of the ESE, a DFT is applied to $y$ to produce a frequency-domain signal vector $Y = [Y[0], Y[1], \cdots, Y[N-1]]^T$ as

$$
Y = FY.
$$
Denote $H = \text{diag}(H[0], H[1], \cdots, H[N-1])$ where $\{H[0], H[1], \cdots, H[N-1]\}$ form the DFT of $h$. From (3), we have

$$Y = HF \text{clp}(x) + W$$

(5)

where $W = Fw$.

The optimal detection strategy follows the maximum a posteriori (MAP) principle [25] to recover $x$ from (5), which generally involves excessive complexity. Subsequently, we will investigate a sub-optimal, iterative soft compensation method.

Let us now consider the SC module in Fig. 2. We model $\text{clp}(x)$ as the sum of the wanted signal $x$ and the additive clipping noise $\text{clpn}(x)$:

$$\text{clp}(x) = x + \text{clpn}(x)$$

(6)

and so (5) becomes

$$Y = HX + HF \text{clpn}(x) + W.$$  

(7)

An unbiased estimate $Z$ can be generated from $Y$ by removing the mean of $\text{clpn}(x)$ as

$$Z = HX + HF(\text{clpn}(x) - E[\text{clpn}(x)]) + W.$$  

(8)

Based on (8), the demapper (see Fig. 2) generates the extrinsic log-likelihood ratio (LLR) for each coded bit as

$$\lambda_k[n] = \ln \left( \frac{\Pr(Z|b_k[n] = 0)}{\Pr(Z|b_k[n] = 1)} \right), k = 1, 2, \cdots, K, n = 0, 1, \cdots, N - 1.$$  

(9)

Here (9) is evaluated ignoring the coding constraints. (See Section V for detailed discussions.)

At the start, without any information about $\text{clpn}(x)$, we simply set its mean $E[\text{clpn}(x)] = 0$ in (8). This is a very rough starting point and $E[\text{clpn}(x)]$ will be refined iteratively as discussed below.

### C. The DEC Principle

The DEC takes the de-interleaved version of $\{\lambda_k[n]\}$ in (9) as inputs and performs standard APP decoding. The extrinsic LLRs produced by the DEC (after interleaving) are denoted by

$$\gamma_k[n] = \ln \left( \frac{\Pr(b_k[n] = 0)}{\Pr(b_k[n] = 1)} \right), k = 1, 2, \cdots, K, n = 0, 1, \cdots, N - 1.$$  

(10)

In contrast to (9), (10) is evaluated based on coding constraints.

### D. The Iterative Process

Based on the DEC feedbacks $\{\gamma_k[n]\}$, $E[\text{clpn}(x)]$ in (8) can be updated. Let $x$ be an entry in $x$. From (1), $x$ is a weighted sum of $N$ random variables. Following the central limit theorem, we model $x$ as a circularly symmetric, complex Gaussian random variable, i.e., $x \sim CN(E[x], \text{Var}[x])$. (This is our first use of the Gaussian approximation. Such Gaussian modeling has been commonly used for the transmitted
signals of an OFDM system.) The mean $E[x]$ and the variance $\text{Var}[x]$ can be estimated from $\{\gamma_k[n]\}$, which will be discussed in detail later. The probability density function of $x$ is given as

$$p(x) = \frac{1}{\pi \text{Var}[x]} \exp \left( -\frac{|x - E[x]|^2}{\text{Var}[x]} \right). \tag{11}$$

After applying the clipping function defined in (2) to $x$, the mean of the clipping noise $\text{clpn}(x) = \text{clp}(x) - x$ can be computed as

$$E[\text{clpn}(x)] = \int_{|x| \geq 0} \text{clpn}(x)p(x)dx. \tag{12}$$

We summarize the ESE function as follows.

(i) The mean $E[X[n]]$ and variance $\text{Var}[X[n]]$ of $X[n]$ (the $n$th entry of $X$) are estimated from $\{\gamma_k[n]\}$ as detailed in Section III.

(ii) Generate the means and variances of the entries of $x$ based on the relationship $x = F^H X$. More specifically, $E[x] = F^H E[X]$ where $E[x]$ and $E[X]$ are respectively the means of $x$ and $X$. The variance of $x[m], \forall m$, is computed as $\text{Var}[x[m]] = 1/N \sum_{n=0}^{N-1} \text{Var}[X[n]]$.

(iii) The mean of $\text{clpn}(x)$ is computed as (12).

(iv) Re-estimate $\{\lambda_k[n]\}$ using (9) based on updated $Z$ in (8).

Steps (i)-(iii) are performed respectively by the soft mapper, IDFT, and SC modules in the ESE. The ESE/DEC operations outlined above can be repeated for a number of iterations. If $E[\text{clpn}(x)] \rightarrow \text{clpn}(x)$, then the clipping effect can be mitigated.

E. The Challenges

Now return to the basic detection model in (8). In what follows, we will focus on a simple strategy by treating $F(\text{clpn}(x) - E[\text{clpn}(x)])$ as an independent AWGN vector. The detection is straightforward using this strategy. However, several issues should be carefully studied to make this strategy work, as listed below.

(i) The residual clipping noise represented by $F(\text{clpn}(x) - E[\text{clpn}(x)])$ should be minimized.

(ii) The correlation between $X$ and $F(\text{clpn}(x) - E[\text{clpn}(x)])$ may be problematic to the receiver.

(iii) How should the demapper outputs (9) be computed efficiently?

(iv) Clipping may cause out-of-band radiation.

(v) Efficient analysis tools are desirable.

We will discuss these issues progressively in Section III-VII.

Note that (12) can be tabulated as a function of $E[x]$ and $\text{Var}[x]$ for on-line computation [22]. In general, this requires a two-dimensional table which increases the storage cost.
III. OPTIMAL MAPPING FOR SOFT COMPENSATION

A. Residual Clipping Noise Power

First, consider issue (i) in Section II-E. Note that $E[\text{clpn}(x)]$ is computed from $\{\gamma_k[n]\}$. We will treat $\{\gamma_k[n]\}$ as independent realizations of a random variable $\gamma$. From (8), define the residual clipping noise power as

$$\sigma^2_D \equiv E_\gamma[||\text{clpn}(x) - E[\text{clpn}(x)]||^2]/N \tag{13}$$

where $|| \cdot ||$ denotes the Frobenius norm of a vector and $E_\gamma[\cdot]$ represents the expectation with respect to the distribution of $\gamma$. (We will return to this in Section III-E.) We can use $\sigma^2_D$ as a measure for the residual clipping effect in (8). Clearly, we want to minimize $\sigma^2_D$.

Similarly, define the residual noise power in estimating $x$ by $E[x]$ as

$$\sigma^2_x \equiv E_\gamma[||x - E[x]||^2]/N = E_\gamma[||X - E[X]||^2]/N. \tag{14}$$

The second equation in (14) holds since $x = FHX$, $E[x] = F^HE[X]$ and $F$ is unitary. From the discussions in Section II-D, $x$ can be approximated as a Gaussian-distributed random vector with mean $E[x]$ and with independent entries. Under this approximation and from the estimation theory [25],

$$\sigma^2_D = E_\gamma \left[ \int_{|x| \geq 0} |\text{clpn}(x) - E[\text{clpn}(x)]|^2 p(x) dx \right], \tag{15}$$

$$\sigma^2_x = E_\gamma [\text{Var}[x]], \tag{16}$$

where notations follow (11) and (12). From (15) and (16), $\sigma^2_D$ is an implicit function of $\sigma^2_x$. We have the following relationship.

*Proposition 1:* $\sigma^2_D$ is a monotonously decreasing function of $\sigma^2_x$; thus, minimizing $\sigma^2_D$ is equivalent to minimizing $\sigma^2_x$.

This proposition is well within expectation: a more accurate estimate of $x$ would lead to a more accurate estimate of $\text{clpn}(x)$. The proof of Proposition 1 is tedious, and so, the details are omitted here. On the other hand, the mapping rule used in generating $X$ can affect $\sigma^2_x$, as discussed below. Our goal is to find the optimal mapping rule that can minimize $\sigma^2_x$ (and so $\sigma^2_D$).

B. Mapping Rule

We now discuss the statistical measures related to a constellation. For notational simplicity, we ignore the symbol index $n$ temporarily.
Let \( b = (b_1, b_2, \cdots, b_K) \) be a binary \( K \)-tuple with \( b_k \in \{0,1\} \) and \( B \) the set of \( 2^K \) such \( K \)-tuples. Let \( s \) be the image of \( b \) (in the complex plane) and \( S \) be a constellation of \( 2^K \) such points\(^2\). We will call \( b \) the “label” of \( s \). Denote by \( B \rightarrow S \) a mapping from \( B \) to \( S \). Some examples of \( B \rightarrow S \) for BICM-ID can be found in [16]-[18]. Another example is the SCM [20]-[22] that generates \( s \) as a superposition of \( K \) antipodal signals,

\[
s = \sum_{k=1}^{K} \beta_k (-1)^{b_k},
\]

where the weighting factors \( \{\beta_k\} \) are complex constants. The operation in (17) is also referred to as “sigma mapping” in [21], [23]. In order to avoid the use of too many acronyms in this paper, we will simply call the operation in (17) “SCM mapping”. We can apply the SCM mapping to the transmitter in Fig. 2, which will result in a special case of BICM-ID [23]. We will derive the advantages of such a BICM-ID scheme in this paper for clipping effect compensation.

Let \( b = (b_1, b_2, \cdots, b_K) \) be random variables characterized by a set of a priori LLRs \( \{\gamma_k\} \):

\[
\gamma_k = \ln \left( \frac{\Pr(b_k = 0)}{\Pr(b_k = 1)} \right), \quad k = 1, 2, \cdots, K,
\]

\[
\Pr(b_k = 0) = 1 - \Pr(b_k = 1) = \frac{e^{\gamma_k}}{1 + e^{\gamma_k}}, \quad k = 1, 2, \cdots, K.
\]

Clearly, after mapping, \( s \) is also a random variable. The mean and variance of \( s \) can be found as follows:

Let \( s = [s_0, s_1, \cdots, s_{2^K-1}]^T \) be the vector of the signaling points in \( S \) and \( p = [p_0, p_1, \cdots, p_{2^K-1}]^T \) the vector of the probabilities associated with the entries in \( s \). We express the mapping \( B \rightarrow S \) in a functional form as \( s = \mu(b) \) and order the elements in \( s \) and \( p \) using the natural binary expression of their labels \( b = (b_1, b_2, \cdots, b_K) \) as below.

\[
s \equiv \begin{bmatrix}
s_0 \\
s_1 \\
\vdots \\
s_{2^K-1}
\end{bmatrix} = \begin{bmatrix}
\mu(b_1 = 0, b_2 = 0, \cdots, b_K = 0) \\
\mu(b_1 = 0, b_2 = 0, \cdots, b_K = 1) \\
\vdots \\
\mu(b_1 = 1, b_2 = 1, \cdots, b_K = 1)
\end{bmatrix},
\]

\[
p \equiv \begin{bmatrix}
p_0 \\
p_1 \\
\vdots \\
p_{2^K-1}
\end{bmatrix} = \begin{bmatrix}
\Pr(b_1 = 0) \Pr(b_2 = 0) \cdots \Pr(b_K = 0) \\
\Pr(b_1 = 0) \Pr(b_2 = 0) \cdots \Pr(b_K = 1) \\
\vdots \\
\Pr(b_1 = 1) \Pr(b_2 = 1) \cdots \Pr(b_K = 1)
\end{bmatrix},
\]

\(^2\)In this paper, we allow overlapping of the signaling points in \( S \). (This is a special case of the multiple labeling scheme in [24].) For example, for the SCM mapping defined in (17), let \( K = 2 \) and \( \beta_1 = \beta_2 = 1 \). Then \( b = (0,1) \rightarrow s = 0 \) and \( b' = (1,0) \rightarrow s' = 0 \). In this case, we still regard \( s \) and \( s' \) as two different points in \( S \), even if they are identical. In other words, each point in \( S \) is distinguished by its label \( b \) rather than by its position in the complex plane. This treatment is useful when we introduce symmetric conditions. See Footnote 3.
where we have assumed that \( \{b_1, b_2, \cdots, b_K\} \) are independent. Then, the mean and variance of \( s \) can be expressed in vector forms as,

\[
E[s] = \sum_{m=0}^{2^K-1} p_m s_m = p^H s, \\
\text{Var}[s] = E[|s - E[s]|^2] = \sum_{m=0}^{2^K-1} p_m |s_m - p^H s|^2 \\
= \sum_{m=0}^{2^K-1} p_m |s_m|^2 - s^H pp^H s.
\]

(22)

C. Local and Global Statistics

Note that \( E[s] \) and \( \text{Var}[s] \) in (22) are computed for fixed \( \{\gamma_k\} \). Similar to the treatment in Section III-A, assume that \( \{\gamma_k\} \) are independent realizations of a random variable \( \gamma \) and denote by \( E_{\gamma}[\cdot] \) the mathematical expectation over the distribution of \( \gamma \). In particular, we are interested in \( E_{\gamma}[\text{Var}[s]] \) (as shown below).

\[
E_{\gamma}[\text{Var}[s]] = E_{\gamma} \left[ \sum_{m=0}^{2^K-1} p_m |s_m|^2 - s^H pp^H s \right] \\
= \sum_{m=0}^{2^K-1} E_{\gamma}[p_m] |s_m|^2 - s^H E_{\gamma}[pp^H] s.
\]

(23)

We will call \( E[s] \) and \( \text{Var}[s] \) as local mean and variance, respectively, and call \( E_{\gamma}[\text{Var}[s]] \) as global variance.

D. Minimum Global Variance

**Assumption 1**: The mapping \( B \rightarrow S \) is unbiased and with unit average power:

\[
2^{-K} \sum_{m=0}^{2^K-1} s_m = 0, \quad (24a) \\
2^{-K} \sum_{m=0}^{2^K-1} |s_m|^2 = 1. \quad (24b)
\]

**Assumption 2**: The elements of \( \{\gamma_k\} \) are independent, identically distributed (i.i.d.). The global statistics of the \( a \text{ priori} \) probabilities are symmetric, which implies that

\[
E_{\gamma}[\text{Pr}(b_k = 0)] = E_{\gamma}[\text{Pr}(b_k = 1)] = 1/2, \quad k = 1, 2, \cdots, K, \quad (25)
\]

there is a constant \( \eta \) such that

\[
\eta = E_{\gamma}[\text{Pr}^2(b_k = 0)] = E_{\gamma}[\text{Pr}^2(b_k = 1)], \quad k = 1, 2, \cdots, K, \quad (26)
\]
and the elements in $S$ have equal occurrence probabilities\(^3\):

$$E_\gamma[p_m] = 2^{-K}, \quad m = 0, 1, \cdots, 2^K - 1. \quad (27)$$

Based on these two assumptions,

$$E_\gamma[\text{Var}[s]] = 1 - s^H E_\gamma[pp^H] s. \quad (28)$$

**Theorem 1:** Under Assumptions 1 and 2, the minimum global variance

$$\min_s E_\gamma[\text{Var}[s]] = 2 - 4\eta \quad (29)$$

where the minimization is over all possible selections of $s$ (i.e., all possible mapping rules $B \to S$).

**Proof:** See Appendix I-A. \(\square\)

**Theorem 2:** Under Assumptions 1 and 2, for arbitrary $K$ and arbitrary $\{\beta_k\}$, the SCM mapping defined by (17) achieves the minimum global variance.

**Proof:** Given the a priori probability of $b_k$, the local mean and variance of the variable $(-1)^{b_k}$ can be written as

$$E[(-1)^{b_k}] = (-1)^0 \Pr(b_k = 0) + (-1)^1 \Pr(b_k = 1) = 2 \Pr(b_k = 0) - 1, \quad (30a)$$

$$\text{Var}[(-1)^{b_k}] = 1 - E^2((-1)^{b_k}). \quad (30b)$$

From (17) and Assumptions 1 and 2, we have

$$E_\gamma[\text{Var}[s]] = \sum_{k=1}^{K} |\beta_k|^2 E_\gamma[1 - (2 \Pr(b_k = 0) - 1)^2] \quad (31a)$$

$$= 4 \sum_{k=1}^{K} |\beta_k|^2 (E_\gamma[\Pr(b_k = 0)] - E_\gamma[\Pr^2(b_k = 0)]) \quad (31b)$$

$$= 2 - 4\eta \quad (31c)$$

where (31c) follows the average power constraint $2^{-K} \sum_{m=0}^{2^K-1} |s_m|^2 = \sum_{k=1}^{K} |\beta_k|^2 = 1. \quad \square$

Some intuitive explanations of the above theorems are included in Appendix I-B. From Theorem 2, the values of $\{\beta_k\}$ have no impact on the global variance achieved by the SCM mappings. However, as shown in Appendix I-B, they may still affect system performance from the point of view of mutual information.

\(^3\)According to footnote 2, two points $s_m$ and $s_{m'}$, $m \neq m'$, may be identical but their occurrence probabilities are counted separately.
E. Residual Clipping Noise Power Again

Consider the time index $n$. Let $\{\gamma_k[n], k=1, 2, \cdots, K, n=0, 1, \cdots, N-1\}$ be $K \times N$ independent realizations of $\gamma$ and $\{X[n], n=0, 1, \cdots, N-1\}$ be $N$ independent realizations of $s$. The local statistics, $E[X[n]]$ and $\text{Var}[X[n]]$, are computed using (22) from $\{\gamma_k[n]\}$ for a fixed $n$. From (14) and (23), we have

$$\sigma^2_x = E_{\gamma}[\text{Var}[s]].$$

(32)

Thus, Theorems 1 and 2 indicate that the SCM mapping is an optimal solution to minimize $\sigma^2_x$ (and so $\sigma^2_D$ ) among all possible mapping rules. (See the discussion in Section III-A.)

It is important to note that the optimality here is with respect to the clipping noise compensation only. The conclusion can be different for a different objective, e.g., the BER performance optimization. To elaborate on this point, we return to Fig. 2. The SCM mapping is optimal for the link from point A to point B in the iterative receiver. However, other parts of the receiver are also affected by the choice of mapping rule for which the SCM mapping may not be optimal. For example, the SCM mapping may not be the optimal choice when the demapper is considered. (See also the discussions following Fig. 4.) Nevertheless, in cases where clipping is deep and/or transmission rate is high, the clipping effect may dominate the BER performance and the SCM does yield lower BER compared to other alternatives. We will verify this by numerical results in Section VII.

In the above discussion, we have assumed that $\{\gamma_k[n]\}$ are i.i.d.. This is approximately true for the BICM-ID schemes [16], [17] where the coded bits are interleaved by an overall interleaver. When the SCM mapping is considered, we can also adopt the multi-layer SCM schemes in [20]-[22], which usually employ unequal power allocations and multiple individual interleavers. In the latter case, $\{\gamma_k[n]\}$ are no longer i.i.d.. However, we will show in Section VII and Appendix II that such SCM schemes are advantageous for clipping effect mitigation.

IV. An Improved Detection Method

This section addresses issue (ii) in Section II-E, i.e., the correlation problem between the residual clipping noise $F(\text{clpn}(x) - E[\text{clpn}(x)])$ and the useful signal $X$ in (8). It can be verified that with the above soft compensation method, $F(\text{clpn}(x) - E[\text{clpn}(x)])$ and $X$ are correlated. This problem may potentially affect the efficiency of the detector. We adopt a de-correlation strategy to alleviate this problem.

Return to the basic system model (5): $Y = HF\text{clp}(x) + W$. This model is non-linear with respect to $x$. Following the common treatment, we can model it using a linear system

$$Y = \alpha HF x + HF\text{clpn}_\alpha(x) + W$$

(33)

where $\alpha$ is a constant scalar and

$$\text{clpn}_\alpha(x) = \text{clp}(x) - \alpha x.$$  

(34)

Then (33) can be written as

$$Y = \alpha HX + W'.$$

(35)
where \( W' = HF_{\text{clpn}_\alpha(x)} + W \). We find \( \alpha \) such that \( \alpha HX \) and \( W' \) are uncorrelated, i.e., \( E[\alpha HX W'^H] = E[\alpha HX]E[W'^H] \). Interestingly, in this case, \( E[||W'||^2] \) is also minimized. Note that (i) \( W \) is independent of \( \text{clpn}_\alpha(x) \) and \( x \), (ii) \( H \) is diagonal with i.i.d. diagonal entries, and (iii) \( F \) is unitary. It can be shown that these facts imply that minimizing \( E[||W'||^2] \) is equivalent to minimizing \( E[||\text{clp}(x) - \alpha x||^2] \). Following [4], [7], the solution to this problem is given by

\[
\alpha = \frac{E[x^H \text{clp}(x)]}{E[||x||^2]}.
\] (36)

After de-correlation, \( F(\text{clpn}_\alpha(x)) \) (and so \( W' \)) is uncorrelated with \( HX \). The value of \( \alpha \) involved in this de-correlation strategy can be evaluated numerically.

The above de-correlation strategy can be combined with the soft compensation process discussed in Section II. Similar to the discussion in Section II, we can apply soft compensation to (33) as

\[
Z = \alpha HX + HF(\text{clpn}_\alpha(x) - E[\text{clpn}_\alpha(x)]) + W.
\] (37)

In contrast to (8), the residual clipping noise \( F(\text{clpn}_\alpha(x) - E[\text{clpn}_\alpha(x)]) \) in (37) is uncorrelated with \( X \), and therefore, the performance of the detector based on (37) is generally better than that based on (8). For this reason, we will consider soft compensation based on (37) only in the remainder of this paper.

V. THE DEMAPPER FUNCTION

We are now ready to address issue (iii) listed in Section II-E.

A. Gaussian Approximation

Define a residual clipping noise vector \( D = [D[0], D[1], \ldots, D[N-1]]^T \) as

\[
D = F(\text{clpn}_\alpha(x) - E[\text{clpn}_\alpha(x)]).
\] (38)

The average power of \( \{D[n]\} \) is given by

\[
\sigma_D^2 = E[||D||^2]/N.
\] (39)

Note that \( \sigma_D^2 \) here is different from that in (13), which is caused by the different signal models (6) and (34). Since such difference is clear from context and for simplicity, we use the same notations for them. In general, \( \sigma_D^2 \) is a function of \( \sigma_x^2 \) that can be written as

\[
\sigma_D^2 = \phi(\sigma_x^2)
\] (40)

where \( \sigma_x^2 \) is defined in (14).

Some details are as follows: The function \( \phi(\cdot) \) can be calculated using the Monte-Carlo method offline and stored in a one-dimensional look-up table. In our simulation study, \( \sigma_x^2 \) is computed as \( \sigma_x^2 \approx 1/N \sum_{n=0}^{N-1} \text{Var}[X[n]] \) with \( \text{Var}[X[n]] \) generated using (22). Then \( \sigma_D^2 \) can be estimated online using the stored table for \( \phi(\cdot) \).

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The signal after soft compensation in (37) can be rewritten as

\[ Z = \alpha H X + HD + W. \]  

(41)

Since \( H \) is a diagonal matrix, we rewrite (41) in a symbol-by-symbol manner as

\[ Z[n] = \alpha H[n] X[n] + W'[n] \]  

(42)

where \( X[n] \) is a point in a \( 2^K \)-ary constellation \( S \) and \( W'[n] = H[n]D[n] + W[n] \). According to the discussions in Section IV, \( D[n] \) (and so \( W'[n] \)) is uncorrelated with \( X[n] \). Therefore, we can approximate \( W'[n] \) as an AWGN with mean zero and variance \( |H[n]|^2 \sigma_D^2 + \sigma^2 \), where \( \sigma_D^2 \) is given by (40) and \( \sigma^2 \) is the channel noise power. (This is our second use of the Gaussian approximation.) Then, (42) becomes a standard scalar fading channel model. The standard MAP demapping method [16], [26] can be applied to compute the demapper outputs \( \{ \lambda_k[n] \} \) in (9). The complexity of the MAP demapping method is \( O(2^K) \), which can be a concern when \( K \) is large.

B. SCM Demapping

When the SCM mapping is used, the signal model is similar to that of the interleave-division multiplexing-access (IDMA) systems. In this case, the Gaussian approximation-based demapping method studied for IDMA in [27], [28] can be applied to evaluate \( \{ \lambda_k[n] \} \). The related complexity is only \( O(K) \). The details of the Gaussian-approximation demapping method are included in Appendix III-A.

C. About Gaussian Approximations

We have applied the Gaussian approximation in two different places: First, in Section II-D, we applied it to the unclipped time-domain signal \( x \). Second, in Section V-A, we applied it to the residual clipping noise \( D \). As shown above, the Gaussian approximation can greatly simplify the system design and analysis.

In an OFDM system, clipping and detection are performed in two different domains separated by the DFT. The presence of the DFT makes any optimal solution too costly. On the other hand, it makes the Gaussian approximation more reasonable. For example, it is quite accurate to model the entries of \( D \) in (41) as Gaussian variables due to the summations in the matrix multiplication involving \( F \) (according to the central limit theorem). With Gaussian modeling, the residual clipping noise can be simply characterized by a single parameter, i.e., the average power \( \sigma_D^2 \) in (39). The SCM mapping is optimal in this setting since it minimizes \( \sigma_D^2 \). When the system becomes more complicated, it will become more difficult to find the optimal solution; it will also be more justifiable to apply the Gaussian approximation to obtain sub-optimal solutions.
VI. Iterative Clipping and Filtering

In practice, the oversampling technique is widely used for the OFDM. Consider an OFDM system with $N'$ sub-carriers. With oversampling, only a subset of $N$ sub-carriers is used to carry information. The rest $N' - N$ sub-carriers are not used. These sub-carriers are usually located in the high frequency end. Such arrangement eases the requirement on the analog filter after the digital-to-analog conversion since the unused sub-carriers can be used for the transition band of the analog filter. The oversampling factor is defined as $L = N'/N$.

Clipping an oversampled OFDM signal may cause out-of-band radiation [3] (i.e., non-zero component may be created in the zero-band), leading to issue (iv) as listed in Section II-E. We can adopt an iterative clipping and filtering (ICF) process [7] outlined below at the transmitter to handle this problem.

(i) Clip the oversampled time-domain signal $x$ using (2).
(ii) Remove the out-of-band radiation due to (i) by band-pass filtering and update $x$ by the filter output.

The above clipping and filtering process is repeated, for example, for $J$ iterations, to achieve satisfactory PAPR reduction. We use the notations $\text{clp}(x)$ and $\text{clpn}_\alpha(x)$ again to denote the transmitted signal and the clipping noise, respectively. The discussions in Sections II to V can be easily extended when the ICF process is involved.

We will assume below that ideal band-pass filtering is used and the clipping ratio is the same for all ICF iterations. The resultant PAPR can be evaluated using the method in [3], [4]. For example, using the ICF scheme ($L = 4, J = 2, N = 256$ and $\text{CR} = 1$ dB) as shown in Fig. 3 below, the achieved PAPR is less than 4.7 dB for 99.9% of the OFDM blocks.

VII. Numerical Results

In this section, we present numerical results to verify the above analysis. We take the OFDM systems based on the BICM-ID [16], [17] and the multi-layer SCM schemes [20]-[22], for examples. For the SCM mappings below, we assume that $K$ is an even number and $\beta_{k-1} = i\beta_k$ with $\beta_{k-1}$ being a real number for $k = 2, 4, \ldots, K/2$. $\{\beta_k\}$ are optimized by an exhaustive search for small $K$ and by the linear programming technique in [27] for large $K$, assuming that there is no clipping noise. The MAP and the Gaussian-approximation demapping methods mentioned in Sections V-A and V-B are applied to the BICM-ID and the SCM schemes, respectively.

A. EXIT Charts Analysis

We first address issue (v) listed in Section II-E and analyze the overall system performance. We assume infinite frame lengths and employ the EXIT chart technique [29] to investigate the asymptotic convergence behavior of the iterative decoding when different mapping rules are applied.

Define the average mutual information between the coded bits $\{b_k[n]\}$ and the extrinsic LLRs as $I_\lambda \equiv I(b_k[n], \lambda_k[n])$ and $I_\gamma \equiv I(b_k[n], \gamma_k[n])$, where $I(\cdot)$ represents the mutual information function.
The EXIT functions $I_\lambda = T_{\text{ESE}}(I_\gamma, E_b/N_0)$ and $I_\gamma = T_{\text{DEC}}(I_\lambda)$ are then used to characterize the ESE and DEC, respectively, where $E_b/N_0$ denotes the ratio of energy per bit to the noise power spectral density. They are generated by the Monte-Carlo simulation, similar to [29]. In general, a larger $I_\lambda$ (or $I_\gamma$) implies that the extrinsic LLRs produced by the ESE (or the DEC) are more reliable. Note that $I_\lambda$ characterizes the joint effect of the soft compensation and demapping.

The $\sigma^2_x$ achieved by three 16-ary mapping rules, namely, the 16-QAM Mixed and MSP mappings\(^5\) discussed in [16] and an SCM mapping with $K = 4$, are compared in Fig. 3. From Section III, to minimize the clipping effect with soft compensation, $\sigma^2_x$ in (14) should be minimized.

Each curve in Fig. 3 can be seen as a transfer function from point A to point B in Fig. 2, for which the SCM mapping is clearly the optimal choice. However, if we consider the whole ESE link from point A to point C in Fig. 2, the situation is somehow different, as shown by the EXIT curves in Fig. 4. The SCM curve is the highest at low $I_\gamma$ but the other two methods take over at high $I_\gamma$. This observation can be explained as follows: Recall that soft compensation is used in all methods. Suppose that feedback information is perfect. The clipping effect can then be fully mitigated. In this case, the performance comparison follows the unclipped scenario that the Mixed and the MSP mappings are optimized. The

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\(^5\)The reasons for considering these mappings are as follows. First, the MSP mapping (and the related set-partitioning (SP) mapping) can yield good performance for BICM-ID with ideal transmission [16]. Second, it is shown in [12] that the Mixed mapping can potentially outperform other mappings for clipped transmission.
SCM mapping is only a special choice among all possible mapping rules for the BICM-ID, and so, it may not be the best one. However, when clipping is present, the SCM mapping yields a better solution, as demonstrated by the EXIT curves in Fig. 4 as well as by the simulation results in Fig. 6.

The observation that the SCM curve in Fig. 4 is relatively low at the high $I_\gamma$ range also suggests a possible error floor problem. We have also observed that this problem is only noticeable in AWGN channels. In fading channels, relatively higher power is required to ensure satisfactory performance and errors are mostly caused by deep fades. In the EXIT chart, a deep fade can be viewed as an event of the ESE curve moving downwards, which causes the closure of the decoding tunnel between the ESE and DEC curves. Then the performance is mostly determined by the width of the decoding tunnel at relatively low $I_\gamma$ values. From Fig. 4, we can see that the SCM mapping is advantageous in this aspect.

In the above, a single parameter $I_\gamma$ is used to measure the average performance of the DEC. The underlying assumption here is that the DEC feedbacks $\{\gamma_k[n]\}$ are i.i.d., which is approximately true for the BICM-ID schemes [16], [17]. However, as mentioned in Section III-E, for the multi-layer SCM schemes [20]-[22], this assumption is not true and the above EXIT chart analysis cannot obtain sufficient accuracy for performance prediction since it ignores the asymmetry of $\{\gamma_k[n]\}$. For the latter case, we can employ a more effective alternative, namely, the signal-to-noise ratio (SNR) evolution technique as outlined in Appendix III-B.

B. Simulation Results

We now present the simulation results to demonstrate the impact of the detection methods, mapping rules, and transmission rates on the BER performance.
Fig. 5. Comparison of different detection methods for a BICM-ID-OFDM scheme employing an SCM mapping \((K = 4, \beta_1 = i\beta_2 = 1, \beta_3 = i\beta_4 = 1.05)\). For the ICF, \(L = 4, J = 2, \) and \(N = 256\). The rate-1/2 convolutional code \((23, 35)\) is used as the component code; the system throughput is \(R = 2\) bits/symbol; and the frame length is 4096. AWGN channels are assumed. The total number of iterations is 10 for all the curves. (a). \(CR = 2\) dB. The resultant PAPR is less than 5.1 dB for 99.9% of the OFDM blocks. (b). \(CR = 1\) dB. The resultant PAPR is less than 4.7 dB for 99.9% of the OFDM blocks.

1) The Impact of Detection Methods: We first show the effectiveness of the proposed soft compensation method by comparing it with the following two alternative methods.

- Method I refers to the method in [4] that treats the clipping noise \(\text{clpn}_\alpha(x)\) in (33) as a zero-mean AWGN independent of the wanted signal.
- Method II refers to the soft-decision-aided clipping noise cancellation method in [14], which estimates \(\text{clpn}_\alpha(x)\) as \(\text{clpn}_\alpha(E[x])\) and then cancels it from the received signal.

A BICM-ID-OFDM system employing the SCM mapping in Fig. 3 is considered for the AWGN channels. The simulation results in Fig. 5 show that the proposed soft compensation method can significantly
Fig. 6. Comparison of BICM-ID-OFDM schemes with different mappings when the ICF and the soft compensation are used. For the ICF, \( L = 4, J = 2, N = 256 \) and \( \text{CR} = 1 \) dB. (The resultant PAPR is less than 4.7 dB for 99.9% of the OFDM blocks.) The frame length is 4096. \( R = 2 \) bits/symbol. AWGN channels are assumed. The total number of iterations is 10 for all the curves.

outperform the other two approaches, especially when the clipping ratio is small. It is because the proposed method estimates \( \text{clpn}_\alpha(x) \) as its mean \( \text{E}[\text{clpn}_\alpha(x)] \), which is optimal when the a priori distribution of \( x \) is available.

2) The Impact of Mapping Rules: We show next the impact of mapping rules. The BICM-ID-OFDM schemes employing the three mappings considered in Fig. 3 are simulated. For all the schemes, the same convolutional code \((23, 35)_8\) is used. The simulation results are presented in Fig. 6. We can make the following observations:

- Without clipping, the Mixed and the MSP mappings outperform the SCM mapping in the relatively high SNR range. The required \( E_b/N_0 \) to achieve BER = 10\(^{-5}\) are about 4.5, 4.6, and 5 dB for the Mixed, the MSP, and the SCM mappings, respectively.
- With clipping, the SCM mapping is more robust against the clipping effect than its alternatives, which confirms the observation on Figs. 3 and 4.

3) SCM with Unequal Power Allocations: Finally, we show examples of multi-layer SCM schemes with unequal power allocations. We assume a fully-interleaved Rayleigh fading channel where \( \{H[n]\} \) are independent, identically Rayleigh-distributed [16] with \( \text{E}[|H[n]|^2] = 1 \). The SCM and the BICM-ID schemes at rates \( R = 2 \) and 4 bits/symbol are compared.

For the SCM schemes, the component code is constructed by concatenating the rate-1/2 convolutional code \((23, 35)_8\) with a length 4 repetition code [27]; \( K = 8R \). An unequal power allocation among the layers is applied and the values of \( \{\beta_k\} \) are tabulated in Table I. For the BICM-ID schemes, the 16-QAM
TABLE I

PARAMETERS OF THE SCM SCHEMES IN Fig. 7

| $R$ | $K$ | $\{|\beta_k|\}$                  |
|-----|-----|----------------------------------|
| 2   | 16  | $1 \times 12, 1.44 \times 4$    |
| 4   | 32  | $1 \times 12, 1.58 \times 4, 2.10 \times 6, 2.49 \times 2, 2.73 \times 2, 3.58 \times 4, 3.93 \times 2$ |

Fig. 7. Comparisons of the SCM-OFDM and the BICM-ID-OFDM when the ICF and the soft compensation are used. $L = 4, J = 2, N = 256$, and $CR = 3$ dB. (The resultant PAPR is less than 5.6 dB for 99.9% of the OFDM blocks.) The frame length is 4096. Fully-interleaved Rayleigh fading channels are assumed. The total number of iterations is 10 for all the curves.

MSP mapping is used for $R = 2$ and the 64-QAM set-partitioning (SP) mapping [30] for $R = 4$. The above convolutional code is directly used for $R = 2$ and punctured to rate 2/3 for $R = 4$.

The MAP and the Gaussian-approximation demapping methods are applied to the BICM-ID and the SCM schemes, respectively. For a $2^K$-ary constellation, the complexities of the MAP and the Gaussian-approximation methods grow linearly with $2^K$ and $K$, respectively. Hence, for each rate, the schemes compared have comparable complexity, despite the fact that the SCM schemes use much larger constellations. The simulation results for $L = 4, J = 2, N = 256$ and $CR = 3$ dB are presented in Fig. 7.

Notice that at $R = 2$, the two schemes are not very sensitive to the clipping effect, with the BICM-ID performing better. This is because, at this relatively low rate, the channel noise is the dominating factor

---

$^6$The SP and MSP mappings are adopted because we observed that they can yield better performance for the BICM-ID over fading channels compared to other options available in the literature.
affecting performance when working SNR is low.

However, when the rate is high (e.g., $R = 4$), the SCM scheme can significantly outperform the BICM-ID scheme. In this case, the iterative compensation is not very effective for the BICM-ID scheme with SP mapping; hence, its performance is dramatically degraded by the clipping effect. Clearly, soft compensation still works well with the SCM mapping in this case.

We observed that the advantage of the SCM becomes more evident as the rate increases. This is because for a higher rate, the SNR value at the working point becomes higher. In other words, the channel noise power level becomes relatively lower. Thus, the clipping induced distortion, which increases with signal power, becomes a dominant factor.

VIII. CONCLUSION

Clipping can alleviate the high PAPR problem in the OFDM systems without incurring rate loss. However, the clipping-induced distortion can cause serious performance degradation when not treated properly. In this paper, we therefore propose a solution to this problem. Our main findings are summarized below.

- The performance loss caused by the clipping distortion can be largely recovered using an iterative soft compensation process together with the SCM mapping.
- An improved detection strategy, which is based on a de-correlated signal model and the Gaussian approximation, can be applied to further enhance the performance of the SCM scheme.
- Both the analytical and numerical results show that the SCM scheme can outperform other alternatives, especially when clipping is severe and/or transmission rate is high.

APPENDIX I

PROOF OF THEOREM 1

A. Proof of Theorem 1

Define $P = E_\gamma [pp^H]$. Then (28) can be rewritten as

$$E_\gamma [\text{Var}[s]] = 1 - s^H Ps.$$ 

(43)

Clearly, $E_\gamma [\text{Var}[s]]$ is determined by two factors: $s$ and $P$. We first derive a simple expression for $P$.

Denote by $\otimes$ the Kronecker product. Based on Assumption 2, we can decompose $p$ as

$$P = \begin{bmatrix} \Pr(b_1 = 0) & \otimes & \Pr(b_2 = 0) & \otimes & \cdots & \otimes & \Pr(b_K = 0) \\
\Pr(b_1 = 1) & & \Pr(b_2 = 1) & & & & \Pr(b_K = 1) \end{bmatrix}. \quad (44)$$

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When $K = 1$,

\[
P = E_\gamma \begin{bmatrix} \Pr(b_1 = 0) \\ \Pr(b_1 = 1) \end{bmatrix} \begin{bmatrix} \Pr(b_1 = 0) & \Pr(b_1 = 1) \\ E_\gamma[\Pr[b_1 = 0]\Pr[b_1 = 1]] & E_\gamma[\Pr[b_1 = 0]\Pr[b_1 = 1]] \end{bmatrix}
\]

From Assumption 2

\[
E_\gamma[\Pr(b_k = 0)\Pr(b_k = 1)] = E_\gamma[\Pr(b_k = 0)(1 - \Pr(b_k = 0))] = 1/2 - \eta, \quad \forall k.
\]

Therefore, for $K = 1$,

\[
P = \begin{bmatrix} \eta & 1/2 - \eta \\ 1/2 - \eta & \eta \end{bmatrix}.
\]

For $K = 2$, from the mixed product property of matrixes [31] [i.e., $(A \otimes B)(C \otimes D) = AC \otimes BD$], we have

\[
P = E_\gamma[pp^H] = E_\gamma \begin{bmatrix} \eta & 1/2 - \eta \\ 1/2 - \eta & \eta \end{bmatrix} \otimes E_\gamma \begin{bmatrix} \eta & 1/2 - \eta \\ 1/2 - \eta & \eta \end{bmatrix}
\]

where we used Assumption 2. The result can be easily generalized for $K > 2$

\[
P = E_\gamma[pp^H] = \begin{bmatrix} \eta & 1/2 - \eta \\ 1/2 - \eta & \eta \end{bmatrix} \otimes^K
\]

where we used the notation $\otimes^K$ for the power of Kronecker product: $A^{\otimes 1} = A$, $A^{\otimes 2} = A \otimes A$ and $A^{\otimes 3} = A \otimes A \otimes A$, etc. Incidentally,

\[
1/4 \leq \eta \leq 1/2,
\]

where $1/4 \leq \eta$ follows $E_\gamma[\Pr(b_k = 0)] = 1/2$ and $E_\gamma[(\Pr(b_k = 0) - E_\gamma[\Pr(b_k = 0)])^2] \geq 0$, and $\eta \leq 1/2$ follows $Pr(b_k = 0) \geq 0$, $Pr(b_k = 1) \geq 0$ and (45).

Proof: From (43), i.e., $E_\gamma[\Var[s]] = 1 - s^H P s$, minimizing $E_\gamma[\Var[s]]$ is equivalent to maximizing $s^H P s$. We may consider using $s = 2^{K/2} u_{max}$ for this purpose, where $u_{max}$ is the eigenvector for the maximum eigenvalue $\lambda_{max}$ of $P$. [$u_{max}$ is scaled by $2^{K/2}$ to satisfy the power constraint in (24b).]
However, as shown below, \( u_{\text{max}} \) does not meet the unbiased condition in (24a); hence, we have to resort to the eigenvector of the second largest eigenvalue.

For \( K = 1 \), \( P \) in (46) can be decomposed as

\[
P = \begin{bmatrix}
\eta & 1/2 - \eta \\
1/2 - \eta & \eta
\end{bmatrix} = \begin{bmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{bmatrix} \begin{bmatrix}
1/2 \\
(4\eta - 1)/2
\end{bmatrix} \begin{bmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{bmatrix}.
\]  

(48)

For \( K > 1 \), from (48) and the mixed product property of matrixes, the eigendecomposition of \( P \) is given by

\[
P = E_\gamma [pp^H] = U\Lambda U^T
\]  

(49)

where

\[
U = \begin{bmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{bmatrix} \otimes^K \text{ and } \Lambda = \begin{bmatrix}
1/2 \\
(4\eta - 1)/2
\end{bmatrix} \otimes^K.
\]  

(50)

Note that \( U \) is an orthogonal matrix with \( UU^T = I \), where \( I \) is the identity matrix. The eigenvalues of \( P \) are given by the diagonal entries of \( \Lambda \):

\[
\Lambda = 2^{-K} \text{diag} \left( 1, 4\eta - 1, \ldots, \underbrace{4\eta - 1, (4\eta - 1)^2, \ldots, (4\eta - 1)^2}_{\text{K times}}, \ldots, \underbrace{(4\eta - 1)^K}_{\text{K(\text{K-1})/2 times}} \right).
\]  

(51)

Since \( 1 > 4\eta - 1 > 0 \) from (47), the maximum eigenvalue \( \lambda_{\text{max}} = 1/2^K \) and the corresponding eigenvector \( u_{\text{max}} = 2^{-K/2}[1 1 \cdots 1]^T \).

Now, return to the original problem of minimizing \( E_\gamma [\text{Var}[s]] \). At first glance, from the matrix theory, we may select \( s = 2^{K/2}u_{\text{max}} \) for this purpose, but this violates the unbiased constraint in (24a). It can be verified that \( 1^T s \neq 0 \) when \( s = 2^{K/2}u_{\text{max}} \) and \( 1^T s = 0 \) when \( s \) is selected from (scaled versions of) other columns of \( U \), where \( 1 \) is the all-one vector. (Note: The entries of every column of \( U \) in (50), except \( u_{\text{max}} \), consist of equal numbers of \( -2^{-K/2} \) and \( 2^{-K/2} \).) Therefore, condition (24) indicates that \( s \) cannot be \( 2^{K/2}u_{\text{max}} \) and must be orthogonal to \( u_{\text{max}} \).

We therefore select \( s \) using the eigenvector for the second largest eigenvalue \( 2^{-K}(4\eta - 1) \). After proper scaling of \( s \) to satisfy (24b) and plugging it into \( E_\gamma [\text{Var}[s]] = 1 - s^H Ps \), we obtain the minimum global variance

\[
\min_s E_\gamma [\text{Var}[s]] = 1 - 2^K \cdot 2^{-K}(4\eta - 1) = 2 - 4\eta.
\]

\( \Box \)

**B. Discussions**

Here are some intuitive explanations on the above proof of Theorem 1. Let us return to (49): \( P = U\Lambda U^T \). Suppose that the \( 2^K \) diagonal entries in \( \Lambda \) are decreasingly ordered. Denote by \( V \) the submatrix
Fig. 8. Examples of mapping rules, where \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) denote the real and imaginary parts of a complex number, respectively.

of \( U \) formed by columns 2 to \( K + 1 \) of \( U \). We take \( K = 2 \) as an example. From (50), we have
\[
\Lambda = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \text{diag}\left(\frac{1}{4}, \frac{4\eta - 1}{4}, \frac{4\eta - 1}{4}, \frac{(4\eta - 1)^2}{4}\right),
\]
(52)
\[
U = [u_0, u_1, u_2, u_3] = \frac{1}{2} \begin{bmatrix}
+1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1
\end{bmatrix},
\]
(53)
\[
V = [u_1, u_2] = \frac{1}{2} \begin{bmatrix}
+1 & +1 \\
-1 & +1 \\
+1 & -1 \\
-1 & -1
\end{bmatrix}.
\]
(54)

In this case, \( u_{\max} = u_0 \) is the eigenvector that corresponds to the largest eigenvalue \( (\lambda_0 = 1/4) \) of \( \mathbf{P} \). Recall that \( s \) contains the collection of points in \( S \). Some special choices of \( s \) are discussed below.
(i) \( s = 2^{K/2}u_0 = 2^{K/2}u_{\max} = [1 1 1 1]^T \). This means that all the singling points in \( S \) overlap each other (i.e., \( s_0 = s_1 = s_2 = s_3 = 1 \)). In this case, the four input bit combinations \( \{ (b_1, b_2) = (00), (01), (10), (11) \} \) are all mapped to the signaling point \( s = 1 \), as shown in Fig. 8(a). The global variance is indeed minimized, as \( E_\gamma[\text{Var}[s]] = 1 - s^H P s = 0 \). Notice that there is only one distinct point, which is obvious. The best estimate is simply this point. However, this choice violates the unbiased constraint in (24a). Furthermore, it is obviously a bad choice when viewed from the information theory as it cannot deliver any information. If we consider a memoryless AWGN channel with inputs drawn from \( s \), then the mutual information between the channel input and output is zero when \( s = 2^{K/2}u_0 \). This agrees with the well-known observation that a scheme which achieves lower MMSE may lead to lower mutual information and vice versa [32].

(ii) \( s = 2^{K/2}u_1 = [1 -1 1 -1]^T \). In this case, \( E_\gamma[\text{Var}[s]] \) is increased (compared with (i)) to \( 2 - 4\eta \). Now, the four points in \( S \) have two distinct values, i.e., \( s_0 = s_2 = 1 \) and \( s_1 = s_3 = -1 \) (see Fig. 8(b)), and mutual information can achieve at most 1 bit/symbol. In general, this is not a good choice also for high-rate transmissions, since it is simply the BPSK signaling in nature.

(iii) \( s = \beta_1 u_1 + \beta_2 u_2 \) where \( \beta_1 \) and \( \beta_2 \) are non-zero constants. An example of \( \beta_1 = \sqrt{2}, \beta_2 = i\sqrt{2} \) is shown in Fig. 8(c)), which results in QPSK signaling. In this case, \( E_\gamma[\text{Var}[s]] \) is still \( 2 - 4\eta \) since both \( u_1 \) and \( u_2 \) correspond to the second largest eigenvalue \( \lambda_1 = \lambda_2 = (4\eta - 1)/4 \) of \( P \). This choice of \( s \) can achieve higher mutual information than the BPSK signaling in case (ii).

(iv) \( s = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 \) where \( \beta_3 \) is non-zero. Based on the discussions above, \( E_\gamma[\text{Var}[s]] \) is larger than \( 2 - 4\eta \) but the achievable mutual information may potentially be increased compared with (i)-(iii).

Clearly, choice (i) is an extreme case that cannot carry any information at all. For choices (ii) and (iii), \( s \) can be written in the form of

\[
s = V \beta^T
\]

where \( \beta = [\beta_1 \beta_2]^T \). From (54), each row of \( V \) can be seen as the binary expression (over \( \{+1, -1\} \)) of a signaling point label, and the \( 2^K \) rows of \( V \) list the labels of all the \( 2^K \) points in \( s \). Therefore, choices (ii) and (iii) are instances of the SCM mapping defined in (17). Both of them can achieve the minimum global variance under the unbiased constraint in (24a), but (iii) is better since it can achieve higher mutual information. Choice (iv) results in a larger \( E_\gamma[\text{Var}[s]] \) so it is a worse choice from the criterion of minimum global variance. However, we should be cautious when considering the overall system performance as (iv) may also potentially lead to higher mutual information. Finding the optimal solution is a very complicated issue, which is beyond the scope of this paper. However, we can justify the asymptotic optimality of the SCM mapping using the following arguments:

(a) For the SCM mappings, \( s \) can be selected as a linear combination of the columns of \( V \). (For general \( K \), \( V \) consists of \( K \) linearly independent column vectors.) The SCM mappings always achieve the minimum global variance condition under the unbiased constraint in (24a).
(b) When $K$ is large, from the central limit theorem, the transmit signal is close to the Gaussian signal that maximizes the mutual information for memoryless AWGN channels [33]. For practical systems, minimizing the BER is a more useful criterion than maximizing mutual information. For the SCM mapping, we can improve the BER performance by optimizing the weighting factors $\{\beta_k\}$ using the methods in [21], [27].

APPENDIX II

THE IMPACT OF INTERLEAVING AND POWER ALLOCATION

Given an SCM mapping with unequal $\{|\beta_k|\}$, we can interleave the coded bits $\{b_k[n]\}$ using either an overall interleaver or an individual interleaver for each given $k$. The former leads to the BICM-ID scheme [16] while the latter leads to the multi-layer SCM scheme. (Detailed discussions of the SCM scheme can be found in [21], [22]. We refer to $\{b_k[0], b_k[1], \cdots, b_k[N-1]\}$ as the $k$th layer of the SCM scheme.) We now show that individual interleaving is potentially a better choice.

The discussions below follow Section III-D. Without loss of generality, consider an SCM mapping with

$$|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_K|. \quad (56)$$

For each $k$, define the global variance of the bits $\{(-1)^{b_k[n]}\}$ as

$$\mathbb{E}_\gamma[\text{Var}((-1)^{b_k[n]})] = \mathbb{E}_\gamma[1 - \tanh^2(\gamma_k[n]/2)] = 2 - 4\eta_k, \quad k = 1, 2, \cdots, K, \quad (57)$$

where $\eta_k$ denotes the global mean of $\{\text{Pr}^2(b_k[n] = 0), n = 0, 1, \cdots, N-1\}$

$$\eta_k = \mathbb{E}_\gamma[\text{Pr}^2(b_k[n] = 0)] = \mathbb{E}_\gamma[\text{Pr}^2(b_k[n] = 1)], \quad k = 1, 2, \cdots, K. \quad (58)$$

In an individual interleaving scheme, from (56) and the discussions above, it is reasonable to assume that

$$1/2 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_K \geq 1/4 \quad (59)$$

because a larger $|\beta_k|$ value implies a higher power for layer-$k$, and so, more reliable feedbacks from the decoder for this layer. In this case, the global variance defined by (23) can be written as

$$\mathbb{E}^{\text{individual}}_\gamma[\text{Var}[s]] = \sum_{k=1}^K |\beta_k|^2(2 - 4\eta_k). \quad (60)$$

Now, consider an overall interleaving scheme. In this case, we can no longer distinguish the statistics of individual coded bits; hence,

$$\mathbb{E}^{\text{overall}}_\gamma[\text{Var}[s]] = \sum_{k=1}^K |\beta_k|^2(2 - 4\bar{\eta}). \quad (61)$$

where

$$\bar{\eta} = \frac{1}{K} \sum_{k=1}^K \eta_k. \quad (62)$$
Then from the Chebyshev’s inequality [34], it can be verified that
\[
E_\gamma^{\text{individual}}[\text{Var}[s]] \leq E_\gamma^{\text{overall}}[\text{Var}[s]].
\] (63)

This means that the individual interleaving is advantageous, at least when the global variance is the only consideration. Note that there can be other considerations. For example, when fixing the total number of information bits in a frame, the overall interleaver implies the use of a long code. On the other hand, multiple individual interleavers imply the use of several relatively short codes. The latter may result in lower coding gain when turbo-type codes are involved. A comprehensive discussion on the impact of the different coding and interleaver schemes is beyond the scope of this paper.

APPENDIX III

DEMAPPING AND ANALYSIS OF SCM

A. Gaussian-Approximation Demapping

We first outline a demapping method for the SCM based on the Gaussian approximation. From (17) and given \(\{\beta_k\}\), an SCM signal can be expressed as
\[
X[n] = \sum_{k=1}^{K} \beta_k X_k[n]
\]
where each \(X_k[n]\) is a BPSK modulated signal, i.e., \(X_k[n] = (-1)^{b_k[n]}\). We refer to \(\{X_k[n], n = 0, 1, \cdots, N-1\}\) as the \(k\)th layer. We focus on a particular \(X_k[n]\) and rewrite (42) as
\[
Z[n] = \alpha H[n] \beta_k X_k[n] + \zeta_k[n]
\] (64)
where
\[
\zeta_k[n] = \alpha H[n] \beta_k X_k[n] + H[n] D[n] + W[n]
\] (65)
is the distortion component. For simplicity, we assume that \(\alpha, \{\beta_k\}\) and \(H[n]\) in (64) are real numbers. (The treatments for complex cases are similar. See [28].) Then, \(\lambda_k[n]\) can be calculated from the real part of \(Z[n]\). The distortion component in \(\text{Re}(Z[n])\) is given by \(\text{Re}(\zeta_k[n])\) with\(^7\)
\[
\text{E}[\text{Re}(\zeta_k[n])] = \alpha H[n] \sum_{m \neq k} \beta_m \text{E}[X_m[n]]
\] (66a)
\[
\text{Var}[\text{Re}(\zeta_k[n])] = |\alpha H[n]|^2 \sum_{m \neq k} |\beta_m|^2 \text{Var}[X_m[n]] + \frac{|H[n]|^2 \sigma_D^2 + \sigma^2}{2}.
\] (66b)

\(^7\)In detecting an individual symbol \(X_k[n]\), a local statistics such as \(\text{Var}[X_k[n]]\) carries more information than the corresponding global statistics such as \(\text{E}_\gamma[\text{Var}[X_k[n]]]\), since the latter is the average of the former. Therefore, whenever possible, we should use local statistics for detection. We resort to global statistics, for example \(\sigma_D^2\) and \(\sigma^2\), only when the local statistics are not available. In (66), we used \(\sigma_D^2\) since \(D[n]\) is a summation of \(N\) variables and its local statistics are difficult to compute.
The statistics of \( \{X_m[n]\} \) in (66) can be found using the DEC feedbacks \( \{\gamma_m[n]\} \) as follows [27]:

\[
E[X_m[n]] = \tanh(\gamma_m[n]/2), \tag{67a}
\]
\[
\text{Var}[X_m[n]] = 1 - E^2[X_m[n]]. \tag{67b}
\]

Now, we treat (64) as a binary-input system and model the distortion \( \zeta_k[n] \) as an independent Gaussian variable. Then \( \lambda_k[n] \) is calculated as

\[
\lambda_k[n] = \frac{2\alpha\beta_k H[n]}{\text{Var}[\text{Re}(\zeta_k[n])]} \left( \text{Re}(Z[n]) - E[\text{Re}(\zeta_k[n])] \right). \tag{68}
\]

**B. The SNR Evolution Technique**

We next introduce an SNR evolution technique to analyze the performance for the iterative detector based on (68). This technique was first proposed for analyzing the IDMA systems over the AWGN channels [27]. Here, we extend it to the clipped SCM-OFDM systems over fading channels. Substituting (64) into (68), we can write the ESE output as

\[
\lambda_k[n] = \frac{2\alpha\beta_k H[n]}{\text{Var}[\text{Re}(\zeta_k[n])]} \left( \alpha H[n] \beta_k X_k[n] + \text{Re}(\zeta_k[n]) - E[\text{Re}(\zeta_k[n])] \right). \tag{69}
\]

In (69), \( \text{Re}(\zeta_k[n]) - E[\text{Re}(\zeta_k[n])] \) represents the interference-noise component. Its average power can be measured using the global variance as follows

\[
E_\gamma[\text{Var}[\text{Re}(\zeta_k[n])]] = E_\gamma[|\text{Re}(\zeta_k[n]) - E[\text{Re}(\zeta_k[n])]|^2] = |\alpha\beta_k|^2 \left( |H[n]|^2 P_{I,k} + P_{W,k} \right)
\]

where \( P_{I,k} \) and \( P_{W,k} \) are the relative power (normalized by \( |\alpha\beta_k|^2 \)) of the interference and noise components, respectively, as shown below:

\[
P_{I,k} = \sum_{m \neq k} \frac{|\beta_m|^2}{|\beta_k|^2} E_\gamma[\text{Var}[X_m[n]]] + \frac{\sigma_D^2}{2|\alpha\beta_k|^2}, \tag{70}
\]
\[
P_{W,k} = \frac{\sigma^2}{2|\alpha\beta_k|^2}. \tag{71}
\]

Then, the SNR for (69) with respect to the wanted signal \( X_k[n] \) is

\[
\text{snr}_k[n] = \frac{|\alpha\beta_k H[n]|^2}{E_\gamma[\text{Var}[\text{Re}(\zeta_k[n])]]} = \frac{|H[n]|^2}{|H[n]|^2 P_{I,k} + P_{W,k}}. \tag{72}
\]

We make three assumptions:

- The inputs to the DEC at different time indexes are uncorrelated. Similarly, the feedbacks from the DEC at different time indexes are also uncorrelated.
- The input sequence of the DEC is characterized by \( \{\text{snr}_k[n]\} \) in (72). (This holds true when Gaussian assumption is applied to \( \{\zeta_k[n]\} \).)
- The distribution of \( \{H[n]\} \) is given. (A typical case is that \( \{H[n]\} \) follows Rayleigh distribution.)

Note that the first assumption above can be ensured through the use of random interleavers. It also implies infinitely long codeword length. Based on the above three assumptions and from (72), the pair...
\((P_{I,k}, P_{W,k})\) fully determines the DEC performance. In the iterative decoding process, \(P_{W,k}\) is a constant number but \(P_{I,k}\) decreases as the iteration proceeds. We discuss below how to track \(P_{I,k}\).

For notational brevity, let \(V_k = E\left[\text{Var}[X_k[n]]\right]\), \(k = 1, 2, \ldots, K\). Now \(\sigma^2 = \sum_{k=1}^{K} |\beta_k|^2 V_k\). Therefore, from (40), \(\sigma^2_D\) can be found from \(\{V_k\}\) as

\[
\sigma^2_D = \phi \left( \sum_{k=1}^{K} |\beta_k|^2 V_k \right)
\]

(73)

where \(\phi(\cdot)\) is given by (40). From (70) and (73), \(P_{I,k}\) is fully determined by \(\{V_k\}\). Since \(V_k\) is the variance of the DEC feedback of the \(k\)th layer, it is a function of \(P_{I,k}\) and \(P_{W,k}\) that characterize the inputs to the DEC of the \(k\)th layer. We write this function as

\[
V_k = f(P_{I,k}, P_{W,k}).
\]

(74)

In general, \(f(\cdot)\) cannot be expressed in a closed form, but can be characterized by a look-up table created by the Monte Carlo simulation. The block diagram of the simulation is depicted in Fig. 9, where we have used an equivalent channel model as

\[
Z_k[n] = H[n](X_k[n] + I_k[n]) + W_k[n]
\]

(75)

where \(X_k[n] \in \{+1, -1\}\) is the coded BPSK signal, \(I_k[n] \sim N(0, P_{I,k})\) and \(W_k[n] \sim N(0, P_{W,k})\), respectively, represent the (normalized) interference and channel noise. \(V_k\) is estimated using the average of the DEC outputs [after the operation in (67)]. Similarly, the BER performance of the DEC can be characterized by a function as

\[
\text{BER}_k = g(P_{I,k}, P_{W,k}).
\]

(76)

To summarize, we can characterize the iterative decoding process using the following procedure. We use superscript \(^{(q)}\) for the iteration index and \(Q\) for the maximum number of iterations.

(i) Initialization: Set \(q = 1\) and \(V^{(q)}_k = 1\), \(P_{W,k} = \frac{\sigma^2}{2|\alpha|\beta_k|^2}\), \(k = 1, 2, \ldots, K\).

(ii) For the \(q\)th iteration:

- Find the normalized interference power for the ESE:

\[
P^{(q)}_{I,k} = \sum_{m \neq k} \frac{|\beta_m|^2}{|\beta_k|^2} V_m^{(q)} + \frac{\phi \left( \sum_{m=1}^{K} |\beta_m|^2 V_m^{(q)} \right)}{2|\alpha|\beta_k|^2}, k = 1, 2, \ldots, K.
\]

(iii) Recursion: If \(q < Q\), set \(q \leftarrow q + 1\) and go to (ii); otherwise, go to (iv).

(iv) Output the BER: \(\text{BER}_k = g(P^{(q)}_{I,k}, P_{W,k}), k = 1, 2, \ldots, K\).
The above SNR evolution technique can be extended to general cases such as those with complex \( \{\beta_k\} \) and \( \alpha \). For the SCM schemes, this technique leads to more accurate results than the EXIT chart technique. (Note: We may use multi-dimensional EXIT functions [35] to improve the EXIT chart technique, but this can be very complicated for large \( K \). The SNR evolution technique simplifies the problem due to the closed-form transfer functions in (72).)

Fig. 10 compares the simulation results and the BER predicted by the SNR evolution technique outlined above. The SCM-OFDM scheme with \( R = 4 \) used in Fig. 7 is considered. A relatively large frame length of 16384 is employed here since we have assumed infinite frame length in SNR evolution. From Fig. 10, the evolution and simulation results agree well for both the clipped and unclipped cases. This clearly demonstrates the efficiency of the SNR evolution technique.

A useful application of the SNR evolution technique is to find the optimized values of \( \{\beta_k\} \) for the SCM scheme. This results in optimized unequal power allocation for the different layers the SCM. A similar strategy has been studied in [27] for optimizing IDMA performance. With the SNR evolution technique outlined above, we can quickly assess the performance for each given set of \( \{\beta_k\} \) values and
employ a searching strategy to accomplish the optimization task.

REFERENCES


