

# Code-shift Division Multiple Access for Cellular Mobile

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## Abstract

This paper presents a new scheme to treat the multiple access interference (MAI) problem in CDMA systems. A cyclic prefix technique is introduced to confine the intra-cell MAI in a multipath environment to a small dimension so that it can be resolved by a common equalizer. The proposed scheme combines some advantages of TDMA and CDMA, namely, same-cell-user interference avoidance and other-cell-user interference diversity. Its receiver complexity is considerably lower than other interference cancellation methods.

## I. Introduction

Multiple access interference (MAI) imposes a major restriction on the capacity and performance of CDMA (Code Division Multiple Access) mobile communication systems [1-4]. Adopting orthogonal waveforms generally cannot solve the problem, since it is very difficult to maintain orthogonality in the presence of multipath delays. Substantial progress has been made in joint detection for MAI cancellation [1,4]. However the application of joint detection is still limited by its high complexity.

This paper presents a CsDMA (Code-shift Division Multiple Access) scheme to eliminate the same-cell-user MAI in multipath environments. Intra-cell orthogonality is established under the ideal circumstances. A cyclic prefix technique is introduced to confine the distortion and interference caused by multipath delay to a small dimension, so that it can be resolved by a common equalizer. The receiver complexity of CsDMA is considerably lower than other interference cancellation methods [4].

In the following we will first introduce the basic principles of CsDMA based on poly-phase codes [5,6]. We will later show that binary codes can also be used, which incurs slightly more overhead but allows an efficient realization based on the FHT (Fast Hadamard Transform) algorithm.

## II. Cyclic orthogonality and the prefix technique

The prefix technique introduced below explores the cyclic property of the spreading codes. Similar technique has also been used in OFDM (Orthogonal Frequency Division Multiplexing) systems to maintain orthogonality in multipath environments [7].

Let  $c = c_0 = \{c_{0,0}, c_{0,1}, c_{0,2}, \dots, c_{0,N-1}\}$  be a length  $N$  code that can be of complex values. Denote by  $c_i = \{c_{i,0}, c_{i,1}, c_{i,2}, \dots, c_{i,N-1}\}$  the cyclic shifts of  $c$  towards right by  $i$  positions. We will say that  $c$  is cyclicly orthogonal if,

$$\sum_{k=0}^{N-1} c_{i,k} \overline{c_{j,k}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (1)$$

where " $\overline{\phantom{x}}$ " denotes the conjugate operation. Equation (1) indicates that the code set  $\{c_i \mid i=0,1,\dots,N-1\}$  is orthogonal. Such

codes can be constructed using the method in [5,6]. A simple example is (0.5, -0.5, 0.5, 0.5).

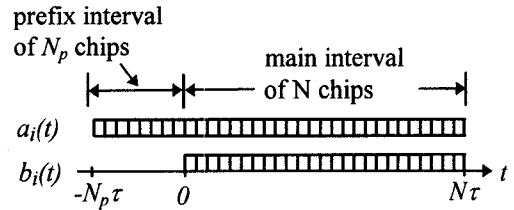


Fig.1 Waveform structure with the prefix technique.

From every  $c_i$  a pair of spreading waveforms  $a_i(t)$  and  $b_i(t)$  are constructed, each consisting of  $N_p + N$  chips with chip duration  $\tau$ , Fig.1. Two intervals are defined,

- In the main interval  $0 \leq t < N\tau$ :

$$a_i(t) = c_{i,n} \quad b_i(t) = \overline{c_{i,n}} \quad \text{for } n\tau < t \leq (n+1)\tau, \quad n=0,1,2,\dots,N-1 \quad (2a)$$

- In the prefix interval  $-N_p\tau \leq t < 0$ :

$$a_i(t) = a_i(t+N\tau) \quad b_i(t) = 0 \quad (2b)$$

We will refer to  $a_i(t)$  and  $b_i(t)$  as a TS (transmitted signature) and a RS (reference signature), respectively. For  $c$  cyclicly orthogonal,  $a_i(t)$  and  $b_i(t)$  are orthogonal to each other when  $i \neq j$ . Furthermore, due to the use of the prefix, their correlation in the presence of a delay term  $\delta$  can be expressed in a very regular form provided  $0 \leq \delta \leq N_p\tau$ ,

$$\int_0^{N\tau} a_j(t-\delta) b_i(t) dt = \begin{cases} \tau - \mu & \text{if } i - j = \lambda \text{ mod } N \\ \mu & \text{if } i - j = \lambda + 1 \text{ mod } N \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where  $\lambda = \lceil \frac{\delta}{\tau} \rceil$ , i.e., the smallest integer  $\leq \frac{\delta}{\tau}$  and  $\mu$  is the residual of  $\frac{\delta}{\tau}$ . We notice that  $a_i(t-\delta)$  and  $b_i(t)$  are orthogonal

except for a small range of  $(i-j-1)\tau < \delta < (i-1+1)\tau$ .

Equation (3) also applies if we re-define  $a_j(t) = g_j(t)$ ,  $b_i(t) = g_i(t)$  and drop the modulo  $N$  operation, where

$$g_i(t) = \begin{cases} 1 & \text{for } i\tau < t \leq (i+1)\tau \\ 0 & \text{elsewhere} \end{cases} \quad (4)$$

is the simple rectangular pulse typically used in TDMA. This implies the equivalent transmission characteristics between CsDMA and TDMA, as discussed in the next Section.

The overhead due to the use of prefix can be estimated by  $N_p/N$ .  $N$  is limited by the coherent time of the channel (in the order of 10 to 100 ms [8]).  $N_p$  is determined by the maximum delay spread (in the order of 1 to 20  $\mu$ s). Therefore the overhead can be made negligible in a practical system.

### III. Code-shift division multiple access scheme

#### A. CsDMA system architecture

For simplicity we will restrict our discussion to the equivalent baseband model. The proposed CsDMA system is essentially a multi-code CDMA, Fig.2. Data are delivered in a framed structure. Frames from users in the same cell are loosely synchronized. Any frame alignment error can be treated in the same way as for random delay. Each frame of a user contains  $m$  information symbols  $\{x_0, x_1, \dots, x_{m-1}\}$ , spread by TS' with consecutive shifts  $\{a_M(t), a_{M+1}(t), \dots, a_{M+m-1}(t)\}$ , which produces

$$s(t) = \sum_{i=0}^{m-1} x_i a_{M+i}(t) \quad (5)$$

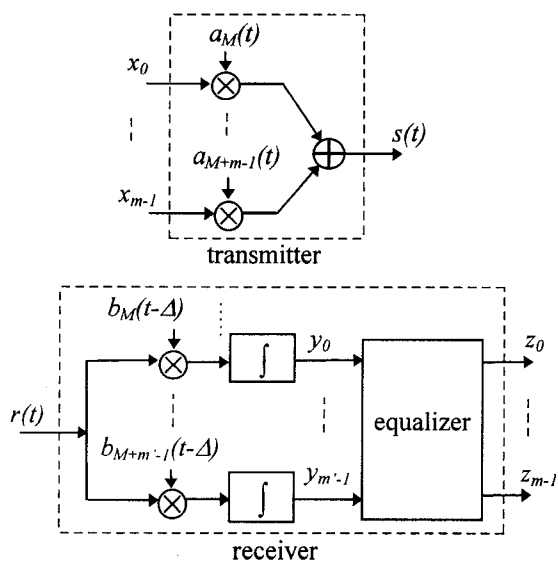


Fig.2 CsDMA transmitter and receiver structures. Every user is assigned with  $m$  spreading sequence pairs.

At the receiver, the received signal  $r(t)$  is correlated by the RS set  $\{b_M(t-\Delta), b_{M+1}(t-\Delta), \dots, b_{M+m-1}(t-\Delta)\}$ , where  $\Delta$  is a fixed delay offset. The outputs of the correlation bank  $\{y_1, y_2, \dots, y_m\}$  are used to estimate the original information symbols. The TS and RS are constructed from a cyclically orthogonal code  $c$  according (2). We will call  $c$  the master code. The offset  $M$  is used for the purpose of multiple access. It determines the appropriate offset position of code shift for each user. The choice of  $m'$  will become clear later.

#### B. The tapped-delay-line model for CsDMA

We adopt the multipath channel model below,

$$r(t) = \sum_k \alpha_k s(t - \Delta_k) + \eta(t) \quad (6)$$

where  $\alpha_k$  and  $\Delta_k$  are, respectively, the fading factor and delay associated with the  $k$ -th path and  $\eta(t)$  is the noise component including other cell interference.

We will concentrate in a frame transmitted during  $(-N_p\tau, N\tau)$ . From (2), (5) and (6), the output of the  $i$ -th correlator in Fig. 2 is,

$$y_i = \int_{\Delta}^{\Delta+N\tau} r(t) b_i(t-\Delta) dt = \sum_{j=0}^{m-1} c_{i,j} x_j + \eta_i \quad (7)$$

with

$$c_{i,j} = \sum_{k=0}^{N\tau} \alpha_k \int_0^{N\tau} a_j(t - \delta_k) b_i(t) dt \quad (8a)$$

$$\eta_i = \int_{\Delta}^{\Delta+N\tau} \eta(t) b_i(t - \Delta) dt \quad (8b)$$

where  $\delta_k = \Delta_k - \Delta$  is the relative delay of the  $k$ -th path. With proper tracking we assume that  $\delta_k$  is non-negative and bounded by the maximum delay spread  $\delta_{max}$ ,

$$0 \leq \delta_k \leq \delta_{max} \quad (9)$$

This implies that in (3)

$$0 \leq \lambda \leq L \quad (10)$$

with  $L = \lceil \frac{\delta_{max}}{\tau} \rceil + 1$ .

From (3),  $c_{i,j}$  in (8a) depends only on  $i-j \bmod N$ , so we can write  $c_{i,j} = c_{i-j \bmod N}$ . From (3) and (10),  $c_{i,j} = 0$  for  $i-j < 0$  or  $i-j > L$ , so (7) can be rewritten as,

$$y_i = \sum_{j=0}^L c_j x_{i-j \bmod N} + \eta_i \quad (11)$$

Equation (11) can be represented by the tapped-delay-line model in Fig.3. The required number of the receiver correlators is  $m' = \min\{N, m+L\}$  as the multipath delay may produce a 'tail' of maximum  $L$  symbols.

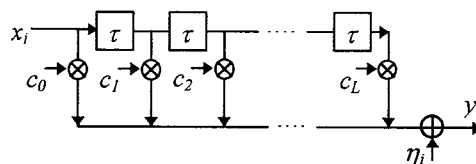


Fig.3 The tapped delay line model for (11).

#### C. Equivalence between CsDMA and TDMA: intra-cell interference avoidance

If we ensure that the sets of code shifts used by different users in a cell do not overlap, then the system is free from same-cell-user interference under the ideal circumstances without multipath delay, due to the cyclic orthogonality of the master code.

Intra-cell interference may occur in the presence of multipath delay characterized by (11). Since (11) is of the same form as the well known multipath model for TDMA [9], the equalization technique commonly used in TDMA can be employed to resolve the multipath effect for CsDMA. This is a direct consequence of the equivalence between the waveforms respectively defined in (2) and (4). Therefore CsDMA can achieve intra-cell interference free transmission in a multipath environment.

#### D. Difference between CsDMA and TDMA: inter-cell interference diversity

Equation (11) is valid for CsDMA as long as the master code is cyclically orthogonal. Without affecting intra-cell characteristics, we can carefully select master codes for neighboring cells so that they are approximately random to each other, which mitigates the worst case other-cell-user interference problem by the processing gain  $N/m$ . This homogeneous inter-cell interference property distinguishes CsDMA from TDMA.

For the selection of master codes, poly-phase codes with good cross correlation properties have been studied in [6]. A binary code approach will be discussed in Section V.

#### IV. System design issues

##### A. Code domain and code burst

It is convenient to discuss the principles of CsDMA in the so-called code domain diagram, in which the abscissa indicates code shift index and the ordinate represents the information symbol carried by the code with the corresponding shift. An example is illustrated in Fig.4. Every user is assigned with a burst of codes with consecutive subscripts  $\{M, M+1 \dots M+m-1\}$ . The code bursts from different users should not overlap, which is controlled by offset  $M$ . This is equivalent to the time domain pulse burst method used in TDMA.

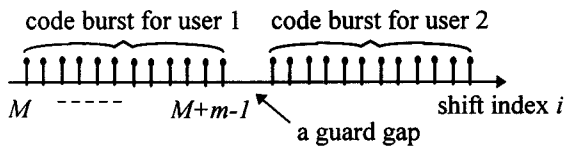


Fig.4 An illustration of the CsDMA signal structure in the code domain. A guard gap is formed by the unused codes with consecutive indices. Notice that every abscissa point above represents a code (or a shift of the master code point).

##### B. Capacity

We will only discuss up-link capacity since there is a very concise formula available [3]. Denote the chip rate of a CsDMA by  $W$ . Excluding the influence of same-cell-user interference, the up-link capacity can be calculated as [3],

$$N_u = 3.4 \frac{W/R}{E_b/N_0} \quad (12)$$

where  $N_u$  is the user number per sector,  $R$  the user rate and  $E_b/N_0$  the required signal to noise ratio. Given a typical required BER (bit error rate) of  $10^{-3}$ , simulation results show that this can be achieved by CsDMA with  $E_b/N_0 = 6.3$  (8dB), (with QPSK and rate 1/2 coding, see Section VI), resulting in

$$N_u = 0.54 W/R \quad (13)$$

or a total user rate per sector of  $0.54 W$ . This represents a potential capacity gain of about 2.2 times compared with a random waveform CDMA [3].

##### C. Guard gaps in the code domain

A single cell system employing QPSK (every TS/RS pair carries two coded bits) and rate 1/2 coding can provide a maximum total data rate per sector of  $W$ . The discussion above implies that a CsDMA can only use up to about half of the available code positions in a multi-cell environment. This ratio may be further reduced when various non-ideal factors are considered [10].

The unused codes can be used constructively to form guard gaps between code bursts carrying information, see Fig.4. It is very useful in the following situations,

- For the down-link, power control results in different power levels among signals directed to users at different distances from the base. The receiver performance may deteriorate due to the near-far effect. Guard gaps of unused code positions can be inserted between code bursts to reduce interference between them and mitigate the near-far effect.
- For the up-link with power control, the signal levels are relatively close. However guard gaps are still useful to prevent severe overlapping of different user frames due to synchronization error. It will relax the requirement on synchronization accuracy.

##### D. Channel sounding

Guard gaps can also be used to improve accuracy of path parameter estimation for coherent detection. This is illustrated in Fig.5. We use one TS, say  $p_0(t)$ , to carry a pilot whose amplitude, say  $P$ , is known to the receiver. The TS  $\{p_1(t), p_2(t), \dots, p_J(t)\}$  form a guard gap and they are not used by any transmitter. From (11),

$$y_i = c_i P + \eta_i \quad \text{for } i=0, 1, \dots, J \quad (14)$$

Clearly  $\{c_i\}$  can be directly estimated from (14).

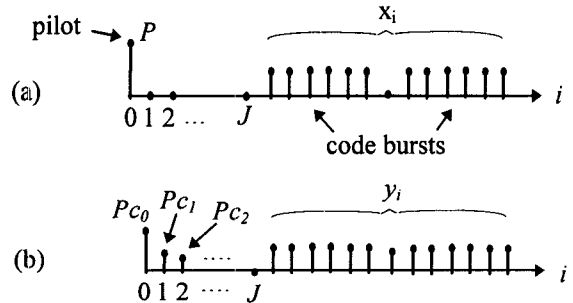


Fig. 5. (a) Signal structure in the code domain where  $p_0(t)$  is used to carry the pilot. The codes with indices from 1 to  $J$  form a guard gap and they are not used by the transmitter.

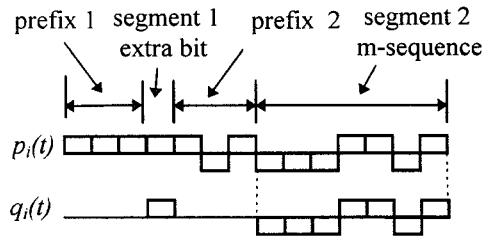
(b) The received signal in the code domain. The first  $J+1$  components are proportional to  $\{c_i\}$ .

#### V. Binary spreading sequences

Besides some very simple cases, most known cyclically orthogonal sequences with good cross correlation properties are not binary. We now introduce a modified scheme which allows the use of binary codes that are not completely cyclically orthogonal.

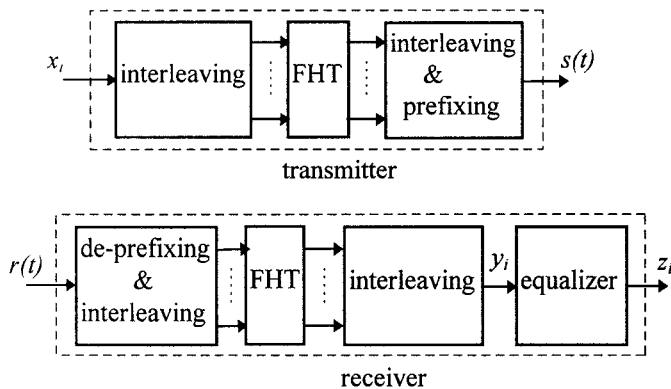
An m-sequence is derived from a maximum length code using mapping  $0 \rightarrow 1$  and  $1 \rightarrow -1$ . Let  $f(n)$  be an m-sequence augmented with an extra bit of 1. Denote by  $f_i(n)$  the sequence obtained by cyclically shifting the m-sequence segment of  $f(n)$  towards right by  $i$  positions and keep the extra bit unchanged, Appendix A. It can be shown that the set  $\{f_i(n)\}$  formed by all such shifts is orthogonal, which is equivalent, up to a interleaving, to the set of all the non-zero codewords of a Hadamard code [11] (also equivalent to the Walsh code). The required interleaving scheme is derived in Appendix A. We now construct a pair of sequences  $p_i(n)$  and  $q_i(n)$  as illustrated in Fig.6. Two prefixes are inserted in front of two segments respectively. It can be verified that (3) applies to the resulted

sequence pair set  $\{p_i(n), q_i(n)\}$  and therefore CsDMA can also be developed based on  $f(n)$ .



**Fig.6. Waveform structure for a binary code based scheme. Prefix 1 and 2 are the cyclic extensions of the extra bit and the m-sequence respectively. (The cyclic extension of the extra bit simply repeats itself.)**

An alternative transmitter/receiver structure based on FHT (Fast Hadamard Transform) can be devised, Fig.7. It is more efficient than the correlation bank scheme in Fig.2.



**Fig. 7 Transmitter and receiver structures based on FHT. The interleaving scheme is derived in Appendix A.**

The interleaving scheme in Fig.7 is not unique (see Appendix A). Different interleavings result in different m-sequences. This is very useful since different m-sequences are approximately random to each other [12]. They can be assigned to the neighboring cells to achieve inter-cell interference diversity.

## VI. Simulation results

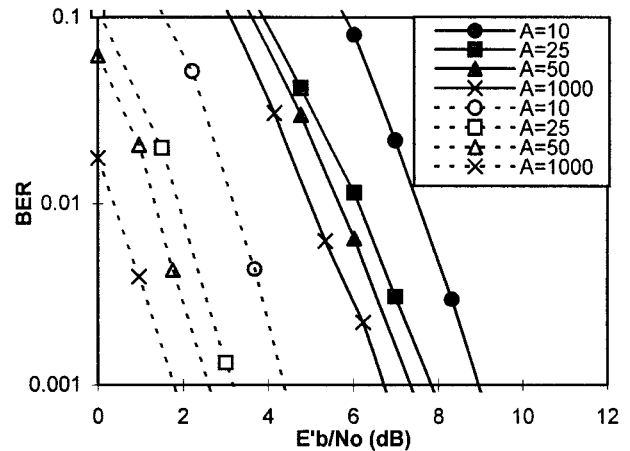
The following parameters are used in simulation,

- chip rate: 997.5 Mbps
- carrier frequency: 1 GHz
- fading characteristic: Rayleigh
- delay spread distribution: negative exponential
- mean delay spread: 3  $\mu$ s
- tap number in the FIR model: 5
- underlying modulation: coherent QPSK
- m-sequence length: 511
- prefix length: 10 chips
- total length of spreading waveforms: 532 chips
- guard gap length following a pilot: 9 chips
- guard gap length between code bursts: 4 chips

- error protection code: rate  $\frac{1}{2}$ , constrain length 9 convolutional code
- code burst length of each user: 16
- equalizer algorithm: soft output algorithm [13]

The full channel rate is 30 kbps and it is gated down to 10 kbps. The maximum user number is 51 per sector for 10 kbps user rate, or  $N_u=0.51W/R$ . This is slightly short of the capacity of  $0.54W/R$  indicated in (13). The reason is that the pilots and guard gaps occupy about half of the code shift positions.

Fig.8 shows the BER performance against  $E_b/N_o$  of CsDMA with different pilot levels. The overhead due to prefix is included in calculating bit energy but that due to pilot is not. We can see that increasing pilot power can significantly improve performance until the pilot energy to information bit energy ratio reaches about 50.



**Fig.8 CsDMA performance for a multipath channel with vehicle speed = 60k/h. A is the pilot energy to information bit energy ratio. The overhead due to prefix is included in calculating bit energy but that due to pilot is not.**

**Solid line: without antenna diversity.**

**Dashed line: with dual antenna diversity.**

For the up-link, every user sends a pilot. We can include it in the calculation of  $E_b/N_o$  as,

$$E_b / N_o = (1 + \frac{A}{n}) E'_b / N_o \quad (15)$$

where  $E_b$  and  $E'_b$  are energy per information bit including and excluding the pilot, respectively, and  $n$  is the number of information bits carried in a burst (=16 in the above simulation). Dual antenna diversity is assumed for the up-link and in this case  $BER=10^{-3}$  at  $E'_b/N_o = 2.8$  (4.5 dB) with  $A=10$ , resulting in  $E_b/N_o = 4.5$  (6.5 dB) in (15). This justifies the use of  $E_b/N_o=6.3$  (8dB) in Section IV, leaving about 1.5 dB margin.

## VII. Conclusions

It has been shown that CsDMA can achieve intra-cell interference free transmission in a multipath environment. Frame synchronisation is required but its accuracy can be flexible since frame alignment error can be treated in the same way as for multipath delay. The receiver complexity of the proposed scheme is considerably lower than other interference cancellation methods.

## VIII. Acknowledgments

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## Appendix A

In the following we will deal with codes in GF[2]. The spreading sequences are obtained by mapping  $\{0 \rightarrow 1, 1 \rightarrow -1\}$ .

Denote by  $i_k$  the  $k$ -th bit in the binary representation of an integer  $i$ , i.e.,

$$i = \sum_k i_k 2^k \quad (A1)$$

Consider a  $2^m \times 2^m$  Hadamard matrix  $H$  in GF[2], whose  $(i, j)$ -th entry is given by

$$H_{i,j} = \left( \sum_{k=0}^{m-1} i_k j_k \right) \text{ mod } 2 \quad (A2)$$

where  $i_k$  and  $j_k$  are the  $k$ th bits in the binary representations of  $i$  and  $j$  respectively.

Except for the first row, every row in the following  $2^m \times 2^m$  matrix is a cyclic shift of a length  $2^m-1$  maximum length code augmented with a zero in front,

$$F = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \text{tr}(\alpha^0) & \text{tr}(\alpha^1) & \dots & \text{tr}(\alpha^{2^m-1}) \\ 0 & \text{tr}(\alpha^{2^m-1}) & \text{tr}(\alpha^0) & \dots & \text{tr}(\alpha^{2^m-2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \text{tr}(\alpha^1) & \text{tr}(\alpha^2) & \dots & \text{tr}(\alpha^0) \end{pmatrix} \quad (A3a)$$

or

$$F_{i,j} = \begin{cases} \text{tr}(\alpha^{j-i}) & \text{for } ij \neq 0 \\ 0 & \text{for } ij = 0 \end{cases} \quad (A3b)$$

where  $\alpha$  is a primitive root in GF[2<sup>m</sup>] and  $\text{tr}(\cdot)$  is the trace function[11]. We now derive the row and column permutations (the interleaving scheme in Fig.7) transforming  $H$  into  $F$ .

Any element  $\alpha^i$  of GF[2<sup>m</sup>] can be expressed by,

$$\alpha^{-i} = \sum_{k=0}^{m-1} b_{i,k} \alpha^k \quad (A4)$$

with  $b_{i,k} \in GF[2]$ . Based on this we define an one to one mapping  $\phi(i)$  from  $(0, 1, \dots, 2^m-2)$  to  $(1, \dots, 2^m-1)$

$$\phi(i) = \sum_{k=0}^{m-1} b_{i,k} 2^k \quad i \neq 0 \quad (A5)$$

Notice that  $b_{i,k}$  is used as an ordinary integer of 0 or 1 above. It can be verified that

$$\alpha^{-\phi^{-1}(i)} = \sum_{k=0}^{m-1} i_k \alpha^k \quad (A6)$$

Now define another one to one mapping  $\gamma(i)$  from  $(0, 1, \dots, 2^m-2)$  to  $(1, \dots, 2^m-1)$ ,

$$\gamma(i) = \sum_{k=0}^{m-1} \text{tr}(\alpha^{i+k}) 2^k \quad (A7)$$

The value of the trace function above is also used as an ordinary integer of 0 or 1. It can also be verified that,

$$\text{tr}(\alpha^{\gamma^{-1}(i)+k}) = i_k \quad i \neq 0 \quad (A8)$$

From (A3), (A6) and (A8), for  $ij \neq 0$

$$\begin{aligned} F_{\phi^{-1}(i)+1, \gamma^{-1}(j)+1} &= \text{tr}(\alpha^{\gamma^{-1}(j)-\phi^{-1}(i)}) = \text{tr}\left(\sum_{k=0}^{m-1} i_k \alpha^{\gamma^{-1}(j)+k}\right) \\ &= \sum_{k=0}^{m-1} i_k \text{tr}(\alpha^{\gamma^{-1}(j)+k}) = \sum_{k=0}^{m-1} i_k j_k = H_{i,j} \end{aligned} \quad (A9)$$

Therefore the required permutations are,

$$\text{Row permutation: } 0 \rightarrow 0 \quad (A10a)$$

$$i \rightarrow \phi^{-1}(i)+1 \quad \text{for } i \neq 0 \quad (A10b)$$

$$\text{Column permutation: } 0 \rightarrow 0 \quad (A11a)$$

$$j \rightarrow \gamma^{-1}(j)+1 \quad \text{for } j \neq 0 \quad (A11b)$$

Different selections of the primitive element  $\alpha$  result in different mappings, which correspond to different maximum length codes in (A3).