Doped Accumulate LT Codes

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***Abstract*— We introduce a family of rateless codes, namely the doped accumulate LT (DALT) codes, that are capacity-approaching on a binary erasure channel (BEC) with bounded encoding and decoding complexity. DALT codes can be either systematic or non-systematic. Non-systematic DALT codes are very similar to raptor codes, except that joint optimization on the full coding graph can be applied to DALT codes, and thus they can be optimized to have better asymptotic performance than raptor codes. Systematic DALT codes, with the proposed protocol, exhibit better performance and lower complexity than their non-systematic counterparts.**

I. Introduction

LT codes [1] form a family of universal rateless codes designed for communication over the binary erasure channels (BEC). Universal rateless codes can provide reliable information delivery without the knowledge of channel erasure rate at the transmitter, and a corresponding receiver can work at channel capacity (or close to channel capacity) regardless of the channel erasure patterns. Raptor codes [2] are a family of enhanced LT codes with linear encoding and decoding complexity.

In many practical situations, systematic codes are preferred, since if the channel is perfect without loss, decoding at the receiver is not necessary. This can greatly reduce decoding cost when the event of information loss is rare. However, LT codes and raptor codes are, in their straightforward forms, non-systematic. A technique to design systematic version of Raptor codes is discussed in [2], but it entails considerably increased encoding and decoding complexity.

In this paper, we first propose accumulate LT (ALT) codes, formed by the concatenation of an accumulate pre-code with LT codes. The introduction of the accumulate pre-code can make the systematic ALT codes as efficient as LT codes. Furthermore, as an alternative to the pre-coding approach in designing raptor codes, we dope the parity bits from the LT encoder with those generated by a modified semi-random (SR) low-density-parity-check (LDPC) encoder [4]. The doping bits can help the decoder to reliably remove the residue loss rate, i.e., the fraction of unrecovered information bits after belief propagation (BP) decoding. The resulting codes are called doped ALT (DALT) codes. One major advantage of this approach is that the LT and the doping components can be easily jointly optimized, and thus DALT codes can be designed with better asymptotical performance than raptor codes.

The proposed systematic DALT codes can achieve near-capacity performance with the transmission protocol below.

Protocol I

Step 1: The information bits are first transmitted through the channel;
Step 2: The decoder feeds back the erasure rate of the information bits to the encoder;
Step 3: The encoder chooses a proper degree distribution to generate parity bits for further transmission;
Step 4: Once the decoder collects enough bits for reliable decoding, the transmission terminates.

Note that LDPC codes [5-11] can also be used in the above protocol. Suppose that the transmitter has the knowledge of the erasure rate δ (of the information bits). The transmitter can select an LDPC code optimized based on δ for the generation of parity bits. Such a scheme would be optimal if the erasure rate of the parity bits is also δ. Otherwise, certain performance loss would occur. However, as will be shown, DALT codes are nearly optimal regardless of the erasure pattern of the parity bits suffered in Step 4 of Protocol I. Protocol I can also be compared with the rateless transmission protocol below.

Protocol II

Step 1: The encoder continuously generates parity bits;
Step 2: Once the decoder collects enough parity bits for reliable decoding, the transmission terminates.

The key difference between the two protocols is the necessity in a feedback on the percentage of lost information bits. Apart from this extra constraint, DALT codes can provide better performance at a lower decoding cost than LT or Raptor codes.

II. Accumulate LT Codes

In this section, we present our ensemble of ALT codes. Density evolution (DE) analysis is applied in designing codes with good asymptotical performance.

A. Description of ALT Codes

ALT codes are constructed by concatenating LT codes with an accumulate pre-code [7]. The encoding process of ALT codes can be divided into two stages. At the first stage, an accumulate encoder generates state bits recursively. More specifically, the value of a current state bit is given by the addition of the current information bit and the previous state bit (which is initialized to 0 at the beginning). At the second stage, LT encoding is applied to the state bits. Let $\{P_1, P_2, \ldots\}$ be a distribution over $\{1, 2, \ldots\}$, which can be succinctly represented by a polynomial $P(x)$ with P_i as the coefficients before x^i. The LT encoder generates parity bits by calculating the addition of randomly selected d state bits, where d is randomly drawn from the distribution $P(x)$. Note that ALT codes can be either non-systematic (where codewords only contain parity bits), or systematic (where codewords contain both information and parity bits). Since non-systematic ALT codes are similar to LT codes, we concentrate on systematic ALT codes from now on.

A typical Tanner-graph representation of an ALT ensemble is given in Fig. 1. Each circle represents a

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variable node[1] and each small square represents a parity check. For each parity check, the value of a connected variable node is equal to the addition of all of the other connected variable nodes. The degree of a variable node (or a parity check) is defined as the number of edges that are connected to it. The nodes in the Tanner graph will be referred to by the labels given in Fig. 1.

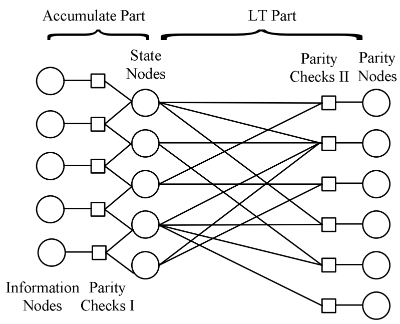

Fig. 1. The Tanner graph for an ALT ensemble.

Consider in Fig. 1 only the edges connecting state nodes and parity checks II. An ALT ensemble can then be characterized by the following degree distributions. $P(x)$, used in the generation of the LT parity bits, can be viewed as the degree distribution (d.d.) of parity checks II. Let $\Lambda(x) \equiv \sum_i \Lambda_i x^i$ be the d.d. polynomial of state nodes, where Λ_i denotes the fraction of state nodes with degree i. Also, let $\lambda(x) \equiv \sum_i \lambda_i x^{i-1}$ and $\rho(x) \equiv \sum_i \rho_i x^{i-1}$ be the d.d. polynomials from the perspective of edges, where λ_i and ρ_i denote the fraction of edges that are connected to state nodes and parity checks II with degree i, respectively. The following relationships naturally hold.

$$\lambda(x) = \frac{\Lambda'(x)}{\Lambda'(1)} \text{ and } \rho(x) = \frac{P'(x)}{P'(1)}$$

Let K, J and E be the number of state bits, parity bits and the connecting edges, respectively. Then,

$$K = E\sum_i \lambda_i / i = E\int_0^1 \lambda(t)dt,$$
$$J = E\sum_i \rho_i / i = E\int_0^1 \rho(t)dt.$$

Define the average state-node degree as $d_l \equiv E/K$. d_l is a measure of encoding/decoding complexity since it is the density of edges in the coding graph. Note that the edges from parity checks II are randomly connected to the state nodes. It is not difficult to show that, for sufficiently large K, $\Lambda(x)$ is actually a Poisson distribution, i.e.,

$$\Lambda(x) = \lambda(x) = e^{d_l(x-1)}. \tag{1}$$

The decoding inefficiency η is defined as the ratio of the coded bits to the information bits. Thus, we have

$$\eta = \frac{K+J}{K} = 1 + \frac{\int_0^1 \rho(t)dt}{\int_0^1 \lambda(t)dt}. \tag{2}$$

Our design objective is to minimize η.

B. Density Evolution of ALT Codes

Consider the transmission of an ALT ensemble on a BEC with erasure rate δ. The received parity bits and the information bits can be represented as in Fig. 1. The density evolution (DE) fixed point of BP decoding can be derived following a *graph reduction* approach [5]. Since degree-1 nodes do not participate in the BP decoding, they can be removed without affecting the decoding performance. Further, the observed information nodes can also be removed, leaving a degree-2 parity check, i.e., the two state nodes connected to each of these checks can be merged into one state node with a degree equal to the addition of the original two. The probability of the event that k consecutive information nodes are observed and the immediate next is erased is $(1-\delta)^k\delta$. This event results in the mergence of the degrees of k+1 state nodes. Thus, the d.d. of the state nodes after mergence, referred to as *merged state nodes*, is given by

$$\tilde{\Lambda}(x) = \frac{\delta\Lambda(x)}{1-(1-\delta)\Lambda(x)}, \tag{3}$$

and correspondingly the new edge d.d. is given by

$$\tilde{\lambda}(x) = \frac{\tilde{\Lambda}'(x)}{\tilde{\Lambda}'(1)} = \frac{\delta^2\lambda(x)}{(1-(1-\delta)\Lambda(x))^2}. \tag{4}$$

Then the residue ensemble becomes an equivalent LT code with d.d. pair $\tilde{\lambda}(x)$ and $\rho(x)$, and the DE fixed point is

$$\tilde{\lambda}(1-\rho(1-x)) = x. \tag{5}$$

By defining

$$\tilde{\rho}(x) \equiv 1-\tilde{\lambda}^{-1}(1-x), \tag{6}$$

(5) can be rewritten as $\rho(x) = \tilde{\rho}(x)$. Considering (1), (4) and (6), we obtain

$$\tilde{\rho}(x) = \frac{1}{d_l}\ln\left(1-\delta+\frac{\delta^2}{2(1-x)}+\sqrt{\left(1-\delta+\frac{\delta^2}{2(1-x)}\right)^2-(1-\delta)^2}\right) \tag{7}$$

Now we discuss how to optimize the ALT codes based on the DE fixed point. Let P_e be the residual loss rate, and $\tilde{P}_e$ the residual loss rate of the merged state nodes. The probability that the decoder fails to recover a lost information node is equal to the probability of the event that either of the two connected merged state nodes fails to be recovered, i.e., $1-(1-\tilde{P}_e)^2$. By considering the fact that the fraction of lost information bits is δ, we obtain

$$P_e = \delta(1-(1-\tilde{P}_e)^2), \tag{8}$$

or equivalently $\tilde{P}_e = 1-\sqrt{1-P_e/\delta}$. Based on the analysis in [9], if $\rho(x) > \tilde{\rho}(x),\ 0 \le x < 1-\tilde{P}_e$, then the decoder can reliably recover $(1-\tilde{P}_e)K$ or more information bits. Thus, we can formulate the optimization problem as

$$\begin{aligned} &\min_{\{\rho_i\}} \int_0^1 \rho(x)dx \\ \text{s.t.}\quad &\rho(x) > \tilde{\rho}(x),\ x \in \left[0, \sqrt{1-P_e/\delta}\right), \\ &\rho(1) = 1, \rho_i \ge 0, D \ge i \ge 1. \end{aligned} \tag{9}$$

Note that to minimize the cost function in (9) is equivalent to minimizing η in (2). For constructing practical codes, we limit the maximum degree of $\rho(x)$ to D. Also, it is assumed $P_e < \delta$ in (9) since otherwise coding is not necessary. We can replace $\rho(x) > \tilde{\rho}(x)$ in (9) by a set of inequalities obtained by letting $\rho(x) > \tilde{\rho}(x)$ hold on discretized x values. Then (9) can be solved efficiently by linear programming.

Fig. 2 shows the contour of decoding inefficiency η obtained by solving (9) with $\delta = 0.5$, $D = 80$, and various values of P_e and d_l. From Fig. 2, if $P_e = 0.01$, the minimum $\eta \approx 0.996$ appears at $d_l \approx 4.4$. In general η can be less than

[1] Throughout this paper, "node" and "bit" actually have the same meaning, and thus can be interchanged freely.

1 since we only require reliable decoding of a fraction $1-P_e$ of the information bits. For comparison, Fig. 3 shows the corresponding contour of η for LT codes. It can be seen that, for a given P_e, the achievable minimum η for optimal systematic ALT codes is almost the same as that for LT codes, but the former requires smaller d_l (i.e., lower complexity) than the later. This will be discussed in more detail below.

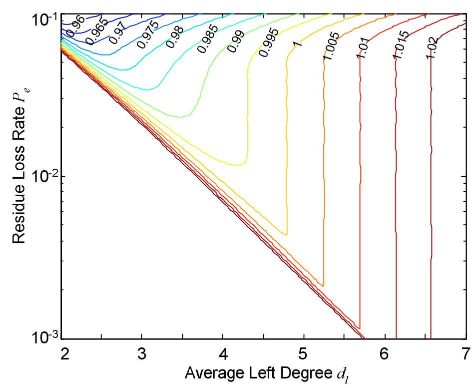

Fig. 2. The contour of decoding inefficiency η with respect to P_e and d_l for systematic ALT codes with $\delta = 0.5$ and $D = 80$.

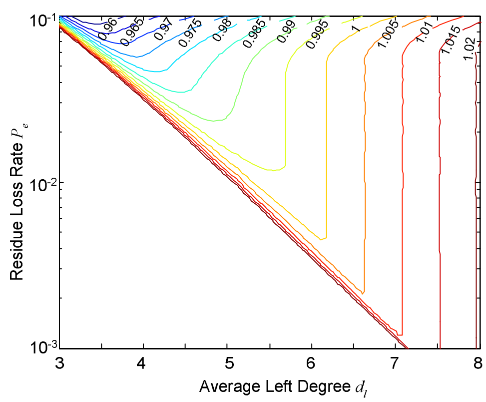

Fig. 3. The contour of decoding inefficiency η with respect to P_e and d_l for LT codes with $D = 80$.

C. Complexity of ALT Codes

Now we consider the encoding and decoding complexity of ALT codes. The following proposition establishes the relationship between d_l, δ and P_e.

Proposition I: If an ALT decoder can reliably decode at least a fraction $1-P_e$ of the information bits, d_l satisfies

$$d_l \ge \ln(1-\delta+(\delta^2/P_e)(1+\sqrt{1-P_e/\delta})) . \quad (10)$$

Proof of Proposition I: From (1) and (3), there is a fraction

$$\tilde{\Lambda}(0)=\frac{\delta\Lambda(0)}{1-(1-\delta)\Lambda(0)}=\frac{\delta e^{-d_l}}{1-(1-\delta)e^{-d_l}} \quad (11)$$

of merged state nodes with degree-0. Thus, $\tilde{P}_e \ge \tilde{\Lambda}(0)$. By combining this inequality with (8) and (11), and after some straightforward manipulations, we obtain (10). □

Although (10) only provides a lower bound on d_l, it gives a good estimate of the required degree density d_l for optimized ALT codes. For a given P_e, the right hand side of (10) monotonically increases with δ and tends to zero when δ reduces to P_e. This implies that the required complexity of optimized ALT codes may decrease gradually to zero when δ reduces to P_e, which is also verified by the numerical results not given here.

III. Doped Accumulate LT Codes

ALT codes can reliably recover a constant fraction of information bits. However, in most situations we need to recover all of the information bits, not just a constant fraction. To reduce the residual loss left by the ALT decoder, we propose DALT codes in which the parity bits from the ALT encoder are doped with those generated by a modified SR-LDPC encoder [4] (that serves a role similar to the LDPC precoder as to raptor codes).

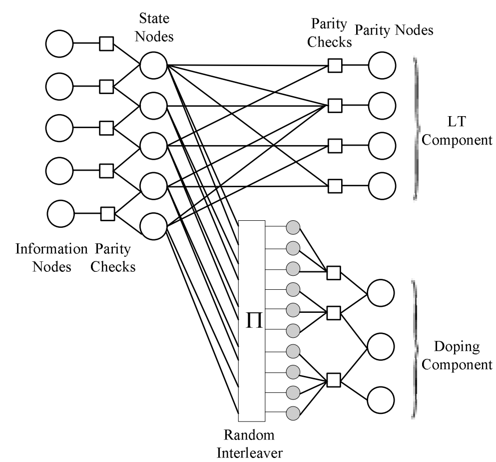

Fig. 4. The Tanner graph for a DALT ensemble.

A. Description of DALT Codes

The Tanner graph of a DALT ensemble is given in Fig. 4. The encoding process of DALT codes is outlined as follows. Similar to ALT codes, accumulate pre-coding is first applied to the information bits to generate state bits. Let p be the dope ratio. Then a parity bit is generated with probability $1-p$ by an LT encoder and with probability p by a modified SR-LDPC encoder. The modified SR-LDPC encoder will be detailed in the next subsection. One advantage of DALT codes over raptor codes is that the d.d. of the LT component can be easily optimized based on the entire coding graph, instead of only on the LT part as in the case of raptor codes. As a consequence, DALT codes can be designed with better asymptotic performance.

B. Description of the Doping Component

The doping component is basically a left-regular version of the SR-LDPC codes [4]. We modify the encoding scheme a little to make the generation of parity bits independent from each other (as required by rateless codes). The encoding process is outlined as follows. We first repeat each of the K information bits by n times, and randomly scramble the nK *repeat*s to form an encoding line. A *parity position* is defined as the position between two adjacent repeats in the encoding line that parity bits can be inserted into. There are nK different parity positions (including the end of the encoding line) along the line. Now we can generate parity bits. Each time we choose with equal probability a parity position along the encoding line to insert a parity bit. The value of each parity bit is given by the sum of all of the repeats located before it, as can be easily accomplished using an accumulate encoder.

The state-node d.d. for the doping component is $\Lambda_d(x) = x^n$, and the corresponding edge d.d. is $\lambda_d(x) = x^{n-1}$. Let the degree of a doping parity bit be the length of the segment of consecutive repeats located immediately before this bit. Let $p' \equiv (J+J_d)p/K$, where J is the number of parity bits of the LT component, and J_d that of the doping component. It is shown in Appendix I that the asymptotical d.d. of the doping parity bits is given by

$$P_d(x) = 1 - \frac{n}{p'}(1 - e^{-p'/n}) + \frac{n(1 - e^{-p'/n})^2}{p'(1 - xe^{-p'/n})}. \quad (12)$$

Thus, the corresponding edge d.d. is given by

$$\rho_d(x) = \frac{P_d'(x)}{P_d'(1)} = \frac{(1 - e^{-p'/n})^2}{(1 - xe^{-p'/n})^2}. \quad (13)$$

C. Evolution Analysis of DALT Codes

Let $\Lambda(u)$ and $\Lambda_d(v)$ be the state-node d.d. polynomials of the LT and the doping component, respectively. The joint state-nodal d.d. is given by $\Lambda(u)\Lambda_d(v)$. By following the graph reduction approach used in obtaining (3), the joint nodal d.d. after graph reduction is given by

$$\tilde{\Lambda}(u,v) = \frac{\delta \Lambda(u)\Lambda_d(v)}{1 - (1-\delta)\Lambda(u)\Lambda_d(v)}. \quad (14)$$

Thus, the joint edge d.d. from the perspective of the LT component is given by

$$\tilde{\lambda}(u,v) = \frac{d\tilde{\Lambda}(u,v)}{du} \Bigg/ \left.\frac{d\tilde{\Lambda}(u,v)}{du}\right|_{u=v=1} = \frac{\delta^2 \lambda(u)\Lambda_d(v)}{(1-(1-\delta)\Lambda(u)\Lambda_d(v))^2}, \quad (15)$$

and from the perspective of the doping component

$$\tilde{\lambda}_d(u,v) = \frac{d\tilde{\Lambda}(u,v)}{dv} \Bigg/ \left.\frac{d\tilde{\Lambda}(u,v)}{dv}\right|_{u=v=1} = \frac{\delta^2 \Lambda(u)\lambda_d(v)}{(1-(1-\delta)\Lambda(u)\Lambda_d(v))^2}, \quad (16)$$

DE analysis shows that the fixed point is given by

$$1 - v = \rho_d(1 - \tilde{\lambda}_d(u,v)), \quad (17a)$$

$$1 - u = \rho(1 - \tilde{\lambda}(u,v)). \quad (17b)$$

where $\rho(x)$ is the parity-check d.d. polynomial for the LT part from the perspective of edges. The following proposition establishes the relationship between the DE fixed point and the decoding inefficiency η for systematic DALT codes. The proof is given in Appendix II.

Proposition II: If $\tilde{\lambda}(\cdot)$, $\tilde{\lambda}_d(\cdot)$, $\rho(\cdot)$ and $\rho_d(\cdot)$ satisfies (17), then the decoding inefficiency of the residue ensemble is

$$\eta' = \frac{J + J_d}{\delta K} = \frac{\int_0^1 \rho(u)du}{\int_0^1 \tilde{\lambda}(u,1)du} + \frac{\int_0^1 \rho_d(v)dv}{\int_0^1 \tilde{\lambda}_d(1,v)dv} = 1.$$

And hence, for the ensemble before graph reduction,

$$\eta = ((1-\delta)K + J + J_d)/K = 1 - \delta + \delta\eta' = 1.$$

Since $\tilde{\lambda}(\cdot)$, $\tilde{\lambda}_d(\cdot)$ and $\rho_d(\cdot)$ is known, we need to find the optimal $\rho(\cdot)$ that minimizes η under the constraint of reliable decoding. Let $\hat{\rho}(x) \equiv 1 - \hat{\lambda}^{-1}(1-x)$, where $\hat{\lambda}(u) \equiv \tilde{\lambda}(u, v_u)$, and v_u is the fixed point of (17a) for a given u. "Reliable decoding" requires that $\rho(x) > \hat{\rho}(x)$ for $0 \le x < 1$. Thus, this optimization problem can be formulated as

$$\begin{aligned} &\min_{\{\rho_i\}} \int_0^1 \rho(x)dy \\ \text{s.t.}\quad &\rho(x) > \hat{\rho}(x),\ x \in [0,1), \\ &\rho(1) = 1, \rho_i \ge 0, D \ge i \ge 0. \end{aligned} \quad (18)$$

Note that to minimize the cost function in (18) is, based on Proposition II, equivalent to minimizing η. Similarly to (9), (18) can also be solved by linear programming. The optimal LT generation distribution $\{P_i\}$ can then be obtained from $\{\rho_i\}$.

Table I shows optimized degree distributions for various δ, p, n and d_l. It can be seen that the designed η for DALT codes is very close to the channel capacity. For example, the asymptotic gap away from the channel capacity is only 0.0011 for Code III. It can also be seen that the complexity of DALT codes (measured by d_l) reduces with the decrease of the channel erasure rate.

TABLE I. DEGREE DISTRIBUTIONS FOR VARIOUS VALUES OF δ, p, n AND d_l

Code I		Code II		Code III	
δ	1	δ	0.5	δ	0.15
p	0.00979	p	0.01675	p	0.04667
n	4	n	4	n	5
d_l	5.448	d_l	4.192	d_l	2.0
η	1.0020	η	1.0017	η	1.0011
P_1	0.008001	P_1	0.008001	P_1	0.050000
P_2	0.460778	P_2	0.292762	P_2	0.101144
P_3	0.270578	P_3	0.274687	P_3	0.393669
P_6	0.115605	P_6	0.202698	P_8	0.172923
P_7	0.046850	P_7	0.000809	P_9	0.052670
P_{14}	0.035912	P_{11}	0.069095	P_{22}	0.067925
P_{15}	0.022356	P_{12}	0.037829	P_{23}	0.070730
P_{30}	0.007295	P_{24}	0.032242	P_{80}	0.090940
P_{31}	0.018887	P_{25}	0.048725		
P_{79}	0.013738	P_{80}	0.033152		

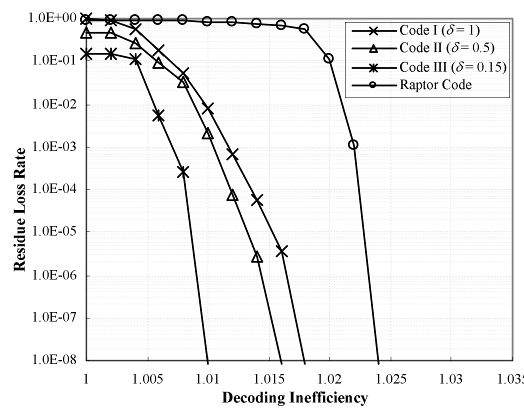

Fig. 5. Performances of DALT codes (designed for various channel erasure rates) with information length 524288. The performance curve of raptor codes is also included for comparison.

Fig. 5 shows the performance of the DALT ensembles given in Table I with information length 524288. We can see that the smaller the channel erasure rate, the better the performance of the DALT codes. The performance curve for the raptor code [3] is also included for comparison. Unlike the conventional LDPC codes [5][10] that suffer from severe error-floor problems, our proposed DALT ensembles exhibit a very low error-floor. The analysis of the error-floor behavior of DALT codes is in consideration.

D. Design of Finite-Length DALT Codes

DALT codes designed based on DE have good performance when the code length is sufficiently large. For moderate code lengths (such as in the tens of thousands), special treatment is necessary to design good codes.

The idea of the design of finite-length DALT codes is borrowed from Luby [1] and Shokrollahi [2]: replace x in (7) with $x + c\sqrt{(1-x)/K}$, and then solve (9) again for suitable c and K to obtain the optimized d.d.. A heuristic explanation of this choice can be found in [2].

Table II shows code distributions for various δ and d_l. Note that Code IV is in fact non-systematic, and thus can share the same d.d. of the raptor code in [3]. The performance of the DALT ensembles in Table II is given in Fig. 6. It is still shown that, with the decrease of the channel erasure rate, DALT codes can achieve better performance at a lower cost.

TABLE II. DEGREE DISTRIBUTIONS FOR VARIOUS VALUES OF δ, p, n AND d_l

Code IV		Code V		Code VI	
δ	1	δ	0.5	δ	0.15
K	65536	K	65536	K	65536
P_e	0.01	P_e	0.01	P_e	0.01
p	0.015	p	0.03	p	0.04667
n	4	n	5	n	5
d_l	5.9	d_l	4.6	d_l	2.0
P_1	0.007971	P_1	0.010233	P_1	0.050000
P_2	0.493570	P_2	0.328860	P_2	0.101144
P_3	0.166220	P_3	0.155316	P_3	0.393669
P_4	0.072646	P_4	0.118828	P_8	0.172923
P_5	0.082558	P_6	0.124061	P_9	0.052670
P_8	0.056058	P_7	0.031734	P_{22}	0.067925
P_9	0.037229	P_{11}	0.040399	P_{23}	0.070730
P_{19}	0.055590	P_{12}	0.076585	P_{80}	0.090940
P_{65}	0.025023	P_{25}	0.025504		
P_{66}	0.003135	P_{26}	0.047908		
		P_{76}	0.010506		
		P_{77}	0.030066		

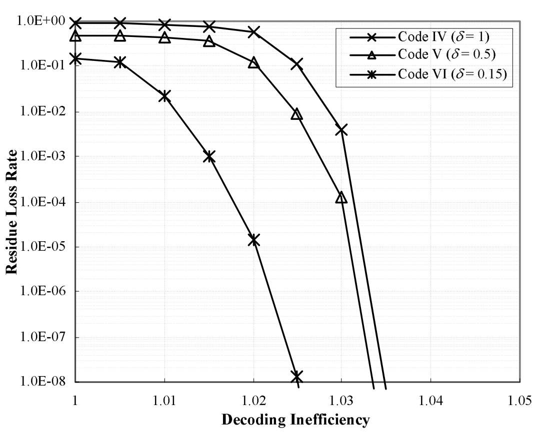

Fig. 6. Performances of DALT codes (designed for various channel erasure rates) with information length 65536.

APPENDIX I. PROOF OF (12)

The length of the encoding line is nK, and there are $p'K$ parity bits to be inserted. Let the degree of a parity position be the number of parity bits inserted to this position. It can be shown that for sufficiently large K the d.d. polynomial of the parity positions is given by $e^{(p'/n)(x-1)}$. Thus, the probability of a randomly chosen parity position with degree-0 is $e^{-p'/n}$, and the opposite is $1-e^{-p'/n}$. Merge the parity bits inserted to the same position into one. Then the d.d. polynomial of the parity bits is given by $(1-e^{-p'/n})/(1-xe^{-p'/n})$. By considering the fact that there are a fraction of $(n/p')(1-e^{-p'/n})$ parity bits that have been merged, the overall d.d. of parity bits is given by (12). □

APPENDIX II. PROOF OF PROPOSITION II

From (17a), we have

$$\int_0^1 \rho_d(v)dv = 1-\int_0^1 \tilde{\lambda}_d^{-1}(u,v)dv = \int_0^1 \tilde{\lambda}_d(u,v)dv$$

where the last equality uses the fact that for a given u,

$$\int_0^1 \tilde{\lambda}_d(u,v)dy + \int_0^1 \tilde{\lambda}_d^{-1}(u,v)dv = 1 .$$

Note that the inverse of $\tilde{\lambda}_d(\cdot)$ above is taken with respect to v. Similarly, it can be shown that

$$\int_0^1 \rho(u)du = \int_0^1 \tilde{\lambda}(u,v)du .$$

Then, η' can be calculated as

$$\begin{aligned}\eta' &= \frac{\int_0^1 \tilde{\lambda}(u,v)du}{\int_0^1 \tilde{\lambda}(u,1)du} + \frac{\int_0^1 \tilde{\lambda}_d(u,v)dv}{\int_0^1 \tilde{\lambda}_d(1,v)dv} \\ &= \frac{\int_0^1 \frac{\partial \tilde{\Lambda}(u,v)}{\partial u}du \Big/ \left.\frac{\partial \tilde{\Lambda}(u,v)}{\partial u}\right|_{u=v=1}}{\int_0^1 \frac{\partial \tilde{\Lambda}(u,1)}{\partial u}du \Big/ \left.\frac{\partial \tilde{\Lambda}(u,v)}{\partial u}\right|_{u=v=1}} + \frac{\int_0^1 \frac{\partial \tilde{\Lambda}(u,v)}{\partial v}dv \Big/ \left.\frac{\partial \tilde{\Lambda}(u,v)}{\partial v}\right|_{u=v=1}}{\int_0^1 \frac{\partial \tilde{\Lambda}(1,v)}{\partial v}dv \Big/ \left.\frac{\partial \tilde{\Lambda}(u,v)}{\partial v}\right|_{u=v=1}} \\ &= \int_0^1 \frac{\partial \tilde{\Lambda}(u,v)}{\partial u}du + \int_0^1 \frac{\partial \tilde{\Lambda}(u,v)}{\partial v}dv \\ &= \int_0^1 d\tilde{\Lambda} = 1.\end{aligned}$$

Thus, the conclusion holds. □

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