On the Capacity Gain from Antenna Correlation in Multi-User MIMO Systems

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Abstract—In this paper, we consider multi-user multiple-input multiple-output (MIMO) systems over either multiple access channels (MACs) or broadcast channels (BCs). Contrary to common views, we show that antenna correlation at the mobile unit (MU), the base station (BS) or both sides can be potentially beneficial for such systems. Both asymptotic analysis and numerical results indicate that antenna correlation can provide a significant capacity gain in multi-user environments.

I. INTRODUCTION

A multiple-input multiple-output (MIMO) system employs multiple antennas at both the transmitter and receiver sides to enhance the performance of wireless communication [1]-[4]. In practice, antenna correlation [3]-[9] can be caused by limited physical sizes of the transmitters/receivers or by the lack of sufficient scatterers in the propagation environments. Antenna correlation is commonly regarded as a negative factor in a conventional single-user MIMO system since it may result in reduced degrees of freedom (DOF) [3]-[4].

The impact of antenna correlation at the base station (BS) has been studied in [5]-[6] for the capacity of a multiple-input single-output (MISO) broadcast channel (BC). Literature [6] considers a MISO BC with up to three mobile units (MUs). It is shown in [6] that antenna correlation at the BS can actually increase the upper bound of the capacity\(^1\) and the related capacity improvement is verified by numerical results. This interesting finding provides motivation to exploit the potential advantage of antenna correlation.

This paper is concerned with the capacities of MIMO multiple access channels (MACs) or BCs. Our focus of theoretical analysis is on the asymptotic situation when the number of MUs (denoted by \(K\) below) is sufficiently large. (Numerical results will also be provided for the situation of finite MUs.) We will study the impact of antenna correlation at both the MU and BS sides and quantify the capacity gain brought by antenna correlation analytically in the limiting case of \(K \to \infty\). It will be shown that antenna correlation is potentially beneficial in a multi-user environment.

It is interesting to compare our work with references [7]-[9] on the beamforming gain\(^2\) related to antenna correlation. We show that, in the presence of multiple MUs, it is possible to exploit such a gain without losing DOF, as the different locations of multiple MUs naturally provide sufficient DOF.

The notations used in this paper are as follows. For any matrix \(A, A^*\) and \(|A|\) are the conjugate transpose and 2-norm of \(A\), respectively. For a Hermitian matrix \(A\), \(\text{tr}(A), A^{1/2}\) and \(\lambda_{\text{max}}(A)\) denote, respectively, its trace, principal square root and maximum eigenvalue. \(I_M\) represents an \(M \times M\) identity matrix. 

\(E\{\cdot\}\) is the expectation operator.

II. MACS

A. Preliminary

Consider a \(K\)-user MIMO system over a quasi-static flat fading MAC with \(M\) antennas at the BS and \(N\) antennas at each MU. Denote by \(H_k\) and \(x_k\) the channel matrix and the transmitted vector of MU \(k\). The received vector \(y\) is given by

\[
y = \sum_{k=1}^{K} H_k x_k + n
\]

where \(n\) is a vector of complex additive white Gaussian noise (AWGN) samples with zero mean and unit variance. \(\{H_k\}\) in (1) are independent among different MUs and perfectly known at both the transmitters and the receiver.

Following the common Kronecker correlation model in [4][7], we can write \(H_k\) as

\[
H_k = \sqrt{s_k} R_k^{1/2} W_k T_k^{1/2} \tag{2}
\]

where \(s_k\) is a scalar denoting the gain of path loss and lognormal fading\(^3\), and \(W_k\) is a Rayleigh fading matrix whose entries are independent and identically distributed complex circular symmetric Gaussian variables with zero mean and unit variance. \(T_k\) and \(R_k\) in (2) are semi-positive definite matrices that characterize correlation effects of antennas at the transmitter and the receiver, respectively, for the link between MU \(k\) and the BS. The normalizations \(\text{tr}(T_k) = N\) and \(\text{tr}(R_k) = M\) are assumed throughout this section.

For MACs, we want to maximize the system sum rate under individual power constraints for each channel realization [2]. The problem (i.e., calculating the system sum-rate capacity) is formulated as

\[
C = \max_{\{Q_k\}} \log \det \left( I_M + \sum_{k=1}^{K} H_k Q_k H_k^* \right) \tag{3}
\]

subject to \(\text{tr}(Q_k) \leq p_k, \forall k\)\(^4\), where \(Q_k = E\{x_k x_k^*\}\) and \(p_k\) are the input covariance matrix and the maximum transmitted power of MU \(k\), respectively.

\(^1\) Since equal power allocation is assumed, the discussions in [6] are only applicable to the capacity of BC in the high signal-to-noise ratio region.

\(^2\) Beamforming gain, a common term for antenna arrays, is used in this paper to refer to the advantage of the effect that the transmitted power is concentrated in some particular directions.

\(^3\) We assume equal path loss and lognormal fading for all the antenna links seen by a particular MU.

\(^4\) The problem is convex and can be solved in polynomial time.
**B. Asymptotic Analysis**

In general, there is no closed-form solution to (3), although it can be evaluated numerically [2]. In this subsection, we analyze the problem in the asymptotic case of $K \to \infty$. The related results can provide useful insights into the problem. Let $g_k \equiv \lambda_{\max}(H_k H_k^*)$ and $u_k$ be the left singular vector corresponding to the maximum singular value of $H_k$. The discussions below show that the system sum-rate capacity is primarily determined by the means of $\{g_k\}$ when $K \to \infty$.

**Lemma 1:** For each channel realization, the sum-rate capacity in (3) is lower-bounded and upper-bounded by

$$C \geq \log \det \left( I_M + \sum_{k=1}^{K} p_k g_k u_k u_k^* \right)$$

and

$$C \leq M \log \left( 1 + \frac{1}{M} \sum_{k=1}^{K} p_k g_k \right),$$

respectively.

The proof of this lemma is given in Appendices A and B. Next, we show that these two bounds converge when $K \to \infty$. Let $\{u_i\}$ be the entries of $u_k$. Since the Kronecker correlation model is used, $\{u_i\}$ have identical distribution with their phases uniformly distributed [4][7]. Then the following can be shown.

$$E \left[ \left| u_{i,k} \right|^2 \right] = 1/M, \forall k,i,$$

and

$$E \left[ u_{i,k}^* u_{j,k} \right] = 0, \forall k \text{ and } i \neq j.$$

Define a matrix $B$ as

$$B = \sum_{k=1}^{K} p_k g_k u_k u_k^*.$$ 

Let $b_{ij}$ be the entry in the $i$th row and $j$th column of $B$. When $K \to \infty$, from the Strong Law of Large Numbers (SLLN) [11, page 27, Theorem C] and (6), we have $\{g_k\}$ are fixed firstly

$$b_{ii} = \sum_{k=1}^{K} p_k g_k u_k u_k^* \to \sum_{k=1}^{K} p_k g_k / M + o(K), \forall i,$$

and

$$b_{ij} = \sum_{k=1}^{K} p_k g_k u_k u_{j,k}^* \to o(K), \forall i \neq j,$$

where $o(K)$ is a term satisfying $o(K)/K \to 0$ as $K \to \infty$. If a random variable $\xi$ converges to a deterministic number $a$, then any continuous function $f(\xi)$ of $\xi$ converges to $f(a)$ [11, page 24]. Then, from (4) in Lemma 1 and (7), we can obtain that for sufficiently large $K$

$$C \geq M \log \left( \frac{1}{M} \sum_{k=1}^{K} p_k g_k + o(K) \right)$$

$$= M \log \left( \frac{1}{M} \sum_{k=1}^{K} p_k g_k \right) + o(1).$$

Combining (8) and (9), we have that when $K \to \infty$

$$C = M \log \left( \frac{1}{M} \sum_{k=1}^{K} p_k g_k \right) + o(1).$$

This theorem indicates that the system sum-rate capacity is only determined by $\bar{g}_k$ when $K$ is sufficiently large. In the proof of Theorem 1 (see Appendix A), we have assumed that MU $k$ only transmits in the maximum eigenmode direction. The optimal solution to (3) generally requires each MU to transmit in multiple eigenmode directions. Thus Theorem 1 implies a simple and asymptotically optimal alternative.

**C. Correlation Gain**

We define the following four extreme scenarios of antenna correlation:

- No correlation (NC): $R_k = I_M$ and $T_k = I_M, \forall k$.
- Full correlation at MUs (FCMU): $T_k$ is a rank-1 matrix and $R_k = I_M, \forall k$.
- Full correlation at the BS (FCBS): $R_k$ is a rank-1 matrix and $T_k = I_M, \forall k$.
- Full correlation (FC): Both $T_k$ and $R_k$ are rank-1 matrices, $\forall k$.

Denote by $\bar{g}_{4,NC}, \bar{g}_{4,FCMU}, \bar{g}_{4,FCBS}$ and $\bar{g}_{4,FC}$, respectively, $\bar{g}_k$ in the scenarios of NC, FCMU, FCBS and FC. To quantify the benefit of antenna correlation, the correlation gain (in rate) is defined as

$$G = C - C_{NC}$$

where $C_{NC}$ is obtained by $C$ in the NC scenario. Substituting (11) into (12), we have that for sufficiently large $K$

$$G = M \log \left( \sum_{k=1}^{K} p_k \bar{g}_k \right) + o(1).$$

Let $\bar{s}_k$ be the mean of $s_k, \forall k$.

**Theorem 2:**

$$\bar{g}_{4,NC} \leq \bar{g}_k \leq \bar{g}_{4,FC} = \bar{g}_{4,FCMU} = \bar{g}_{4,FCBS} = \bar{s}_k \cdot MN.$$ 

The proof of this theorem is shown in Appendix C. Based on (13) and Theorem 2, we have $G \geq 0$. This indicates that when $K \to \infty$, antenna correlation can provide a sum-rate capacity gain for all power range and such a gain grows approximately linearly with $M$ (note that both of $\bar{g}_k$ and $\bar{g}_{4,NC}$ are approximately proportional to $M$ [12]). In particular, the maximum correlation gain is
\[
\max_{\{\tau_k, \nu_k\}} G = M \log \left( \frac{\sum_{i=1}^{K} p_i \sum_{k} \tau_k}{\sum_{i=1}^{K} p_i \sum_{k} \nu_k} / M \right) + o(1) \tag{14}
\]

where the maximum is achieved when \(\tau_k = \bar{\sigma}_{k,\text{FCMU}} = \bar{\sigma}_{k,\text{FCBS}} = \bar{\sigma}_{k,\text{FC}} \cdot MN\) (i.e., the scenario of FCMU, FCBS or FC).

Equation (14) indicates that the maximum correlation gain grows approximately linearly with \(M\) and logarithmically with \(N\).

D. Numerical Illustrations

We now provide some numerical results. Figure 1 shows the average (over the distribution of \(\{H_{ik}\}\)) sum-rate capacities of MIMO systems with \(M = 8\), \(N = 4\) and different \(K\). Specifically, Figs. 1(a), 1(b) and 1(c), respectively, compare NC with FCMU, FCBS and FC.

To construct the rank-1 matrix \(R_k\) in Fig. 1, we use the complex exponential correlation model characterized by \([6][12]\)

\[
R_k = \begin{bmatrix}
1 & e^{-j\theta_k} & \ldots & (e^{-j\theta_k})^{M-1} \\
\vdots & 1 & \ldots & (e^{-j\theta_k})^{M-2} \\
(e^{-j\theta_k})^{M-1} & (e^{-j\theta_k})^{M-2} & \ldots & 1
\end{bmatrix}
\tag{15}
\]

where \(\theta_k\) is randomly generated from a uniform distribution over \([0, 2\pi)\), \(\forall k\). The rank-1 matrix \(T_k\), \(\forall k\) can be constructed similarly. All MUs are independent and uniformly distributed in an edge-length-1 hexagon cell with fourth power path-loss law and the standard deviation of their normalized lognormal fading is 8. We assume a power control scheme that is widely used in code-division multiple-access systems [13], i.e., the maximum transmitted power of MU \(k\) in (3) is controlled by its slow fading (i.e., pass loss and lognormal fading in (2)). The curves for \(K\) of \(K\) are obtained by (11).

It is observed from Fig. 1 that when \(K\) is finite, there is a cross point for each pair of curves representing NC and FCMU/FCBS/FC; below this point, correlation is advantageous and vice versa. The corresponding sum-rate capacities of the cross points increase with \(K\), indicating that the advantage of antenna correlation becomes more noticeable when \(K\) increases. In the limiting case of \(K \to \infty\), FCMU/FCBS/FC always outperforms NC, which agrees with (14). Actually, the capacity gain brought by antenna correlation is significant. For example, as shown in Fig. 1(a), the corresponding gain is about 2.1 bits/symbol for \(K = 4\) when the transmitted sum power is 4 dB.

We explain the above observations as follows. Intuitively, antenna correlation has two consequences. First, it reduces DOF; the related disadvantage is most noticeable in the high rate/power regime. Second, it increases the potential of beamforming gain; the related advantage is most noticeable in the low rate/power regime. These two opposite effects result in a cross point mentioned above. In a conventional single-user MIMO system, such a point occurs at a relatively low information rate [4], so antenna correlation has been generally regarded as a detrimental factor. However, as seen from Fig. 1, antenna correlation can be potentially beneficial in a multi-user environment. In this case, the positive effect of antenna correlation becomes dominant since its negative effect (i.e., loss of DOF) can be compensated by the spatial diversity related to different locations of multiple MUs.

Note that even with full correlation, the locations of multiple MUs can still provide spatial diversity. For FC considered in Fig. 1(c), the loss of DOF becomes serious so that the cross points in Fig. 1(c) are lower than their counterparts in Figs 1(a) and 1(b).
Although only full correlation (at MUs, the BS or both) is compared in Fig. 1, we have observed in our simulations that partial correlation can still provide a capacity gain.

III. BCs

A. Preliminary

We now turn to the scenario of BCs in this section. We still use $K, M$ and $N$ to denote the number of MUs in the system, and the numbers of antennas at the BS and each MU, respectively. We again denote by $H_k$ the channel matrix between the BS and MU $k$ without confusion. The received vector $y_k$ at MU $k$ can be written as

$$y_k = H_k x + n_k$$

(16)

where $x$ is the transmitted vector of the BS and $n_k$ a vector of complex AWGN samples with zero mean and unit variance. Similar to the MAC, $\{H_k\}$ are independent among all MUs and perfectly known at both the transmitter and the receiver sides.

To guarantee fairness among different MUs in BCs, we assume that the large-scale fading of all MUs are the same and normalized to 1 (i.e., $s_k = 1, \forall k$). The common Kronecker correlation model in (2) is still used. Note that in this case, $T_k$ and $R_k$ with $\text{tr}(T_k) = M$ and $\text{tr}(R_k) = N$ represent correlation effects at the BS side and MU side for MU $k$, respectively.

Based on the duality principle between MIMO BCs and MACs [10], the sum-rate capacity of a BC can be written as

$$C = \max_{\{Q_k\}} \log \det \left( I_M + \sum_{k=1}^K H_k^* Q_k H_k \right)$$

subject to

$$\sum_{k=1}^K \text{tr}(Q_k) \leq P$$

(17)

where $P$ is the transmitted sum power constraint for the BS, and $Q_k$ the input covariance matrix of MU $k$ in the dual MAC. Note that the difference between problem (3) and (17) is the constraint. The former is based on individual power constraints; while the later is based on a sum power constraint.

B. Numerical illustrations

Since there is generally no closed-form solution to (17), we first show some numerical results to illustrate the impact of antenna correlation on MIMO BCs. Figure 2 shows the average (over the distribution of $\{H_k\}$) sum-rate capacities of MIMO BCs with $M = N = 4$ and different $K$. Specifically, Figs. 2(a), 2(b) and 2(c) compare NC with FCMU, FCBS and FC, respectively. $\{T_k\}$ and $\{R_k\}$ for these scenarios are constructed using the same method in Fig. 1.

The observations from Fig. 2 are very similar to those from Fig. 1. This indicates that antenna correlation at MUs, BS or both is also potentially beneficial for MIMO BCs. Similar observations (not shown here) can been made for partial correlation.

C. Asymptotic Analysis

Due to the lack of closed-form expressions for (17), we resorted to numerical results in the last subsection. However, in some asymptotic situations for $K \to \infty$, concise closed-form expressions can be obtained as shown below, which may provide more insights into the problem.

**Theorem 3:** When $K \to \infty$, the sum-rate capacities of MIMO BCs in the scenarios of FCMU, FCBS and FC scale like

$$C_{\text{FCMU}} = M \log \frac{NP}{M} + M \log \log K + o(1),$$

(18a)

$$C_{\text{FCBS}} = M \log P + M \log \log K + o(1),$$

(18b)

Fig. 2. Average sum-rate capacities of MIMO systems over BCs with different $K, M = N = 4$. The numbers of MUs $K$ are marked on the curves.
and
\[ C_{\text{NC}} = M \log NP + M \log K + o(1), \]
respectively.

Similar to the proof of Theorem 1, this theorem can also be proved using the bounding technique, where for the lower bound \( M \) selected MUs only transmit in their maximum eigenmode directions. Due to space limitation, we omit the details of the proof.

It is shown in [5] that for a BC in the NC scenario, the sum-rate capacity behaves like
\[ C_{\text{NC}} = M \log \frac{P}{M} + M \log K + o(1). \]

Based on Theorem 3 and (19), the following asymptotic correlation gain for the FCMU, FCBS and FC scenarios can be shown when \( K \to \infty \).
\[
G_{\text{FCMU}} = M \log N + o(1), \quad (20a)
\]
\[
G_{\text{FCBS}} = M \log M + o(1) \quad (20b)
\]
and
\[
G_{\text{FC}} = M \log MN + o(1). \quad (20c)
\]

From (20), we can see that the correlation gain is positive in a multiple antenna system with \( K \to \infty \). This indicates that FCBS/FCMU/FC always outperforms NC when \( K \) is sufficiently large, regardless of the transmitted sum power \( P \). Furthermore, the correlation gain for FC is larger than those for FCMU or FCBS, and such a gain scales like \( \log M \) with \( M \), which is different from the behavior for MACs (as shown in (14)). The reason is explained as follows. Antenna correlation can increase the variance of the channel gain, especially for FC. With the sum power constraint in BCs, user scheduling is available (i.e., the BS transmits to some MUs with better channel conditions), and thus variance increase becomes an advantage (due to the increased probability of the large channel gain) [9].

### IV. RATE CONSTRAINTS

This section is concerned with systems with rate constraints. We assume that each MU must transmit a certain amount of information within a fixed time period, which applies to delay sensitive services such as speech and real-time video [14][15]. In this case, minimizing the system sum power is an appropriate objective [14].

It is shown in [15] that antenna correlation at MUs is potentially advantageous to MIMO systems with rate constraints in multi-user environments. Actually, this advantage can also be observed for the scenario of antenna correlation at the BS. We can define the correlation gain (in power) for the case with rate constraints as
\[ G = P_{\text{NC}} / P. \]

where \( P \) is the minimum transmitted sum power under the rate constraints, and \( P_{\text{NC}} \) is obtained by \( P \) in the NC scenario. The following theorem is an extension of the results in [15].

**Theorem 4:** For multi-user MIMO systems with rate constraints and no correlation at the BS, when \( M \to \infty \), the correlation gain is always larger than or equal to 1. The maximum correlation gain is \( N \), which is achieved by FCMU.

We omit the proof of this theorem due to space limitation.

### V. CONCLUSIONS

This paper shows that antenna correlation can bring a potential performance gain for multi-user MIMO systems over MACs or BCs. Such a gain mainly comes from two aspects. First, the spatial diversity related to user locations compensates the loss of DOF due to antenna correlation. Second, antenna correlation increases the potential of beamforming gain. The findings in this paper may provide useful guidelines for the future communication system design.

### APPENDIX

**A. Proof of (4) in Lemma 1**

Let \( v_k \) be the right singular vector corresponding to the maximum singular value of \( H_k \) in (1). Consider a specific transmission strategy, in which each MU only transmits in the maximum eigenmode direction, i.e.,
\[ x_k = v_k \cdot m_k \]
where \( m_k \) is the transmitted symbol with \( E\{|m_k|^2\} = p_k \). The resultant input covariance matrix for MU \( k \) is given by
\[ Q_k = E\{x_k x_k^*\} = p_k \cdot v_k v_k^*. \]

Based on (A-2), the achievable sum rate for this transmission strategy can be written as
\[ \log \det \left( I_M + \sum_{k=1}^{K} P_k \cdot H_k v_{k,\text{max}}^* H_k^* \right) \]
\[ = \log \det \left( I_M + \sum_{k=1}^{K} p_k g_k u_k u_k^* \right) \leq C. \quad (A-3) \]

The last inequality is due to the fact that the transmission strategy above is a particular realization technique, which completes the proof.

**B. Proof of (5) in Lemma 1**

For the system in (1), denote by \( x_k^{\text{opt}}, p_k \) and \( Q_k^{\text{opt}} \), respectively, the optimal transmitted vector, the corresponding transmitted power and input covariance matrix of MU \( k \), i.e., \( p_k = \| x_k^{\text{opt}} \|^2 \), \( Q_k^{\text{opt}} = E\{x_k^{\text{opt}} (x_k^{\text{opt}})^*\} \) and
\[ C = \log \det \left( I_M + \sum_{k=1}^{K} H_k Q_k^{\text{opt}} H_k^* \right). \quad (A-4) \]

Let \( d_k \) be the maximum singular value of \( H_k \) in (1). Consider the following MAC systems,
\[ \bar{y} = \sum_{k=1}^{K} (d_k I_M) \cdot \bar{x}_k + n. \quad (A-5) \]

In (A-5), MU \( k \) sees \( M \) parallel single-input single-output sub-channels with equal gain \( d_k^2 = g_k \). Then its sum-rate capacity with the transmitted power \( \{p_k\} \) is
\[ \bar{C}(\{p_k\}) = M \log \left( 1 + \frac{1}{M} \sum_{k=1}^{K} p_k g_k \right). \quad (A-6) \]

Based on \( x_k^{\text{opt}} \), construct
\[ \bar{x}_k = (H_k / d_k) \cdot x_k^{\text{opt}}. \quad (A-7a) \]

Then the input covariance matrix and the transmitted power of MU \( k \) for such a transmission strategy are, respectively,
\[ \tilde{Q}_k = E \left( \tilde{x}_k \tilde{x}_k^* \right) = \left( H_k / d_k \right) \cdot Q_{\text{opt}} \cdot \left( H_k / d_k \right)^* \]  \hspace{1cm} (A-7b) \]

and
\[ \tilde{p}_k = \left| \tilde{x}_k \right| \leq \left( \left| H_k \right| / g_k \right), \quad \left| \tilde{x}_k \right| = p_k \]  \hspace{1cm} (A-7c) \]

where the last equality in (A-7c) is due to the fact that \( g_k \) is the maximum eigenmode for MU \( k \). From (A-7b), the achievable sum rate for this strategy is
\[ R\left( \{ \tilde{p}_k \} \right) = \log \det \left( I_{N_k} + \sum_{k=1}^{K} \left( d_k I_{N_k} \right) \tilde{Q}_k \left( d_k I_{N_k} \right)^* \right) \]
\[ = \log \det \left( I_{N_k} + \sum_{k=1}^{K} H_k Q_{\text{opt}} H_k^* \right) = C. \]  \hspace{1cm} (A-8) \]

Following (A-6), we can also obtain the sum-rate capacity for the system in (A-5) with the transmitted power \( \{ \tilde{p}_k \} \):
\[ \tilde{C} \left( \{ \tilde{p}_k \} \right) = M \log \left( 1 + \frac{1}{M} \sum_{k=1}^{K} \tilde{p}_k g_k \right) \]  \hspace{1cm} (A-9) \]

To complete the proof of (5) in Lemma 1, we only need to verify the following two inequalities.
\[ C = R\left( \{ \tilde{p}_k \} \right) \leq \tilde{C} \left( \{ \tilde{p}_k \} \right) \leq \tilde{C} \left( \{ p_k \} \right). \]  \hspace{1cm} (A-10) \]

The first inequality holds because the transmission strategy in (A-7) is a specific realization technique; while the second inequality is based on the fact that \( \tilde{p}_k \leq p_k, \forall k \) (shown in (A-7c)) and \( \log(\cdot) \) is a monotonously increasing function.

C. Proof of Theorem 2

We first prove the left inequality of Theorem 2. Based on the definition of maximum eigenvalue, the following can be shown. For a given \( R_k \),
\[ \overline{g}_k = E \left\{ \lambda_{\max} \left( H_k H_k^* \right) \right\} \]
\[ = E \left\{ s_k \cdot \max_{z} \left( z R_z^2 W_z T_z \left( R_z^2 W_z \right)^* z^* \right) \right\} \]
\[ \geq E \left\{ s_k \cdot w_j \left( I_{N_k} \right) T_j \left. W_k \right| \left. T_k \right| w_j \right\} \]

where \( z \) is an arbitrary unit vector. The last inequality is based on the fact that a proper vector is selected so that \( w_k \) is the maximum singular value of \( R_k^2 W_k \), and \( I_k \) is the corresponding right singular vector (actually, the proper selected vector is the left singular vector of \( R_k^2 W_k \) corresponding to \( w_k \)). Since the amplitudes of the entries of the unit vector \( I_k \) are identically distributed and \( \text{tr}(T_k) = N \), we can obtain
\[ E \left\{ s_k \cdot w_j \left( I_{N_k} \right) T_j \left. W_k \right| \left. T_k \right| w_j \right\} = E \left\{ s_k \cdot w_j^2 \right\} \]
\[ = E \left\{ \lambda_{\max} \left( H_k H_k^* \right) \right\}_{I_{N_k}=I_{N_k}} \]  \hspace{1cm} (A-12) \]

Therefore, for a given \( R_k \), we have
\[ \overline{g}_k \geq E \left\{ \lambda_{\max} \left( H_k H_k^* \right) \right\}_{I_{N_k}=I_{N_k}} \]  \hspace{1cm} (A-13) \]

Similarly, for a given \( T_k \)
\[ \overline{g}_k \geq E \left\{ \lambda_{\max} \left( H_k H_k^* \right) \right\}_{I_{N_k}=I_{N_k}} = E \left\{ \lambda_{\max} \left( H_k H_k^* \right) \right\}_{I_{N_k}=I_{N_k}} \]  \hspace{1cm} (A-14) \]

From (A-13) and (A-14), the following can be shown.
\[ \overline{g}_k \geq E \left\{ \lambda_{\max} \left( H_k H_k^* \right) \right\}_{I_{N_k}=I_{N_k}} = \overline{g}_k, \]  \hspace{1cm} (A-15) \]

This proves the left inequality of Theorem 2.

Next we prove the right inequality of Theorem 2. According to the normalizations \( \text{tr}(T_k) = N \) and \( \text{tr}(R_k) = M \), we have
\[ \sum_k E \{ \lambda_i (H_k H_k^*) \} = E \{ \text{tr}(H_k H_k^*) \} = \overline{g}_k \cdot MN. \]  \hspace{1cm} (A-16) \]

where \( \lambda_i(\cdot) \) denotes the \( i \)th eigenvalue of a Hermitian matrix. Since \( \lambda_i(H_k H_k^*) \geq 0, \forall i, \overline{g}_k \) is bounded by \( \overline{g}_k \cdot MN \). Such a bound is achieved by setting \( T_k \) and/or \( R_k \) to be a rank-1 matrix, i.e.,
\[ \overline{g}_k \leq \overline{g}_k,_{FC} = \overline{g}_k,_{FC} = \overline{g}_k \cdot MN, \]  \hspace{1cm} (A-17) \]

which ends the proof.

ACKNOWLEDGMENT

The authors thank Alcatel-Lucent Shanghai Bell Co. Ltd for supporting the research work leading to this paper and appreciate the beneficial discussions with the researchers in the Research and Innovation Center of Alcatel-Lucent Shanghai Bell.

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