Gain from Antenna Correlation in Multi-User MIMO Systems

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***Abstract*— This paper is concerned with multi-user multiple-input multiple-output (MIMO) systems under correlated fading. Contrary to common views, we show that antenna correlation can be beneficial for multi-user MIMO systems with either power or rate constraints. Both asymptotic analyses and numerical results indicate that antenna correlation can provide a significant performance gain in such environments.**

I. Introduction

Multiple-input multiple-output (MIMO) systems provide a promising solution to enhance the performance of wireless communication systems [1]-[4]. Independent and identically distributed (i.i.d.) fading at antenna elements is a common assumption in the research work related to MIMO systems [1][2]. However, antenna correlation does exist in practice [3][4][6]-[8]. This may be caused by limited physical sizes of transmitters/receivers or by the lack of sufficient scatters in the transmission environments. Antenna correlation is commonly regarded as a negative factor since it may result in reduced degrees of freedom (DOF) [3][4].

This paper is concerned with multi-user MIMO systems. Our focus is on multiple access channels (MACs), but the results can be extended to broadcast channels straightforwardly using the duality principle [5]. We will show numerically that below a certain cross point, antenna correlation can actually lead to a performance improvement. Furthermore, such a point occurs at a rate increasing with the number of mobile units (MUs) (denoted by K below), so the range where antenna correlation is beneficial increases with K. We also quantify this benefit analytically in the limiting case of $K\to\infty$. The results in this paper indicate that antenna correlation can provide a potential gain in a multi-user environment.

Our discussions are based on capacity analysis, which distinguishes this paper from some existing work on related issues. For example, [6], [7] and [8] are for specific transmission strategies in time-division multiple-access format. The advantage of antenna correlation reported in [6]-[8] results purely from beamforming gain[1]; while that reported in this paper results from, besides beamforming gain, the space diversity related to the locations of multiple MUs.

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[1] Beamforming gain, a common term for antenna arrays, is used in this paper to refer to the advantage of the effect that the transmitted power is concentrated in some particular directions.

The notations used in this paper are as follows. For any matrix $\boldsymbol{A}$, $\boldsymbol{A}^*$ and $\|\boldsymbol{A}\|$ are the conjugate transpose and 2-norm of $\boldsymbol{A}$, respectively. For a Hermitian matrix $\boldsymbol{A}$, $\text{tr}(\boldsymbol{A})$, $\boldsymbol{A}^{1/2}$ and $\lambda_{\max}(\boldsymbol{A})$ denote, respectively, its trace, principal square root and maximum eigenvalue. $\boldsymbol{I}_M$ represents an $M \times M$ identity matrix. $\text{E}\{\cdot\}$ is the expectation operator.

II. System Model

Consider a K-user MIMO system over a flat MAC with M antennas at the base station (BS) and N antennas at each MU. Denote by $\boldsymbol{H}_k$ and $\boldsymbol{x}_k$ the channel matrix and the transmitted vector of MU k. The received vector $\boldsymbol{y}$ is given by

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{n}$ is a vector of complex additive white Gaussian noise samples with zero mean and unit variance. $\{\boldsymbol{H}_k\}$ in (1) are perfectly known at both the transmitters and the receiver.

Assumption 1: $\{\boldsymbol{H}_k\}$ are i.i.d. among different MUs.

Following the common Kronecker correlation model in [4][6], $\boldsymbol{H}_k$ can be written as

$$\boldsymbol{H}_k = \sqrt{s_k}\, \boldsymbol{R}_k^{1/2} \boldsymbol{W}_k \boldsymbol{T}_k^{1/2} \tag{2}$$

where s_k is a scalar denoting the gain of path loss and lognormal fading[2], and $\boldsymbol{W}_k$ is a Rayleigh fading matrix whose entries are i.i.d. complex circular symmetric Gaussian variables with zero mean and unit variance. $\boldsymbol{T}_k$ and $\boldsymbol{R}_k$ in (2) are semi-positive definite matrices that characterize correlation effects of antennas at the transmitter and the receiver, respectively, for the link between MU k and the BS. The normalizations $\text{tr}(\boldsymbol{T}_k) = N$ and $\text{tr}(\boldsymbol{R}_k) = M$ are assumed throughout this paper.

III. Power Constraints

We first consider systems with power constraints, where we want to maximize the system sum rate for each channel realization [2]. The problem (i.e., calculating the system sum-rate capacity) is formulated as

$$C = \max_{\{\boldsymbol{Q}_k\}} \log\det\left(\boldsymbol{I}_M + \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{Q}_k \boldsymbol{H}_k^*\right)$$
$$\text{subject to} \qquad \text{tr}(\boldsymbol{Q}_k) \le p_k, \forall k \tag{3}$$

where $\boldsymbol{Q}_k \equiv \text{E}\{\boldsymbol{x}_k \boldsymbol{x}_k^*\}$ and p_k are the input covariance matrix and the maximum transmitted power of MU k, respectively.

[2] We assume equal path loss and lognormal fading for all the antenna links seen by a particular MU.

A. Asymptotic Analysis

In general, there is no closed-from solution to (3), although it can be evaluated numerically [2]. In this subsection, we analyze the problem for the asymptotic case with $K\to\infty$. The results from this special case can provide useful insights into the problem. Let $g_k \equiv \lambda_{\max}(\boldsymbol{H}_k\boldsymbol{H}_k^*)$ and $\boldsymbol{u}_k$ be the left-singular column vector corresponding to the maximum singular value of $\boldsymbol{H}_k$. The discussions below show that the system sum-rate capacity is primarily determined by the means of $\{g_k\}$ when $K\to\infty$.

***Lemma 1**:* For each channel realization, the sum-rate capacity in (3) is lower-bounded and upper-bounded by

$$C \ge \log\det\left(\boldsymbol{I}_M + \sum_{k=1}^{K} p_k g_k \boldsymbol{u}_k \boldsymbol{u}_k^*\right) \tag{4}$$

and

$$C \le M\log\left(1+\frac{1}{M}\sum_{k=1}^{K} p_k g_k\right). \tag{5}$$

respectively.

The proof of this lemma is given in Appendix.

Assumption 2: The entries $\{u_{k,i}\}$ of $\boldsymbol{u}_k$ are identically distributed, their phases are uniformly distributed.

Based on Assumption 2, the following can be shown.

$$\mathrm{E}\left\{\left|u_{k,i}\right|^2\right\} = 1/M, \forall k, i, \tag{6a}$$

and

$$\mathrm{E}\left\{u_{k,i}u_{k,j}^*\right\} = 0, \forall k \text{ and } \forall i \ne j. \tag{6b}$$

Define a matrix $\boldsymbol{B}$ as

$$\boldsymbol{B} = \sum_{k=1}^{K} p_k g_k \boldsymbol{u}_k \boldsymbol{u}_k^*.$$

Let b_{ij} be the entry in the ith row and jth column of $\boldsymbol{B}$. When $K\to\infty$, from the Strong Law of Large Numbers (SLLN) [9, page 27, Theorem C] [3] and (6), we have ($\{g_k\}$ are fixed firstly)

$$b_{ii} = \sum_{k=1}^{K} p_k g_k \left|u_{k,i}\right|^2 \to \sum_{k=1}^{K} p_k g_k / M, \forall i \tag{7a}$$

and

$$b_{ij} = \sum_{k=1}^{K} p_k g_k u_{k,i} u_{k,j}^* \to 0, \forall i \ne j. \tag{7b}$$

If a random variable ξ converges to a deterministic number a, then any function $f(\xi)$ of ξ converges to $f(a)$ [9, page 24]. Then, from (4) in Lemma 1 and (7), we can obtain

$$\lim_{K\to\infty} C \ge \lim_{K\to\infty} \log\det\left(\boldsymbol{I}_M + \left(\frac{1}{M}\sum_{k=1}^{K} p_k g_k\right)\cdot \boldsymbol{I}_M\right)$$
$$= \lim_{K\to\infty} M\log\left(1+\frac{1}{M}\sum_{k=1}^{K} p_k g_k\right). \tag{8}$$

On the other hand, based on (5) in Lemma 1, the following inequality can be shown.

$$\lim_{K\to\infty} C \le \lim_{K\to\infty} M\log\left(1+\frac{1}{M}\sum_{k=1}^{K} p_k g_k\right). \tag{9}$$

Combining (8) and (9), we have

$$\lim_{K\to\infty} C \doteq M\log\left(1+\frac{1}{M}\sum_{k=1}^{K} p_k g_k\right). \tag{10}$$

where $\doteq$ denotes "equal for an asymptotically large K". Let $\bar{g}_k \equiv \mathrm{E}\{g_k\}$. From the SLLN, we can show the following theorem.

***Theorem 1**:* For the system in (1) with power constraints,

$$\lim_{K\to\infty} C \doteq M\log\left(1+\frac{1}{M}\sum_{k=1}^{K} p_k \bar{g}_k\right). \tag{11}$$

This theorem indicates that the system sum-rate capacity is determined by $\bar{g}_k$ only when K is sufficiently large. In the proof of Theorem 1 (see Appendix), we have assumed that MU k only transmits in the maximum eigenmode direction. The optimal solution of (3) generally requires each MU to transmit in multiple eigenmode directions. Thus Theorem 1 implies a simple and asymptotically optimal alternative.

B. Correlation Gain

A consequence of Assumption 1 is that $\{g_k\}$ are i.i.d. Then (11) becomes

$$\lim_{K\to\infty} C \doteq M\log\left(1+\frac{P_{\text{sum}}\bar{g}}{M}\right) \tag{12}$$

where $\bar{g} = \bar{g}_k$, $\forall k$ that is independent of k, and $P_{\text{sum}} \equiv \Sigma_k\, p_k$ is the transmitted sum power of all MUs.

We define the following four extreme scenarios of antenna correlation:

- No correlation (NC): $\boldsymbol{R}_k = \boldsymbol{I}_M$ and $\boldsymbol{T}_k = \boldsymbol{I}_N$, $\forall k$.
- Full correlation at MUs (FCMU): $\boldsymbol{T}_k$ is a rank-1 matrix and $\boldsymbol{R}_k = \boldsymbol{I}_M$, $\forall k$.
- Full correlation at the BS (FCBS): $\boldsymbol{R}_k$ is a rank-1 matrix and $\boldsymbol{T}_k = \boldsymbol{I}_N$, $\forall k$.
- Full correlation (FC): Both $\boldsymbol{T}_k$ and $\boldsymbol{R}_k$ are rank-1 matrices, $\forall k$.

Denote by $\bar{g}_{\text{NC}}$, $\bar{g}_{\text{FCMU}}$, $\bar{g}_{\text{FCBS}}$ and $\bar{g}_{\text{FC}}$, respectively, for $\bar{g}$ in the scenarios of NC, FCMU, FCBS and FC. The so-called correlation gain (in rate) is defined as

$$G = \lim_{K\to\infty}\left(C - C_{\text{NC}}\right) \tag{13}$$

where C_{NC} is obtained by C in the NC scenario. Substituting (12) into (13), we have

$$G \doteq M\log\left(\frac{1+P_{\text{sum}}\bar{g}/M}{1+P_{\text{sum}}\bar{g}_{\text{NC}}/M}\right). \tag{14}$$

Theorem 2: $\bar{g}_{\text{NC}} \le \bar{g} \le \bar{g}_{\text{FC}} = \bar{g}_{\text{FCMU}} = \bar{g}_{\text{FCBS}} = MN$.

We omit the proof of Theorem 2 due to the space limitation. Based on (14) and Theorem 2, $G \ge 0$. This indicates that when $K\to\infty$, antenna correlation can provide a sum-rate capacity gain for all power range and such a gain grows approximately linearly with M.

From Theorem 2, we can obtain

$$\max_{\{\boldsymbol{T}_k\},\{\boldsymbol{R}_k\}} G \doteq M\log\left(\frac{1+P_{\text{sum}}N}{1+P_{\text{sum}}\bar{g}_{\text{NC}}/M}\right) > 0 \tag{15}$$

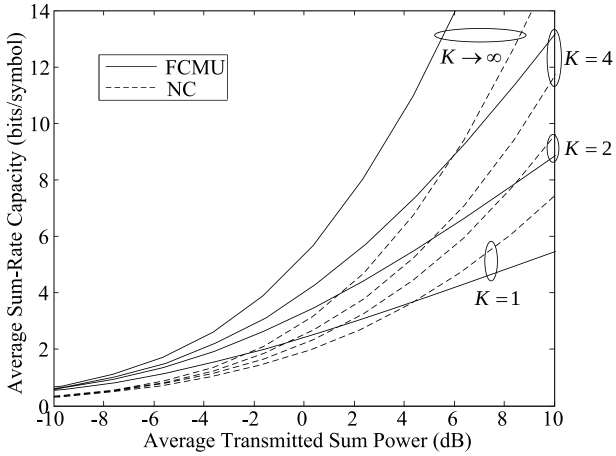
where the maximum is achieved when $\bar{g} = \bar{g}_{\text{FCMU}} = \bar{g}_{\text{FCBS}} = \bar{g}_{\text{FC}} = MN$ (i.e., the scenario of FCMU, FCBS or FC). The last

[3] This theorem states that the sum of independent and non-identically distributed random variables converges to the sum of the means of these variables.

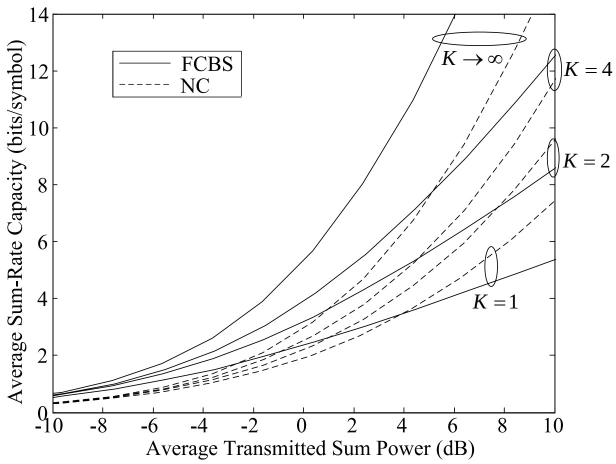
inequality in (15) is due to the fact that $\bar{g}_{\text{NC}} < MN$.

C. Numerical Illustrations

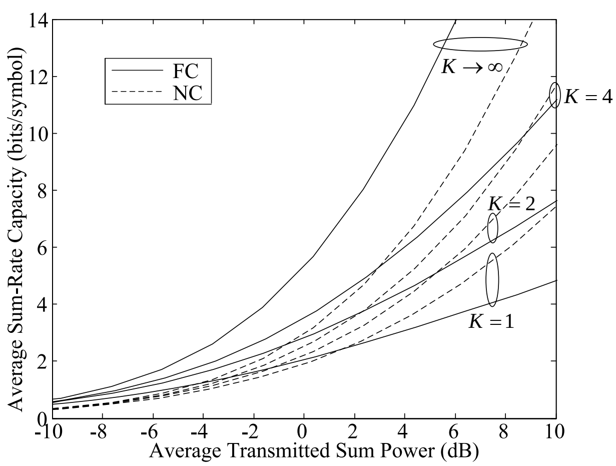
We now provided some numerical results. Figure 1 show the average (over the distribution of $\{\boldsymbol{H}_k\}$) sum-rate capacities of MIMO systems with $M = 8$, $N = 4$ and different K. Specifically, Fig.s 1(a), 1(b) and 1(c), respectively, compare NC with FCMU, FCBS and FC.



(a) Comparison between NC and FCMU



(b) Comparison between NC and FCBS



(c) Comparison between NC and FC

Fig. 1. Average sum-rate capacities of MIMO systems with different K. $M = 8$ and $N = 4$. The numbers of MUs K are marked on the curves.

To construct the rank-1 $\boldsymbol{R}_k$ in Fig. 1, we use the complex exponential correlation model characterized by

$$\boldsymbol{R}_k = \begin{bmatrix} 1 & e^{-j\theta_k} & \dots & (e^{-j\theta_k})^{M-1} \\ e^{j\theta_k} & 1 & & (e^{-j\theta_k})^{M-2} \\ \vdots & & \ddots & \\ (e^{j\theta_k})^{M-1} & (e^{j\theta_k})^{M-2} & \dots & 1 \end{bmatrix}$$

where θ_k is randomly generated from a uniform distribution over $[0, 2\pi)$, $\forall k$. The rank-1 $\boldsymbol{T}_k$, $\forall k$ can be constructed similarly. All MUs are independent and uniformly distributed in an edge-length-1 hexagon cell with fourth power path-loss law and the standard deviation of their normalized lognormal fading is 8. We assume a power control scheme that is widely used in code-division multiple-access systems [10], i.e., the maximum transmitted power of MU k in (3) is controlled by its slow fading (i.e., pass loss and lognormal fading in (2)).

It is observed in Fig. 1 that when K is finite, there is a cross point for each pair of curves representing NC and FCMU/FCBS/FC: below this point, correlation is advantageous and *vice versa*. We can also see that the corresponding sum-rate capacity of the cross point increase with K, i.e., the advantage of antenna correlation becomes more noticeable when K increases. In the limiting case of $K\to\infty$, FCMU/FCBS/FC always outperforms NC, which agrees with (15).

We explain the above observations as follows. Intuitively, antenna correlation has two consequences. First, it reduces DOF; the related disadvantage is most noticeable in the high rate/power regime. Second, it increases the potential of beamforming gain; the related advantage is most noticeable in the low rate/power regime. These two opposite effects result in a cross point mentioned above. In a conventional single-user MIMO system, such a point occurs at a relatively low information rate [4], so antenna correlation has been generally regarded as a detrimental factor, at least in a single-user environment. However, as seen from Fig. 1, antenna correlation can be potentially beneficial in a multi-user environment. In this case, the positive effect of antenna correlation becomes dominant since its negative effect (i.e., loss of DOF) can be compensated by the space diversity related to different locations of multiple MUs.

Note that even with full correlation, the locations of multiple MUs can still provide space diversity. For FC considered in Fig. 1(c), the loss of DOF becomes serious so that the cross points in Fig. 1(c) are lower than their counterparts in Fig.s 1(a) and 1(b).

Although only full correlation (at MUs, the BS or both) is compared in Fig. 1, we have observed in our simulations that partial correlation can still provide a capacity gain.

IV. Rate Constraints

We next consider systems with rate constraints. We assume that each MU must transmit a certain amount of information within a fixed time period, which applies to delay sensitive services such as speech and real-time video [11]. In this case, minimizing the system sum power is an appropriate objective.

Denote by R_k the rate constraint for MU k. The minimum transmitted sum power (MTSP) for each channel realization

$\{\boldsymbol{H}_k\}$ is given by [12]

$$P = \min_{\{\boldsymbol{Q}_k\}} \sum_{k=1}^{K} \mathrm{tr}(\boldsymbol{Q}_k)$$
$$\text{subject to} \quad \boldsymbol{R} \in C_{\mathrm{MAC}}(\{\boldsymbol{H}_k\},\{\boldsymbol{Q}_k\}) \qquad (16)$$

where $\boldsymbol{R} = [R_1, \ldots, R_k, \ldots, R_K]$ and $C_{\mathrm{MAC}}(\{\boldsymbol{H}_k\}, \{\boldsymbol{Q}_k\})$ is the capacity region of a MIMO MAC given $\{\boldsymbol{H}_k\}$ and $\{\boldsymbol{Q}_k\}$ [10].

A. *Asymptotic Analysis*

Similar to the case with power constraints, (16) generally has no closed-form solution but it can be evaluated numerically [12]. To gain insights into the problem, we again resort to asymptotic analysis. Our discussions are based on the following closed-form solution to (16) in the limiting case of $K\to\infty$ (derived in [11]).

$$\lim_{K\to\infty} P = \int_{\varepsilon}^{1} R \ln 2 \cdot 2^{R(t-\varepsilon)/M} \,/\, F^{-1}(t)\, dt \qquad (17)$$

where $F(\cdot)$ denotes the cumulative distribution function of g_k, $F^{-1}(\cdot)$ is its inverse, and $R \equiv \Sigma_k R_k$ is the system sum rate. ε in (17) will be explained in Section IV.C.

Equation (6) indicates that when K is sufficiently large, the performance is determined by the maximum eigenmode, which is similar to the case with power constraints (shown in (11)).

B. *Correlation Gain*

For the case with rate constraints, the correlation gain (in power) is defined as

$$G = \lim_{K\to\infty} \left(P_{\mathrm{NC}} / P\right). \qquad (18)$$

where P_{NC} is obtained by P in the NC scenario. The following theorem of G is for a special case, i.e., antenna correlation is only at MUs and M is sufficiently large.

Theorem 3: For the system in (1) with rate constraints and $\boldsymbol{R}_k = \boldsymbol{I}_M$, $\forall k$, when $M\to\infty$, the correlation gain in (18) is always larger than or equal to one. The maximum correlation gain is N, which is achieved by FCMU.

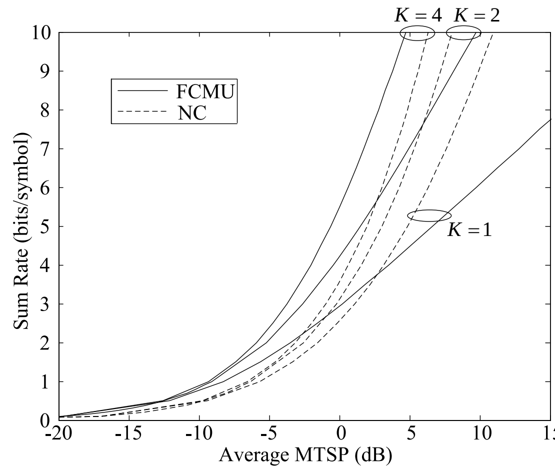
This theorem can be proved using (17) and the limiting distribution of g_k at $M\to\infty$. Due to the space limitation, we omit the details.

C. *Numerical Illustrations*

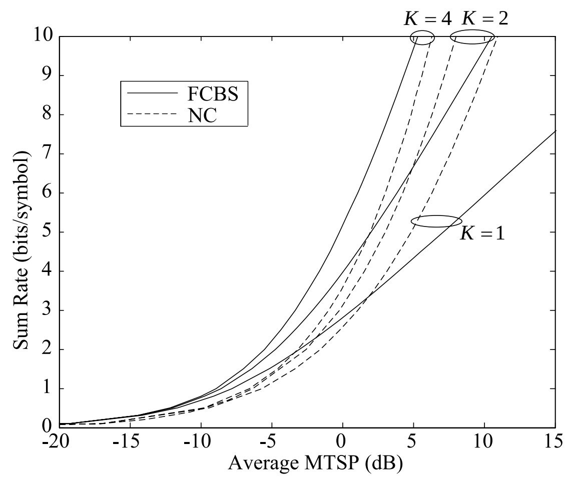
We plot in Fig. 2 the average (over the distribution of $\{\boldsymbol{H}_k\}$) performance of MIMO systems with rate constraints. NC is compared with FCMU and FCBS in Fig.s 2(a) and 2(b), respectively. The rate constraints for all MUs are set to be the same, i.e., $R_k = R/K$, $\forall k$. To avoid extremely large transmission power, a MU doesn't transmit under deep fading. The probability of such an event for each MU is set at $\varepsilon = 0.01$. Other system settings and the parameters of channel fading are the same as those of Fig. 1.

The observations from Fig. 2 are very similar to those from Fig.s 1(a) and 1(b). They indicate that antenna correlation at MUs or the BS is potentially beneficial for multi-user MIMO systems with rate constraints. Similar observations (not shown here) can been made for partial correlation.

In Fig. 3, we show the performance of MIMO systems with $K\to\infty$ for NC, FCMU, FCBS and FC. The system settings and the parameters of channel fading are the same as those of Fig. 2. It is seen that both FCMU and FCBS can bring about significant correlation gains (about 2.8 dB and 2.2 dB at the sum rate R = 4 bits/symbol, respectively).



(a) Comparison between NC and FCMU



(b) Comparison between NC and FCBS

Fig. 2. Average performance of MIMO systems with different K. M = 8, N = 4 and ε = 0.01. The numbers of MUs K are marked on the curves.

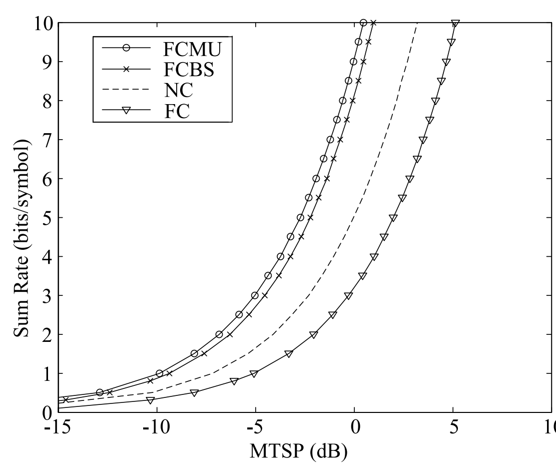


Fig. 3. Performance of MIMO systems with $K\to\infty$ for NC, FCMU, FCBS and FC. M = 8, N = 4 and ε = 0.01.

From Fig. 3, the performance of FC for $K\to\infty$ is very poor, and we have observed in our simulations that its performance is also poor for a finite K (not shown here). This is in contrast to the observation in Fig. 1(c) (i.e., the case with power constraints) where FC is beneficial for $K\to\infty$. The difference of these system behaviors is explained as follows. With power constraints, the performance for $K\to\infty$ is determined by the means of $\{g_k\}$ only. From Theorem 2, antenna correlation results in increased means of $\{g_k\}$, which is advantageous, as seen from Fig. 1(c). However, with rate constants, the performance is not determined by the means of $\{g_k\}$ only but by their distributions. Generally speaking, antenna correlation increases the variances of $\{g_k\}$, which can be disadvantageous (due to the increased probability of very small g_k). For FC, the variance increase of g_k is dominant, which leads to the performance loss in Fig. 3.

V. Conclusions

This paper shows that antenna correlation can bring about a potential gain for multi-user MIMO systems. Such a gain mainly comes from two aspects. First, the space diversity related to user locations compensates the loss of DOF due to antenna correlation. Second, antenna correlation increases the potential of beamforming gain. These findings may provide useful guidelines for the future communication system design.

Appendix

A. Proof of (4) in Lemma 1

Let $\boldsymbol{v}_k$ be the right-singular vector corresponding to the maximum singular value of $\boldsymbol{H}_k$ in (1). Consider a specific transmission strategy, in which each MU only transmits in the maximum eigenmode direction, i.e.,

$$\boldsymbol{x}_k = \boldsymbol{v}_k \cdot s_k \tag{A-1}$$

where s_k is the transmitted symbol with $\mathrm{E}\{|s_k|^2\} = p_k$. Then the resultant input covariance matrix for MU k is given by

$$\boldsymbol{Q}_k = \mathrm{E}\{\boldsymbol{x}_k\boldsymbol{x}_k^*\} = p_k \cdot \boldsymbol{v}_k\boldsymbol{v}_k^* . \tag{A-2}$$

Based on (A-2), the achievable sum rate for this transmission strategy can be written as

$$\begin{aligned}&\log\det\left(\boldsymbol{I}_M + \sum\nolimits_{k=1}^{K} p_k \cdot \boldsymbol{H}_k\boldsymbol{v}_{k,\max}\boldsymbol{v}_{k,\max}^*\boldsymbol{H}_k^*\right)\\ &= \log\det\left(\boldsymbol{I}_M + \sum\nolimits_{k=1}^{K} p_k g_k \boldsymbol{u}_k\boldsymbol{u}_k^*\right) \le C .\end{aligned} \tag{A-3}$$

The last inequality is due to the fact that the transmission strategy above is a particular realization technique, which completes the proof.

B. Proof of (5) in Lemma 1

For the system in (1), denote by $\boldsymbol{x}_k^{\mathrm{opt}}$, p_k and $\boldsymbol{Q}_k^{\mathrm{opt}}$, respectively, the optimal transmitted vector, the corresponding transmitted power and input covariance matrix of MU k, i.e., $p_k = \|\boldsymbol{x}_k^{\mathrm{opt}}\|^2$, $\boldsymbol{Q}_k^{\mathrm{opt}} = \mathrm{E}\{\boldsymbol{x}_k^{\mathrm{opt}}\cdot(\boldsymbol{x}_k^{\mathrm{opt}})^*\}$ and

$$C = \log\det\left(\boldsymbol{I}_M + \sum_{k=1}^{K}\boldsymbol{H}_k\boldsymbol{Q}_k^{\mathrm{opt}}\boldsymbol{H}_k^*\right) . \tag{A-4}$$

Let d_k be the maximum singular value of $\boldsymbol{H}_k$ in (1). Consider the following MAC systems:

$$\tilde{\boldsymbol{y}} = \sum_{k=1}^{K}\left(d_k\boldsymbol{I}_M\right)\cdot\tilde{\boldsymbol{x}}_k + \boldsymbol{n} \cdot \tag{A-5}$$

In (A-5), MU k sees M parallel single-input single-output sub-channels with equal gain $d_k^2 = g_k$. Then its sum-rate capacity with the transmitted power $\{p_k\}$ is

$$\tilde{C}\left(\{p_k\}\right) = M\log\left(1 + \frac{1}{M}\sum\nolimits_{k=1}^{K} p_k g_k\right) . \tag{A-6}$$

Based on $\boldsymbol{x}_k^{\mathrm{opt}}$, construct

$$\tilde{\boldsymbol{x}}_k = \left(\boldsymbol{H}_k / d_k\right)\cdot\boldsymbol{x}_k^{\mathrm{opt}} . \tag{A-7a}$$

Then the input covariance matrix and the transmitted power of MU k for such a transmission strategy are, respectively,

$$\tilde{\boldsymbol{Q}}_k = \mathrm{E}\left\{\tilde{\boldsymbol{x}}_k\tilde{\boldsymbol{x}}_k^*\right\} = \left(\boldsymbol{H}_k / d_k\right)\cdot\boldsymbol{Q}_k^{\mathrm{opt}}\cdot\left(\boldsymbol{H}_k / d_k\right)^* \tag{A-7b}$$

and

$$\tilde{p}_k = \left\|\tilde{\boldsymbol{x}}_k\right\|^2 \le \left(\left\|\boldsymbol{H}_k\right\|^2 / g_k\right)\cdot\left\|\boldsymbol{x}_k^{\mathrm{opt}}\right\|^2 = p_k \tag{A-7c}$$

where the last equality in (A-7c) is due to the fact that g_k is the maximum eigenmode for MU k. From (A-7b), the achievable sum rate for this strategy is

$$\begin{aligned}\tilde{R}\left(\{\tilde{p}_k\}\right) &= \log\det\left(\boldsymbol{I}_M + \sum_{k=1}^{K}\left(d_k\boldsymbol{I}_M\right)\tilde{\boldsymbol{Q}}_k\left(d_k\boldsymbol{I}_M\right)^*\right)\\ &= \log\det\left(\boldsymbol{I}_M + \sum_{k=1}^{K}\boldsymbol{H}_k\boldsymbol{Q}_k^{\mathrm{opt}}\boldsymbol{H}_k^*\right) = C .\end{aligned} \tag{A-8}$$

Following (A-6), we can also obtain the sum-rate capacity for the system in (A-5) with the transmitted power $\{\tilde{p}_k\}$:

$$\tilde{C}\left(\{\tilde{p}_k\}\right) = M\log\left(1 + \frac{1}{M}\sum\nolimits_{k=1}^{K}\tilde{p}_k g_k\right) . \tag{A-9}$$

To complete the proof of (5) in Lemma 1, we only need to verify the following two inequalities.

$$C = \tilde{R}\left(\{\tilde{p}_k\}\right) \le \tilde{C}\left(\{\tilde{p}_k\}\right) \le \tilde{C}\left(\{p_k\}\right) . \tag{A-10}$$

The first inequality holds because the transmission strategy in (A-7) is a specific realization technique; while the second inequality is based on the fact that $\tilde{p}_k \le p_k$, $\forall k$ (shown in (A-7c)) and $\log(\cdot)$ is a monotonously increasing function.

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