The Jointly Gaussian Approach to Iterative Detection in MIMO Systems[1]

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***Abstract*—This paper is concerned with signal estimation over general multiple-input-multiple-output (MIMO) channels based on a jointly Gaussian (JG) approach. We show that the proposed method is equivalent to the LMMSE approach, but is conceptually more concise and computationally more efficient. A 2×2 multi-antenna system with multipath effect is considered as an example to illustrate the advantages of the JG method. Also, a simple SNR-bounding evolution tool, similar to EXIT chart technique, is developed for fast assessment of the JG method. Compared with the optimal MAP algorithm, the JG method can achieve almost the same performance with greatly reduced complexity.**

***Keywords*—Turbo equalization, iterative detection, SNR Evolution, MIMO system.**

I. INTRODUCTION

Consider a general linear multiple-input-multiple-output (MIMO) system described by the following linear equation,

$$\boldsymbol{r} = \boldsymbol{Hx} + \boldsymbol{\eta}, \tag{1}$$

where $\boldsymbol{r}$ is a length-N observation vector, $\boldsymbol{H}$ a channel matrix known at the receiver, $\boldsymbol{x}$ a length-J transmitted information vector with its jth entry $x_j \in \{1,-1\}$, and $\boldsymbol{\eta}$ an additive noise vector with zero mean and covariance $\sigma^2\boldsymbol{I}$. Our discussion is limited to channels over the real field, but the results can be easily extended to complex channels since any complex linear matrix equation similar to (1) has an equivalent real representation obtained by equating the real and imaginary parts separately. The linear model in (1) covers a variety of communication systems, such as equalization [1], multiuser detection [2] and space-time coding systems [3].

We adopt the following system structure. (For example, see Fig.1 below.) The transmitter is composed of the channel encoder (ENC) and the random interleaver (π). The iterative receiver consists of two local processors, the elementary signal estimator (ESE) and the FEC decoder (DEC), respectively, and passes the log-likelihood information between each other iteratively to obtain refined inference. Since FEC decoding (usually based on the *a posteriori* probability (APP) principle) has been well-studied, our focus in this paper will be on the ESE for interference cancellation (i.e., handling the channel effect). The optimal realization of the ESE is the maximum *a posteriori* probability (MAP) detector, which has exponential complexity. The well-known linear minimum-mean-square-error (LMMSE) approach [2] can be used as an alternative realization of the ESE with complexity of quadratic order.

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In this paper, we consider a jointly Gaussian (JG) approach [4] to the realization of the ESE. The basic strategy is very simple: we focus on the detection of a particular transmitted symbol and treat all the other signals (and noise) as a jointly Gaussian random vector. We prove that the JG approach is equivalent to the LMMSE approach. We show that the JG approach offers several advantages:

- The JG approach is conceptually simple and straightforward. Its derivation is much more concise than the LMMSE approaches, as seen by comparing (7)-(8) with the derivation of the LMMSE equalizer in [1] and the LMMSE multiuser detector in [2].
- With a Cholesky factorization technique, the JG approach provides an option for efficient implementation.
- The JG method is easy to analyze. Inspired by the EXIT-chart technique [7], we propose a signal-to-noise-ratio (SNR) evolution bounding technique to predict the performance of the iterative detection process. This technique avoids time-consuming simulation.

The JG method provides a general framework for low-complexity, sub-optimal turbo detection in general linear MIMO systems. We use a multi-antenna system in multipath channels as an example to demonstrate the efficiency of the proposed method. This principle can also be extended to many other systems that can be characterized by the same model in (1).

II. THE LMMSE APPROACH

Assume that the entries of $\boldsymbol{x}$ are uncorrelated. Let the mean and variance of $\boldsymbol{x}$ be denoted by $\bar{\boldsymbol{x}} \equiv \mathrm{E}(\boldsymbol{x}) = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_J]^{\mathrm{T}}$ and $\boldsymbol{V} \equiv \mathrm{Cov}(\boldsymbol{x},\boldsymbol{x}) = \mathrm{Diag}\{v_1, v_2, \ldots, v_J\}$, respectively, where $\bar{x}_j$ is the mean of x_j, v_j the variance of x_j, and $\mathrm{Cov}(\cdot)$ the covariance. Let

$$\boldsymbol{R} \equiv \mathrm{Cov}(\boldsymbol{r},\boldsymbol{r}) = \boldsymbol{HVH}^{\mathrm{T}} + \sigma^2\boldsymbol{I}. \tag{2}$$

Further, we denote $\boldsymbol{R}_j$ as the covariance of $\boldsymbol{\xi}_j = \boldsymbol{r} - \boldsymbol{h}_j x_j$, i.e.,

$$\boldsymbol{R}_j = \boldsymbol{R} - v_j \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}} = \boldsymbol{H}\underline{\boldsymbol{V}}_j\boldsymbol{H} + \sigma^2\boldsymbol{I}, \tag{3}$$

where $\boldsymbol{h}_j$ denotes the jth column of $\boldsymbol{H}$, and $\underline{\boldsymbol{V}}_j$ is the same as $\boldsymbol{V}$ except that its jth diagonal entry is set to 0.

The LMMSE estimator is given by [2]

$$\hat{x}_j = \bar{x}_j + v_j \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}^{-1}(\boldsymbol{r} - \bar{\boldsymbol{r}}). \tag{4}$$

Equ. (4) can be rewritten in a desired-signal-plus-noise form as

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$$\hat{x}_j = u_j x_j + \eta_j \,, \tag{5}$$

where $u_j = v_j \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}^{-1} \boldsymbol{h}_j$ and $\eta_j = \hat{x}_j - u_j x_j$.

<u>Assumption I:</u> η_j in (5) is a Gaussian random variable for given x_j.

Based on the above assumption, the output extrinsic log-likelihood-ratio (LLR) of the ESE for x_j, as derived in [2], can be written as

$$\lambda_j^{\mathrm{MMSE}} = \ln \frac{p(\hat{x}_j \mid x_j = +1)}{p(\hat{x}_j \mid x_j = -1)} = 2 z_j \big/ (1-\mu_j) \tag{6}$$

where $z_j \equiv \boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{R}_j + \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}})^{-1} (\boldsymbol{r} - \bar{\boldsymbol{r}} + \boldsymbol{h}_j \bar{x}_j)$,

$$\mu_j \equiv \boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{R}_j + \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}})^{-1} \boldsymbol{h}_j \,.$$

One can refer to [2] for more details.

III. THE JG APPROACH

We now consider an alternative approach. Let us focus on the estimation of a particular bit x_j and treat all the other bits as interference. The signal model in (1) can be rewritten as

$$\boldsymbol{r} = \boldsymbol{h}_j x_j + \boldsymbol{\xi}_j \,, \tag{7}$$

where $\boldsymbol{h}_j$ is the jth column of $\boldsymbol{H}$ and $\boldsymbol{\xi}_j \equiv \sum_{i \neq j} \boldsymbol{h}_i x_i + \boldsymbol{\eta}$.

<u>Assumption II:</u> The entries of $\boldsymbol{\xi}_j$ in (7) form a set of jointly Gaussian random variables for given x_j.

Under the above assumption, the extrinsic LLR of x_j can be calculated as follows:

$$\begin{aligned}
\lambda_j^{\mathrm{JG}} &= \ln \frac{p(\boldsymbol{r} | x_j = +1)}{p(\boldsymbol{r} | x_j = -1)} \\
&= \ln \frac{\exp\left(-\frac{1}{2} (\boldsymbol{r} - \boldsymbol{h}_j - \mathrm{E}(\boldsymbol{\xi}_j))^{\mathrm{T}} \mathrm{Cov}(\boldsymbol{\xi}_j, \boldsymbol{\xi}_j)^{-1} (\boldsymbol{r} - \boldsymbol{h}_j - \mathrm{E}(\boldsymbol{\xi}_j)) \right)}{\exp\left(-\frac{1}{2} (\boldsymbol{r} + \boldsymbol{h}_j - \mathrm{E}(\boldsymbol{\xi}_j))^{\mathrm{T}} \mathrm{Cov}(\boldsymbol{\xi}_j, \boldsymbol{\xi}_j)^{-1} (\boldsymbol{r} + \boldsymbol{h}_j - \mathrm{E}(\boldsymbol{\xi}_j)) \right)} \\
&= 2 \boldsymbol{h}_j^{\mathrm{T}} \mathrm{Cov}(\boldsymbol{\xi}_j, \boldsymbol{\xi}_j)^{-1} (\boldsymbol{r} - \mathrm{E}(\boldsymbol{\xi}_j)) \\
&= 2 \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} (\boldsymbol{r} - \bar{\boldsymbol{r}} + \boldsymbol{h}_j \bar{x}_j)
\end{aligned} \tag{8}$$

Clearly, the above derivation is very concise (as compared with the derivation of the LMMSE approach in [2]).

IV. EQUIVALENCE BETWEEN JG AND LMMSE APPROACHES

We now prove the equivalence between the JG and LMMSE approaches. From the matrix inversion lemma, we have

$$\boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{R}_j + \alpha \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}})^{-1} = \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} - \frac{\alpha \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1}}{1 + \alpha \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \boldsymbol{h}_j} = \frac{\boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1}}{1 + \alpha \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \boldsymbol{h}_j} \tag{9}$$

where α is an arbitrary constant. Using (9), we have

$$\begin{aligned}
\boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} &= \frac{\boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \big/ (1 + \alpha \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \boldsymbol{h}_j)}{1 - \alpha \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \boldsymbol{h}_j \big/ (1 + \alpha \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \boldsymbol{h}_j)} \\
&= \frac{\boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{R}_j + \alpha \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}})^{-1}}{1 - \alpha \boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{R}_j + \alpha \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}})^{-1} \boldsymbol{h}_j}
\end{aligned} \tag{10}$$

By setting $\alpha = 1$ in (10), we get the equivalence between (6) and (8) as follows:

$$\begin{aligned}
\lambda_j^{\mathrm{MMSE}} &= \frac{2 \boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{R}_j + \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}})^{-1} (\boldsymbol{r} - \bar{\boldsymbol{r}} + \boldsymbol{h}_j \bar{x}_j)}{1 - \boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{R}_j + \boldsymbol{h}_j \boldsymbol{h}_j^{\mathrm{T}})^{-1} \boldsymbol{h}_j} \\
&= 2 \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} (\boldsymbol{r} - \bar{\boldsymbol{r}} + \boldsymbol{h}_j \bar{x}_j) = \lambda_j^{\mathrm{JG}}
\end{aligned} \tag{11}$$

Furthermore, by setting $\alpha = v_j$ in (10), we obtain another useful expression of the extrinsic LLR as

$$\lambda_j^{\mathrm{JG}} = \frac{2 \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}^{-1} (\boldsymbol{r} - \bar{\boldsymbol{r}} + \boldsymbol{h}_j \bar{x}_j)}{1 - v_j \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}^{-1} \boldsymbol{h}_j} \,. \tag{12}$$

The main advantage of (12) over (6) and (8) is that (12) involves the inversion of the same matrix $\boldsymbol{R}$ for different j. This can greatly reduce the computational complexity in some situations, as shown later.

V. APPLICATION IN MULTI-ANTENNA ISI CHANNELS

As an example, we consider the application of the JG method in a multi-antenna communication system with N_t transmitting antennas and N_r receiving antennas over an ISI channel with L tap coefficients.

The upper part of Fig.1 shows the transmitter structure. The information sequence $\boldsymbol{d}$ is first encoded using a binary forward error correction (FEC) code $\boldsymbol{C}$. The coded sequence $\boldsymbol{c} = \{c_j\}$ ($c_j \in (+1, -1)$) is then randomly interleaved and fed into a serial-to-parallel converter, in which the interleaved version of $\boldsymbol{c}$ is segmented into $2 \times N_t$ equal-length sequences to form the real and imaginary parts of $\{ \boldsymbol{x}^n \mid n = 1, \ldots, N_t \}$. Finally, $\{\boldsymbol{x}^n\}$ are transmitted from the N_t antennas simultaneously.

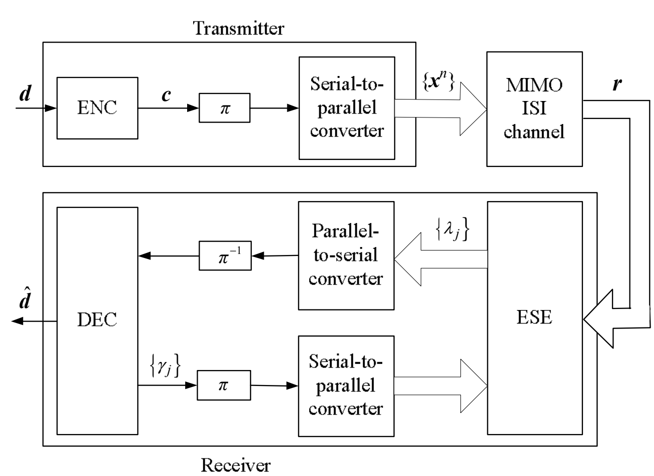

Fig.1 The transmitter and receiver system structures over quasi-static MIMO ISI channels with channel input $\boldsymbol{x} = [\boldsymbol{x}^{1\mathrm{T}}, \ldots, \boldsymbol{x}^{N\mathrm{T}}]^{\mathrm{T}}$, and π and π^{-1} denoting the interleaver and the corresponding deinterleaver.

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The lower part of Fig. 1 shows the structure of an iterative receiver, consisting of a MIMO soft-in/soft-out (SISO) equalizer (ESE) and an *a posteriori* probability (APP) decoder (DEC). The ESE resolves the ISI and cross-antenna interference (CAI), and generates coarse estimates for $\{\boldsymbol{x}^n\}$. The DEC carries out APP decoding of C and generates extrinsic information for $\{c_j\}$. The ESE and the DEC exchange their outputs with each other in a turbo-type manner. The detailed detection system in channels without ISI can be found in [3]. Here we consider the application of the MAP and JG methods, respectively, for the realization of the ESE.

The quasi-static MIMO ISI channel is modeled as follows. We denote by $\{h^n_{m,0},\cdots,h^n_{m,l},\cdots,h^n_{m,L-1}\}$ the Rayleigh fading coefficients between the nth transmit antenna and the mth receive antenna. The signal received by the N_r antennas at time interval k can be represented by

$$\boldsymbol{r}_k = \sum_{l=0}^{L-1} \boldsymbol{H}_l \boldsymbol{x}_{k-l} + \boldsymbol{\eta}_k , \tag{13}$$

where $\boldsymbol{H}_l$ for any $l = 0, \ldots, L\text{-}1$ is a $N_r{\times}N_t$ matrix with its nm-th element $h^n_{m,l}$, $\boldsymbol{x}_k = [x^1_k, x^2_k, ..., x^{N_t}_k]$ is the $N_t{\times}1$ transmitted information vector at time interval k, and $N_r{\times}1$ vectors $\boldsymbol{r}_k$ and $\boldsymbol{\eta}_k$ represent the corresponding received signals and additive noises, respectively. Equ. (13) can be expressed in a real form as

$$\tilde{\boldsymbol{r}}_k = \sum_{l=0}^{L-1} \tilde{\boldsymbol{H}}_l \tilde{\boldsymbol{x}}_{k-l} + \tilde{\boldsymbol{\eta}}_k , \tag{14}$$

where
$$\tilde{\boldsymbol{r}}_k = [\mathrm{Re}(\boldsymbol{r}_k)^{\mathrm{T}}, \mathrm{Im}(\boldsymbol{r}_k)^{\mathrm{T}}]^{\mathrm{T}},$$
$$\tilde{\boldsymbol{x}}_k = [\mathrm{Re}(\boldsymbol{x}_k)^{\mathrm{T}}, \mathrm{Im}(\boldsymbol{x}_k)^{\mathrm{T}}]^{\mathrm{T}},$$
$$\tilde{\boldsymbol{\eta}}_k = [\mathrm{Re}(\boldsymbol{\eta}_k)^{\mathrm{T}}, \mathrm{Im}(\boldsymbol{\eta}_k)^{\mathrm{T}}]^{\mathrm{T}},$$
and
$$\tilde{\boldsymbol{H}}_l = \begin{bmatrix} \mathrm{Re}(\boldsymbol{H}_l) & -\mathrm{Im}(\boldsymbol{H}_l) \\ \mathrm{Im}(\boldsymbol{H}_l) & \mathrm{Re}(\boldsymbol{H}_l) \end{bmatrix}.$$

Consider the transmission of a frame of information signals $\boldsymbol{x} = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_K]$ over the quasi-static channel. The received signals can be written as

$$\begin{bmatrix} \tilde{\boldsymbol{r}}_0 \\ \vdots \\ \tilde{\boldsymbol{r}}_k \\ \vdots \\ \tilde{\boldsymbol{r}}_{K+L-1} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{H}}_0 & & & \boldsymbol{0} \\ \vdots & \tilde{\boldsymbol{H}}_0 & & \\ \tilde{\boldsymbol{H}}_{L-1} & \vdots & \ddots & \\ & \tilde{\boldsymbol{H}}_{L-1} & & \tilde{\boldsymbol{H}}_0 \\ & & \ddots & \vdots \\ \boldsymbol{0} & & & \tilde{\boldsymbol{H}}_{L-1} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_0 \\ \vdots \\ \tilde{\boldsymbol{x}}_k \\ \vdots \\ \tilde{\boldsymbol{x}}_K \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\eta}}_0 \\ \vdots \\ \tilde{\boldsymbol{\eta}}_k \\ \vdots \\ \tilde{\boldsymbol{\eta}}_{K+L-1} \end{bmatrix}. \tag{15}$$

Notice that (15) is a special case of (1) when $\boldsymbol{H}$ takes a block convolution form. Thus, the output extrinsic LLR of the ESE for each entry of $\tilde{\boldsymbol{x}}_k$ ($k = 0,\ldots,K$) can also be calculated by using (12). For simplicity, we use the notations based on the general signal model in (1) instead of the particular one in (15) in the following discussion.

Let us now consider the implementation issues when $\boldsymbol{H}$ in (1) is a convolution matrix. The channel coefficient matrix in (15) is such an example. In this case, the correlation matrix $\boldsymbol{R}$ is a band limited matrix, and we denote its bandwidth as $2B$-1. The cost involved in (12) is mainly related to evaluating

$$\boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}^{-1} (\boldsymbol{r} - \bar{\boldsymbol{r}}) \qquad \text{and} \qquad \boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}^{-1} \boldsymbol{h}_j . \tag{16}$$

Decompose $\boldsymbol{R}$ as $\boldsymbol{R} = \boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}$ where $\boldsymbol{L}$ is a lower-triangular matrix with bandwidth B. The cost for this Cholesky decomposition is $O(JB^2)$. Denote $\boldsymbol{u}_j = \boldsymbol{L}^{-1}\boldsymbol{h}_j$ and $\boldsymbol{w} = \boldsymbol{L}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})$. Then the two values in (16) can be rewritten as $\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{w}$ and $\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{u}_j$. The cost for solving $\boldsymbol{w} = \boldsymbol{L}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})$ is $O(B)$ per entry of $\boldsymbol{x}$. But the cost for solving $\boldsymbol{u}_j = \boldsymbol{L}^{-1}\boldsymbol{h}_j$ is quite high since it should be carried out for every j. To further reduce complexity, we can adopt an extending window technique as described in Appendix I, by which the complexity involved in (12) is $O(B^2)$ per entry of $\boldsymbol{x}$. In particular, the overall complexity is $O(N_r^2L^2)$ per entry of $\boldsymbol{x}$ for the system in (15).

VI. Analysis of the JG-Method

We now propose a simple SNR-bounding evolution technique to track the iterative decoding process of the JG detector and to predict its performance. Some functions involved are determined by simulation, which is similar to the previous work presented in [2][7][8]. For simplicity, we use the notations based on the general signal model in (1) when we present the general principle, but we will discuss the special case in (15) in more details when we consider the low-complexity bounding technique.

A. The SNR-VAR Characterization of the ESE

We write λ_j^{JG} in (8) in a signal-plus-distortion form as

$$\lambda_j^{\mathrm{JG}} = \mu_j x_j + \zeta_j \tag{17}$$

where $\mu_j = 2\boldsymbol{h}_j^{\mathrm{T}}\boldsymbol{R}_j^{-1}\boldsymbol{h}_j$ and $\zeta_j = 2\boldsymbol{h}_j^{\mathrm{T}}\boldsymbol{R}_j^{-1}(\boldsymbol{r} - \bar{\boldsymbol{r}} - \boldsymbol{h}_j(x_j - \bar{x}_j))$. Thus, the received signal power is μ_j^2, and the variance of the distortion term is

$$\mathrm{Var}(\zeta_j) = 4\boldsymbol{h}_j^{\mathrm{T}}\boldsymbol{R}_j^{-1}\boldsymbol{h}_j = 2\mu_j . \tag{18}$$

The signal-to-noise ratio (SNR) of λ_j^{JG} is then given by

$$SNR_j = \mu_j^2 / \mathrm{Var}(\zeta_j) = \boldsymbol{h}_j^{\mathrm{T}}\boldsymbol{R}_j^{-1}\boldsymbol{h}_j . \tag{19}$$

In general, each entry of $\boldsymbol{x}$ has its individual *a priori* variance and *a posteriori* SNR. This causes difficulty in tracking the evolution process. However, we can treat $\{v_j\}$ (the diagonal entries of $\boldsymbol{V}$) as identically distributed random variables [8], i.e.,

$$\mathrm{E}(v_j) = v_{\mathrm{DEC}}, \text{ for all } j, \tag{20}$$

where v_{DEC} is a constant at each stage of iteration, and the subscript '$_{\mathrm{DEC}}$' indicates that the variance is supplied by the DEC. By taking the expectation of (19) with respect to $\{v_j\}$, we obtain

$$\mathrm{E}[SNR_j] = \mathrm{E}[\boldsymbol{h}_j^{\mathrm{T}} \boldsymbol{R}_j^{-1} \boldsymbol{h}_j] . \quad (21)$$

The right side of (21) is still not easy to evaluate, which involves the distribution of the *a priori* variance. For simplicity, we introduce the following bounding technique. It can be shown that SNR_j is a convex function of $[v_1, ..., v_j, …, v_J]^{\mathrm{T}}$. Thus we have

$$\begin{aligned} \mathrm{E}[SNR_j] &= \mathrm{E}[\boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{H}\underline{\boldsymbol{V}}_j \boldsymbol{H}^{\mathrm{T}} + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{h}_j] & (22.a)\\ &\ge \boldsymbol{h}_j^{\mathrm{T}} (\boldsymbol{H}\mathrm{E}[\underline{\boldsymbol{V}}_j]\boldsymbol{H}^{\mathrm{T}} + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{h}_j & (22.b)\\ &= \boldsymbol{h}_j^{\mathrm{T}} (v_{\mathrm{DEC}}(\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} - \boldsymbol{h}_j^{\mathrm{T}}\boldsymbol{h}_j) + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{h}_j & (22.c)\\ &\equiv \underline{SNR}_j & (22.d) \end{aligned}$$

where (22.b) follows the Jensen's inequality. We define the average of $\{\underline{SNR}_j\}$ as the output of the ESE,

$$SNR_{\mathrm{ESE}} \equiv \frac{1}{J} \sum_{j=1}^{J} \underline{SNR}_j . \quad (23)$$

In (23), we need to calculate the $\underline{SNR}_j$ for each j. However, this can be simplified if $\boldsymbol{H}$ is a convolution matrix such as the one in (15) for a quasi-static ISI channel. The signal passing through such a channel can be modeled as a Markov process. This Markov process is stationary when the information length J is infinite. Any statistic of the inference based on this Markov process is also stationary. For practical systems, we assume that J is sufficiently large so that the stationarity approximately holds. Then the statistics related to $\{\tilde{\boldsymbol{x}}_k, k = 1, …, K\}$ (such as the average SNRs in (22)) are invariant to k except for those close to the beginning and end (referred as the termination effect). For sufficiently large J, we can ignore the termination effect and only evaluate the $\underline{SNR}_j$s of a particular $\tilde{\boldsymbol{x}}_{k^*}$ with $k^* \approx K/2$. The values of $\{\underline{SNR}_j\}$ involved in (23) are approximately the repetitions of those for $\tilde{\boldsymbol{x}}_{k^*}$. More specifically, we compute (22.d) for those $\{\boldsymbol{h}_j\}$ related to $\tilde{\boldsymbol{x}}_{k^*}$, where $\boldsymbol{h}_j$ is a column from the following column set

$$[\underbrace{\boldsymbol{0} \ \cdots \ \boldsymbol{0}}_{k^*-1} \tilde{\boldsymbol{H}}_0^{\mathrm{T}} \quad \tilde{\boldsymbol{H}}_1^{\mathrm{T}} \quad \cdots \quad \tilde{\boldsymbol{H}}_{L-1}^{\mathrm{T}} \boldsymbol{0} \quad \cdots \quad \boldsymbol{0}]^{\mathrm{T}}, \quad (24)$$

and each '$\boldsymbol{0}$' is a zero block of the same size as $\tilde{\boldsymbol{H}}_l^{\mathrm{T}}$ for $\forall l$. (Note the usage of indices, j and k, here. Let $\boldsymbol{H}$ be the channel convolution matrix in (15). Then, $\boldsymbol{h}_j$ is the jth column in $\boldsymbol{H}$, and (24) represents the k^*th set of columns in $\boldsymbol{H}$ expressed in a block form.) Thus, we approximate (23) as follows

$$SNR_{\mathrm{ESE}} \approx \frac{1}{2N_t} \sum_{j} \underline{SNR}_j , \quad (25)$$

where the summation is over the $\underline{SNR}_j$ values for $\tilde{\boldsymbol{x}}_{k^*}$. Note that (25) contains $2\times N_t$ items including the real and imaginary parts. The computational cost can be further reduced by half due to the fact that the average SNRs of the real part and imaginary part from the same symbol are equal.

For simplicity of notation, we characterize the input variance and output SNR relationship of the ESE as

$$SNR_{\mathrm{ESE}} = \phi(v_{\mathrm{DEC}}) . \quad (26)$$

B. The Characterization of the DEC

Next we consider the DEC part. Recall from Section V that $\{\lambda_j^{\mathrm{JG}}\}$ are delivered to the DEC for producing the corresponding extrinsic LLR outputs $\{\gamma_j\}$. Similar to [8], the evolution process in the DEC part can be characterized as

$$v_{\mathrm{DEC}} = f(SNR_{\mathrm{ESE}}) . \quad (27)$$

We can perform a hard decoding on the DEC's soft outputs $\{\gamma_j\}$. Since the performance of the DEC is solely determined by the input SNR, the output frame error rate (FER) is a also function of SNR_{ESE}, i.e.,

$$FER = g(SNR_{\mathrm{ESE}}) . \quad (28)$$

Both $f(\cdot)$ and $g(\cdot)$ can be obtained by simulations. Note that we can use BER as the performance indicator in a similar way.

C. The Overall SNR Evolution Process

By combining (26) and (27), we have

$$SNR_{\mathrm{ESE}}^{i\_new} = \phi(f(SNR_{\mathrm{ESE}}^{i\_old})) \quad (29)$$

where $SNR_{\mathrm{ESE}}^{i\_new}$ and $SNR_{\mathrm{ESE}}^{i\_old}$ are, respectively, SNR_{ESE} values after and before the ith iteration. At the beginning, we initialize $f(SNR_{\mathrm{ESE}}^{i\_old}) = 1$, implying no feedback from the DEC. By repeating (29), we can track the iterative SNR evolution process. In the final iteration, we can estimate the FER performance as $FER = g(SNR_{\mathrm{ESE}}^{i\_final})$ in (28). The above analysis is valid for fixed channel coefficients. It can also be applied to quasi-static fading channels by averaging the *FER* in (28) over the channel distributions.

Before ending this section, we give some intuitions about the accuracy of the SNR-bounding technique developed above. The lower bound $\underline{SNR}_j$ given in (22) is a pessimistic estimation of $\mathrm{E}[SNR_j]$. Thus, we expect that the prediction by the SNR evolution should be an upper bound of the simulated system performance. Moreover, it is straightforward to verify that the equality in (22) holds when $v_{\mathrm{DEC}}\rightarrow 1$ (no *a priori* information) and $v_{\mathrm{DEC}}\rightarrow 0$ (perfect *a priori* information). For a given channel realization, $v_{\mathrm{DEC}}\rightarrow 1$ corresponds to the situation that the ESE outputs are not reliable that is usually at the beginning of the iterative detection process; $v_{\mathrm{DEC}}\rightarrow 0$ corresponds to the situation that the ESE outputs are very reliable that is usually at the end of the iterative detection process. Thus, the predictions by the SNR evolution in these two situations are accurate. In other cases, there is a small gap between the prediction and simulation. Observations on the numerical results have verified our analysis.

VII. Simulation Results

In our simulation, we use a MIMO ISI channel with $N_t = 2$ and $N_r = 2$. L is chosen as 2 and 4, respectively, to show the effect of the number of multipath on the system performance. The fading coefficients are modeled as independent and identically distributed complex Gaussian random variables with zero-mean. We assume that the fading coefficients in any antenna link have a unit average power gain:

$$\mathrm{E}\left(\sum_{l=0}^{L-1} |h_{m,l}^{n}|^2\right)=1, \qquad \text{for } \forall m, n. \tag{30}$$

For channel coding, a simple non-recursive convolution code and a punctured rate-1/2 turbo code, respectively, with generators $(7, 5)_8$ and information block length 1024 bits are used as the FEC code. Let R be the overall rate and R_C the rate of C. The total throughput for this system is $R = 4R_C = 2$.

Fig.2 illustrates the FER performance of the system using the JG equalizer over ISI channels with L = 2 and 4. The iteration number is 15, and the interleaver used in the simulation is generated randomly. The FER based on the MIMO-MAP equalizer [6] is also included for L = 2. (For L = 4, the complexity of the MIMO MAP algorithm is too high.) It is shown that the JG equalizer performs close to the MAP equalizer at low to moderate E_b/N_0, but degrade slightly at high E_b/N_0. The complexity of the MAP equalizer ($O(N_r 4^{LNt})$) is much higher than that of the JG equalizer ($O(N_r^2 L^2)$) for large N_t and L. The dashed lines show the performance predicted by the SNR evolution analysis. These curves are generated as follows:

(1) Generate a channel realization from the given distribution of the channel coefficients;

(2) Perform the SNR evolution as described in Section VI;

(3) Repeat Step (1) and (2) for n times;

(4) Take the average of all the FERs obtained at the n times as the prediction of the system performance.

The gaps between the predicted performance and simulation result in Fig.2 are due to the use of the bounding technique in (22).

In Fig.3, with the same channel conditions, we compare the performance of the turbo and convolution FEC coded systems. The convolution code is exactly the same as used in Fig.2. The turbo code uses $(7, 5)_8$ as the generator and punctured to rate 1/2 for fairness of comparison. As shown in Fig.3, the performance can be improved by using a more powerful FEC code.

VIII. Conclusions

In this paper, we have analyzed the iterative decoding in general linear MIMO systems and presented a concise derivation of the JG method. The equivalence between the JG and LMMSE approaches are proven. Also, we have applied the JG method on a MIMO ISI quasi-static fading channel, and designed an SNR-bounding technique for fast assessment of the system performance with reasonable accuracy. Numerical results show that the JG equalizer performs almost as well as the MAP equalizer at a significantly lower cost.

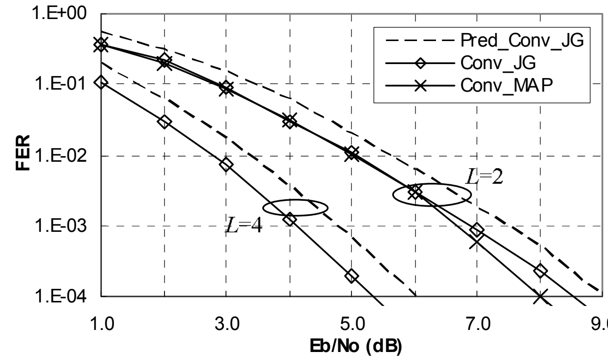

Fig.2 FER performance of the JG and MAP detectors with $(7, 5)_8$ convolution code in 2×2 MIMO ISI Rayleigh fading channels.

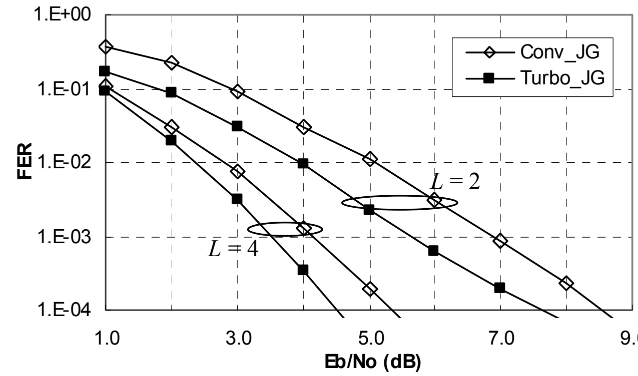

Fig.3 Comparison of $(7, 5)_8$ convolution code and punctured rate 1/2 $(7, 5)_8$ turbo code in 2×2 MIMO ISI Rayleigh fading channels.

Appendix I

Approximate Implementation of the JG Method

Denote $\boldsymbol{u}_j = [u_{j,1}, u_{j,2}, \ldots, u_{j,N}]^{\mathrm{T}}$. Since the first j-1 entries in $\boldsymbol{h}_j$ are zeros and $\boldsymbol{L}$ is lower triangular, the first j-1 entries of $\boldsymbol{u}_j = \boldsymbol{L}^{-1}\boldsymbol{h}_j$ are also zeroes. Thus, the two inner products required in (16) can be written as

$$\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{u}_j = \sum_{m=j}^{N} u_{j,m}^2 \;, \boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{w} = \sum_{m=j}^{N} u_{j,m} w_m \;. \tag{31}$$

We propose to use the following partial sums to approximate the sums in (31).

$$\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{u}_j = \sum_{m=j}^{j+W-1} u_{j,m}^2 \;, \boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{w} = \sum_{m=j}^{j+W-1} u_{j,m} w_m \;, \tag{32}$$

where W is the extended window size (a properly selected positive integer). By using (32), we only need to calculate the jth to $(j+W-1)$th entries of $\boldsymbol{u}_j$. The cost involved is $O(WB)$ per entry.

The approximation (32) is justified as follows. For a given j, let $\boldsymbol{r}'$ be a truncated vector consisting of the first $j+W$-1 entries of $\boldsymbol{r}$. Then the approximate result for λ_j^{JG} obtained using (32)

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is actually the LLR using a reduced set of observations $\boldsymbol{r}'$. The approximation becomes more accurate when W increases. From simulation results not presented here, we have observed that the performance loss is marginal when setting $W \geq 3B$ so that the cost related to $\boldsymbol{u}_k$ becomes $O(B^2)$ per entry. In this paper, the extended window size W is set to $4B$ for the JG equalizer.

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