Multi-User Gain and Maximum Eigenmode Beamforming for MIMO Systems with Rate Constraints

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***Abstract*—We consider the sum-power minimization problem for multi-user multiple-input multiple-output (MIMO) systems with rate constraints. We show that significant performance improvements, qualified by multi-user gain (MUG), can be achieved by allowing multiple users to transmit simultaneously on their individual maximum eigenmodes. We also show that a major part of MUG can be achieved even with quite a small number of users and increasing the number of antennas at the base station is an efficient method for maximizing MUG.**

I. INTRODUCTION

Multi-user multiple-input multiple-output (MIMO) systems have attracted much research interest recently [1][2]. Eigenmode beamforming techniques [3][4][5][6] have been widely studied for the rate maximization problem (with fixed total transmission power).

In delay sensitive multi-user applications such as speech and video, every user must transmit a certain amount of information within a fixed time period. In this case, sum-power (i.e., total power transmitted by all users) minimization is an appropriate target. This issue is also closely related to throughput maximization in cellular systems where less transmission power implies less interference to other cells and so potentially higher cellular capacity. Several algorithms [7][8] have been proposed recently to compute the minimum sum-power (MSP) of a multi-user MIMO system with a given rate constraint for each user. However, these algorithms involve iterative joint optimization on the transmission covariance matrices and decoding order. They become computationally costly even when the number of users, denoted by K below, is only moderately large.

In this paper, we focus on the sum-power minimization problem for multi-user MIMO systems with rate constraints. We will investigate the potential power saving, quantified by multi-user gain (MUG) below, by allowing simultaneous multi-user transmission. It is known that, in fading channels, transmission power can be reduced by allowing multiple users to transmit simultaneously [3]. However, to the best of our knowledge, there is only limited effort to quantify the related gain (e.g., for channel with path loss, lognormal fading and Rayleigh fading). We will, using both analytical and numerical results, show that MUG is very significant in practical environments.

A simple method to realize MUG is the so-called maximum eigenmode beamforming (MEB) on the maximum eigenmode of each user. MEB has lower implementation complexity compared with waterfilling over all eigenmodes of each user. We will show that MEB is asymptotically optimal when K is large and its performance is impressive even for a quite small K (e.g., $K = 2$ or 4).

An interesting finding of this paper is that employing multiple antennas only at the base station is a good strategy in both multiple access channels (MACs) and broadcast channels (BCs). Increasing the number of antennas at the base station has far more impact on the system performance than at the mobile units. This finding has useful practical implications, since it is much easier to install more antennas at the base station than at all mobile units.

If all the mobile units in a multi-user MIMO MAC can cooperate, it results in an equivalent single-user MIMO channel with more transmit antennas than receive antennas when K is large. It is well known that the performance of such an unbalanced single-user MIMO channel is limited by the number (denoted by M) of receive antennas for which the achievable rate R increases linearly with M at fixed average transmission power [9] (when R is large). In this paper, we show that this is also true even when mobile units cannot cooperate.

Our focus is mainly on MIMO MACs. The discussion on MIMO BCs is brief since the duality principle provides a direct connection between them [10].

II. PRELIMINARIES

A. *MIMO MAC Model*

Consider a K-user system over a quasi-static MIMO MAC with M antennas at the base station and N antennas at each user side. Denote by $\boldsymbol{H}_k$ and $\boldsymbol{x}_k$ the channel matrix and transmitted signal for user k, respectively. The received signal at the base station is given by

$$\boldsymbol{y} = \sum\nolimits_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{n} \qquad (1)$$

where $\boldsymbol{n}$ is a vector of complex additive white Gaussian noise (AWGN) samples with zero mean and unit variance. $\{\boldsymbol{H}_k\}$ are assumed to be identically and independently distributed (i.i.d.) and perfectly known at the transmitters.

For simplicity, we assume that the rate of each user is the same at R/K bits/symbol, where R is referred to as the system *sum-rate*. The objective below is to identify the required average (over the distribution of $\{\boldsymbol{H}_k\}$) transmitted MSP, denoted by $P_{N\times M}(K, R)$, to support such a rate requirement.

B. *The MSP for SISO MACs*

For a single-input single-output (SISO) system with $M = N = 1$, $\boldsymbol{H}_k$ in (1) reduces to a scalar h_k. Without loss of

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generality, we assume that $|h_1|^2 \leq \ldots \leq |h_K|^2$ for each channel realization. The MSP of a SISO MAC is achieved by the successive interference cancellation (SIC) strategy with descending decoding order on k (i.e., decoding starts from user K and ends at user 1) [11] at the base station, i.e.

$$P_{1\times 1}(K,R) = \mathrm{E}\left(\sum_{k=1}^{K} \frac{(2^{R/K}-1)\cdot 2^{R(k-1)/K}}{|h_k|^2} \right) \quad (2)$$

where E[·] denotes the mean over the distribution of $\{h_k\}$.

C. The MSP for MIMO MACs

When $K = 1$, (1) reduces to a single-user MIMO system and $P_{N\times M}(1, R)$ can be easily obtained. (See Appendix I.) When $K > 1$, the sum-power minimization problem for computing $P_{N\times M}(K, R)$ involves joint optimization of the transmission covariance matrices and decoding order for each channel realization. Detailed discussions on this issue can be found in [7][8]. The existing methods are highly complicated. However, the following proposition shows a good property of the MSP.

Proposition 1: For a K-user $N\times M$ MIMO system with rate constraint R/K for each user, the average transmitted MSP is a monotonically decreasing function of K, i.e.

$$P_{N\times M}(K,R) \leq P_{N\times M}(K-1,R). \quad (3)$$

The proof of proposition 1 can be found in Appendix II. According to this monotonic property, power saving can be achieved by increasing K, i.e., by allowing more users to transmit simultaneously. This indicates the potential advantage of a multi-user system.

III. MSP BOUNDS

In the following, we derive a lower bound for $P_{N\times M}(K, R)$ that provides many useful insights into the sum-power minimization problem.

A. SISO MACs

We first consider a simple SISO scenario. Recall the assumption $|h_1|^2 \leq \ldots \leq |h_K|^2$ for (2). When $K\to\infty$, in probability 1 we have

$$F_{1\times 1}\left(|h_k|^2\right) = k/K \quad (4)$$

where $F_{1\times 1}(\cdot)$ is the cumulative density function (CDF) of $\{|h_k|^2\}$ and can be obtained using Monte-Carlo method. In this case, the channel gain of user k can be rewritten as

$$|h_k|^2 = F_{1\times 1}^{-1}(k/K) \quad (5)$$

where $F_{1\times 1}^{-1}(\cdot)$ is the inverse of $F_{1\times 1}(\cdot)$. Then (2) becomes

$$\begin{aligned} P_{1\times 1}(\infty,R) &= \lim_{K\to\infty} P_{1\times 1}(K,R) \\ &= \lim_{K\to\infty} K\left(1-2^{-R/K}\right)\sum_{k=1}^{K} \frac{2^{Rk/K}}{F_{1\times 1}^{-1}(k/K)}\cdot\frac{1}{K} \\ &= R\ln 2\int_0^1 \frac{2^{Rt}}{F_{1\times 1}^{-1}(t)}dt. \end{aligned} \quad (6)$$

Combining (6) with the monotonous reducing property of MSP against K, we have the following proposition.

Proposition 2: The MSP for a K-user SISO MAC with rate constraint R/K for each user is lower bounded by

$$P_{1\times 1}(K,R) \geq R\ln 2\int_0^1 \frac{2^{Rt}}{F_{1\times 1}^{-1}(t)}dt. \quad (7)$$

This lower bound is asymptotically tight for large K.

B. MIMO MACs

For each channel realization, let the singular value decomposition (SVD) of channel matrix $\boldsymbol{H}_k$ be

$$\boldsymbol{H}_k = \boldsymbol{U}_k\boldsymbol{D}_k\boldsymbol{V}_k^H \quad (8)$$

where $\boldsymbol{U}_k$ and $\boldsymbol{V}_k$ are unitary matrices and $\boldsymbol{D}_k$ is an $M\times N$ diagonal matrix consisting of all the singular values of $\boldsymbol{H}_k$. Denote by $d_{k,\max}$ the maximum singular value of $\boldsymbol{H}_k$. Let $\boldsymbol{u}_{k,\max}$ and $\boldsymbol{v}_{k,\max}$ be the corresponding singular vectors in $\boldsymbol{U}_k$ and $\boldsymbol{V}_k$, respectively. For the MEB strategy, each user k transmits information only in the direction of $\boldsymbol{v}_{k,\max}$, i.e., $\boldsymbol{x}_k = \boldsymbol{v}_{k,\max}\cdot x_k$. Then (1) becomes

$$\begin{aligned} \boldsymbol{y} &= \sum_{k=1}^{K} \boldsymbol{U}_k\boldsymbol{D}_k\boldsymbol{V}_k^H\boldsymbol{v}_{k,\max}x_k + \boldsymbol{n} \\ &= \sum_{k=1}^{K} d_{k,\max}x_k\boldsymbol{u}_{k,\max} + \boldsymbol{n}. \end{aligned} \quad (9)$$

Assume that $d_{1,\max} \leq \ldots \leq d_{K,\max}$. Apply SIC at the base station with descending decoding order on k. When decoding $\boldsymbol{x}_k$, we correlate the received signal by $\boldsymbol{u}_{k,\max}$. The signal-to-noise ratio (SNR) for user k (denoted by SNR_k) after correlation is

$$\begin{aligned} SNR_k &= \frac{p_k d_{k,\max}^2 |\boldsymbol{u}_{k,\max}^H\boldsymbol{u}_{k,\max}|^2}{1+\sum_{i=1}^{k-1} p_i d_{i,\max}^2 |\boldsymbol{u}_{k,\max}^H\boldsymbol{u}_{i,\max}|^2} \\ &= \frac{p_k d_{k,\max}^2}{1+\sum_{i=1}^{k-1} p_i d_{i,\max}^2 \phi_{k,i}} \end{aligned} \quad (10)$$

where $\phi_{k,i} = |\boldsymbol{u}_{k,\max}^H\boldsymbol{u}_{i,\max}|^2$ and $p_k = \mathrm{E}[|x_k|^2]$ is the transmitted power of user k. Note that in (10), we assumed that the interference from {user i: $i > k$} has been removed by SIC. For each channel realization, we can compute the values of $\{p_k\}$ using (10) recursively according to the channel capacity formula $R/K = \log_2(1+SNR_k)$, $\forall k$. The resultant sum power $\Sigma_k p_k$ (averaged over the fading distribution) is realizable based on a conventional FEC coding scheme. It is also an upper bound for $P_{N\times M}(K, R)$.

Note that the complexity of the above MEB scheme is much lower than that of the optimal multi-user MIMO scheme. The latter involves array processing with water filling and joint optimization of input covariance matrices and decoding order.

The proofs of the following two propositions can be found in Appendix II.

Proposition 3: The MSP for a K-user $N\times M$ MIMO MAC with rate constraint R/K for each user is lower bounded by

$$P_{N\times M}(K,R) \geq R\ln 2\int_0^1 \frac{2^{Rt/M}}{F_{N\times M}^{-1}(t)}dt \quad (11)$$

where $F_{N\times M}(\cdot)$ is the CDF of $\{d_{k,\max}^2\}$. (**Note:** Since $\{\boldsymbol{H}_k\}$ are i.i.d, so are $\{d_{k,\max}^2\}$.)

Proposition 4: The lower bound (11) is asymptotically achievable by MEB for large K.

From propositions 3 and 4, we have the following theorem.

Theorem 1: For a K-user $N\times M$ MIMO MAC with rate constraint R/K for each user, when K becomes infinite, the MSP of the system converges to

$$P_{N\times M}(\infty,R)=R\ln 2\int_0^1 \frac{2^{Rt/M}}{F_{N\times M}^{-1}(t)}dt\,. \tag{12}$$

It is easy to see that (6) and (7) are special cases of (12) and (11), respectively.

C. The Impacts of R, M and N

Some interesting observations can be made from theorem 1. The first one is regarding the asymptotic MSP required for very large R. From (12) we have

$$\begin{aligned}
&\lim_{R\to+\infty}\frac{d\left(P_{N\times M}(\infty,R)\right)_{\mathrm{dB}}}{dR}\\
&=10\log_{10}2\cdot\lim_{R\to+\infty}\frac{d}{dR}\left(\log_2\left(R\ln 2\int_0^1\frac{2^{Rx/M}}{F_{N\times M}^{-1}(x)}dx\right)\right)\\
&=10\log_{10}2\cdot\lim_{R\to+\infty}\frac{d}{dR}\left(\frac{R}{M}+\log_2\left(\int_0^1\frac{2^{-R(1-x)/M}}{F_{N\times M}^{-1}(x)}dx\right)\right)\\
&=\frac{10\log_{10}2}{M}
\end{aligned} \tag{13}$$

where $(A)_{\mathrm{dB}}\equiv 10\log_{10}A$.

Equation (13) shows that the asymptotic slope of $(P_{N\times M}(\infty, R))_{\mathrm{dB}}$ depends on M only and is independent of N and the fading distribution. This indicates that the number of antennas at the base station is the most important factor.

Next we examine the impact of M and N. Recall that $F_{N\times M}(\cdot)$ is the CDF of $\{d_{k,\max}^2\}$. Increasing either M or N leads to reduced $F_{N\times M}(x)$ for $\forall x > 0$ (which indicates an increased mean for $d_{k,\max}^2$) and so reduced MSP. Moreover, increasing M has an additional benefit since it also reduces the numerator inside the integral in (12). This can be quantified based on (13) as

$$\left(P_{N\times aM}(\infty,aR)\right)_{\mathrm{dB}}\approx\frac{10\log_{10}2}{aM}\cdot aR\approx\left(P_{N\times M}(\infty,R)\right)_{\mathrm{dB}} \tag{14}$$

where a is a positive integer and R is sufficiently large. Equation (14) shows that, when R is large, it increases asymptotically linearly with M for a fixed transmission sum-power. This observation has interesting implications in cellular systems. Suppose that the average transmitted sum-power remains unchanged and cross-cell interference is also maintained at a fixed level. Then (14) implies that the cellular capacity increases approximately linearly with M. Generally speaking, a cellular system benefits more from increasing the number of antennas at the base station than at each mobile unit.

We now illustrate the above observations using numerical results. In Figs. 1 and 2, MSPs are plotted for various multiple access systems over a single-cell fading channel involving three factors, namely, Rayleigh fading, normalized lognormal fading with $\sigma_s = 8$ and path loss in an edge-length-1 single hexagon cell with uniform user distribution and fourth power path-loss law. We assume independent Rayleigh fading for every transmit-receive antenna link and equal lognormal fading and pass loss for all the links seen by a particular user. We can clearly see the change of the slope for different M in Fig. 1. We can also see from Figs. 1 and 2 that drastic power saving can be achieved by a multi-user system compared with a single-user one, especially when the sum-rate R is large. Furthermore, increasing M, i.e., the number of antennas at the base station, has a more significant effect (shown in Fig. 1) than increasing N (shown in Fig. 2).

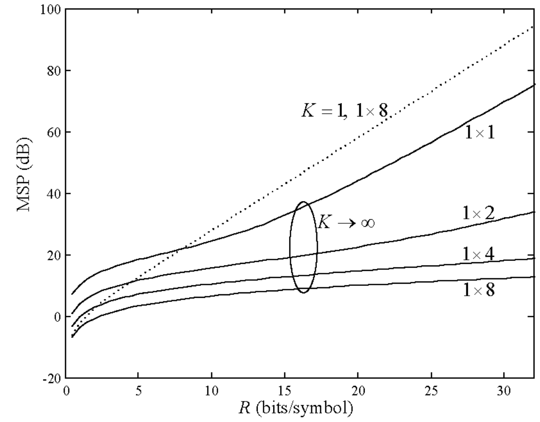

Fig. 1. The MSP versus the sum-rate R for various multiple access systems with different M and the same N over a single-cell fading channel. The outage probability is P_{out} = 0.01. The antenna settings $N \times M$ are marked on the curves.

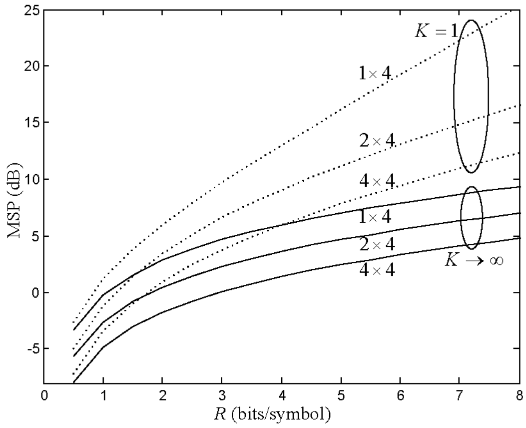

Fig. 2. The MSP versus the sum-rate R for various multiple access systems with the same M and different N over a single-cell fading channel. P_{out} = 0.01. The antenna settings $N \times M$ are marked on the curves.

D. System behavior for finite K

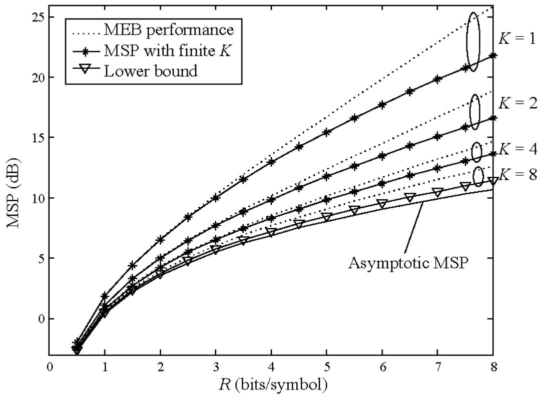

Fig. 3. The MSP versus the sum-rate R for a 2×2 multiple access system with different K over a single-cell fading channel. P_{out} = 0.01.

The lower bound (11) is achievable when $K\to\infty$, which is not a realizable situation. One may therefore ask whether the MEB approach is efficient for systems with finite K. Fig. 3 shows the MEB performance compared with the MSPs for a 2×2 multiple access system with different K over a single-cell fading channel. The MEB performance is generated using the procedure outlined in Section III.B. For K = 1, 2 and 4, the MSP curves are computed using the technique introduced in [8]. For K = 8, computing $P_{2\times2}(8, R)$ is very time consuming. Instead, we use the lower bound given in Appendix II for estimation. From Fig. 3 we can see that for K = 8, the MEB performance is already very close to the asymptotic MSP, indicating that the lower bound (11) can be used as a coarse estimation of the MEB approach provided that K is not too small.

It is also interesting to see that the difference between the MEB performance and the true MSP (or the lower bound) is marginal for $R < 4$. This indicates that the low cost MEB approach is nearly optimal if R is not too high.

IV. MULTI-USER GAIN

From the examples in Section III, we can clearly see the advantage of a multi-user system over a single-user one. This advantage is quantified by the following ratio.

$$G_{N\times M}(K,R)=\frac{P_{N\times M}(1,R)}{P_{N\times M}(K,R)} \tag{15}$$

where the derivations for $P_{N\times M}(1, R)$ are given in the Appendix I. Since such an advantage is due to the presence of multiple simultaneous users in the system, we refer to it as multi-user gain (MUG) in this paper.

Fig. 4 shows the asymptotic (i.e., by setting $K = \infty$ in (15)) MUGs for various multiple access systems over a single-cell fading channel. We can see from Fig. 4 that single-input-multiple-output (SIMO) systems have much higher MUGs than MIMO ones. (The MUGs for multiple-input-single-output (MISO) systems are close to those for MIMO ones and are not shown in Fig. 4 for simplicity.) A SIMO MAC involves multiple antennas at the base station only, which is basically the technique used by most current cellular systems. This indicates the possibility to improve system performance without significantly altering the basic system architecture.

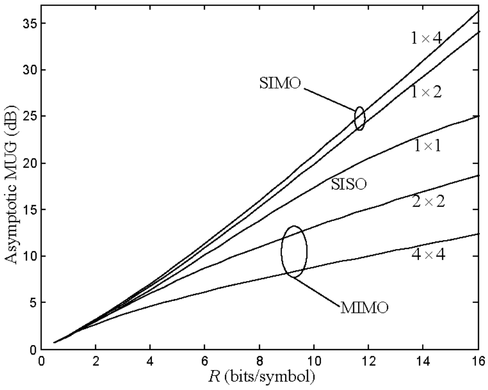

Fig. 4. The asymptotic MUG versus the sum-rate R for various multiple access systems over a single-cell fading channel. P_{out} = 0.01. The antenna settings $N \times M$ are marked on the curves.

V. MIMO BC

Based on the duality principle [10], the dual MIMO BC of (1) still has M antennas at the base station and N antennas at every mobile unit. (In particular, the dual of a SIMO MAC is a MISO BC.) Consequently, all the results above, in particular, all the observations related to Theorem 1 and Figs 1-4, can be directly applied in MIMO BCs. Again, increasing the number of antennas at the base station is a more efficient way to enhance performance than increasing that at the mobile units.

VI. CONCLUSIONS

We now summarize the main findings in this paper.

- In fading channels with rate constraints, multi-user systems have a power advantage over single-user ones, as indicated by MUG in Fig. 4.
- The MEB approach is asymptotically optimal for large K. It is also nearly optimal even for finite K.
- More antennas at the base station leads to greater performance improvements than more antennas at the mobile units.
- The above principles apply to both MACs and BCs.

APPENDIX I: MSP FOR K = 1

In this appendix, we derive the MSP for a single user system. We first consider a SISO system with rate R and K = 1. If there is no fading, the MSP is simply given by 2^R-1. For a fading SISO channel, we can average the above power over the fading distribution as

$$P_{1\times1}(1,R)=\int_0^1\frac{2^R-1}{F_{1\times1}^{-1}(t)}dt. \tag{16}$$

Note that (16) provides the average transmitted MSP for a TDMA system.

For the MIMO case, the MSP is achieved by applying the single-user water-filling algorithm to each user separately for each channel realization. Suppose one user has its channel matrix with rank r and singular values $\{d_1, d_2, \ldots, d_r\}$. The minimum transmitted power is achieved by

$$p=\sum\nolimits_{i=1}^{r}\left[p^*-1/d_i^2\right]^+ \tag{17a}$$

where $[x]^+ = \max\{x, 0\}$ and p^* satisfies

$$\sum\nolimits_{i=1}^{r}\log_2\left(1+d_i^2\left[p^*-1/d_i^2\right]^+\right)=R. \tag{17b}$$

Then $P_{N\times M}(1, R)$ can be obtained by taking the average of p over the channel distribution. The details are omitted here for brevity. For a large R, p^* in (17) is also large such that (17a) reduces to

$$p=\sum\nolimits_{i=1}^{r}\left(p^*-1/d_i^2\right) \tag{18a}$$

where p^* satisfies

$$\sum\nolimits_{i=1}^{r}\log_2\left(d_i^2p^*\right)=R. \tag{18b}$$

Thus we have

$$\lim_{R\to\infty}\frac{d(p)_{\mathrm{dB}}}{dR}=\lim_{R\to\infty}\frac{d(p)_{\mathrm{dB}}}{dp^*}\cdot\frac{dp^*}{dR}=\frac{10\log_{10}2}{r}. \tag{19}$$

It can be shown that for most practical distributions, the channel matrices have full rank $\min(M, N)$ with probability 1, which indicates that

$$\lim_{R\to\infty}\frac{d\left(P_{N\times M}(1,R)\right)_{\mathrm{dB}}}{dR}=\frac{10\log_{10}2}{\min(M,N)}. \tag{20}$$

APPENDIX II: PROOFS OF PROPOSITIONS

This appendix gives the proofs of Propositions 1, 3 and 4.

Proof of Proposition 1: The monotonicity of the MSP can be seen as follows: For a K-user system, we divide the total transmission time during each channel realization into K equal-length slots. We take K-1 users out of K each time and form K different combinations. Let each combination of K-1 users transmit in a distinguished slot in an optimal way. The MSP of such a scheme is clearly equal to that of a (K-1)-user system with optimal transmission (after average over all possible channel realizations). Thus $P_{N\times M}(K, R) \le P_{N\times M}(K\text{-}1, R)$. □

Proof of Proposition 3: Define a new multiple access system as

$$\tilde{\boldsymbol{y}} = \sum\nolimits_{k=1}^{K} d_{k,\max} \boldsymbol{I}_M \cdot \tilde{\boldsymbol{x}}_k + \boldsymbol{n} \tag{21}$$

where $\boldsymbol{I}_M$ is an $M\times M$ identity matrix and $d_{k,\max}$ is the maximum singular value of $\boldsymbol{H}_k$. In such a system, each user k sees M parallel sub-channels with equal gain, one for each receive antenna. For each channel realization $\{\boldsymbol{H}_k\}$, assume that $\{\boldsymbol{x}_k\}$ achieves the MSP to support the target rates for (1). Let

$$\tilde{\boldsymbol{x}}_k = \boldsymbol{H}_k / d_{k,\max} \cdot \boldsymbol{x}_k \,, \tag{22}$$

then the same target rates can be supported in the system (21) (which can be verified by substituting (22) into (21).) with equal or less power since

$$\| \tilde{\boldsymbol{x}}_k \|_2 \le \frac{1}{d_{k,\max}} \| \boldsymbol{H}_k \|_2 \cdot \| \boldsymbol{x}_k \|_2 = \| \boldsymbol{x}_k \|_2 \tag{23}$$

where $||\cdot||_2$ denotes the 2-norm. Hence the MSP of (21) is a lower bound for that of (1), as shown in Fig. 3 for $K = 8$. Furthermore, the MSP of (21) can be achieved with every user transmitting at rate R/MK in every sub-channel (so the sum-rate for each sub-channel is R/M). When $K\to\infty$, the asymptotic MSP of each sub-channel can be computed using (6) with R and $F_{1\times 1}(t)$ replaced by R/M and $F_{N\times M}(t)$, respectively. The asymptotic MSP of the overall system (21) is then M times that of each sub-channel, which leads to (11). □

Proof of Proposition 4: Similarly to the MEB approach discussed in Section III.B, let each user k transmit in the direction of $\boldsymbol{v}_{k,\max}$ and allocate power to user k using the following rule:

$$p_k = \left(2^{R/K} - 1\right)\left(1 + (2^{R/K} - 1)/M\right)^{k-1} \left(1+\delta\right)/ d_{k,\max}^2 \tag{24}$$

where δ is a positive constant. Let ε ($0 < \varepsilon < 1$) be another positive constant. When $K\to\infty$ (and so $\varepsilon K\to\infty$), $d_{k,\max}^2 = F_{N\times M}^{-1}(k/K)$, and for any $k \ge \varepsilon K$, (24) becomes

$$\begin{aligned}
\lim_{K\to\infty} p_k &= \lim_{K\to\infty} \frac{K\left(2^{R/K} - 1\right)\left(1 + (2^{R/K} - 1)/M\right)^{k-1}\left(1+\delta\right)}{F_{N\times M}^{-1}(k/K)} \cdot \frac{1}{K} \\
&= \lim_{K\to\infty} \frac{\left(1+\delta\right) 2^{R(k-1)/MK} R \ln 2}{F_{N\times M}^{-1}(k/K)} \cdot \frac{1}{K} \\
&= \lim_{K\to\infty} g(k/K)/K
\end{aligned} \tag{25}$$

where $g(t) = R\ln 2 \cdot \left(1+\delta\right) 2^{Rt/M} / F_{N\times M}^{-1}(t)$ is a continuous function.

We now prove by induction on k that SIC can be asymptotically successfully applied to the above system when $K\to\infty$. Suppose that the signals of {user i: $i > k$} have been successfully decoded and removed by SIC. Note that in (9), $\phi_{k,i} = |\boldsymbol{u}_k^H \boldsymbol{u}_i|^2$ ($k \ne i$) is the correlation of two independent unit vectors with $\mathrm{E}[|\boldsymbol{u}_k^H \boldsymbol{u}_i|^2] = 1/M$, and $\{p_i\}$ can be rewritten into a continuous form based on (25). When $K\to\infty$ and $k \ge \varepsilon K$, according to the law of large numbers, (9) can be rewritten as

$$\begin{aligned}
\lim_{K\to\infty} SNR_k &= \frac{\left(2^{R/K} - 1\right)\left(1 + (2^{R/K} - 1)/M\right)^{k-1}\left(1+\delta\right)}{1 + \sum_{i=1}^{k-1} \left(2^{R/K} - 1\right)\left(1 + (2^{R/K} - 1)/M\right)^{i-1}\left(1+\delta\right) \frac{1}{M}} \\
&\ge \frac{\left(2^{R/K} - 1\right)\left(1 + (2^{R/K} - 1)/M\right)^{k-1}}{1 + \frac{2^{R/K} - 1}{M} \sum_{i=1}^{k-1} \left(1 + (2^{R/K} - 1)/M\right)^{i-1}} \\
&= 2^{R/K} - 1 \,.
\end{aligned} \tag{26}$$

Let the users with indexes $k < \varepsilon K$ not transmit any information (i.e., they are in outage and the outage probability now is ε). When $K\to\infty$ and with probability 1, the power allocation scheme (24) can support rate R/K for {user k: $k \ge \varepsilon K$} with sum power

$$\begin{aligned}
\lim_{K\to\infty} \sum_{k=\varepsilon K}^{K} p_k &= \lim_{K\to\infty} \sum_{k=\varepsilon K}^{K} \frac{\left(1+\delta\right) 2^{R(k-1)/MK} R \ln 2}{F_{N\times M}^{-1}(k/K)} \cdot \frac{1}{K} \\
&= (1+\delta) R \ln 2 \cdot \int_{\varepsilon}^{1} \frac{2^{Rt/M}}{F_{N\times M}^{-1}(t)} dt \,.
\end{aligned} \tag{27}$$

The sum-rate of the active users is (1-ε)R. Since both δ and ε can be arbitrarily small, the limit of (27) is an upper bound of $P_{N\times M}(\infty, R)$ with an arbitrarily small outage probability ε. □

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