Performance analysis of superposition coded modulation

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A B S T R A C T
This paper presents a comprehensive analysis of superposition coded modulation (SCM). Two types of SCM schemes, i.e., the single-code SCM (SC-SCM) and multi-code SCM (MC-SCM), are analyzed. The basic features of SCM are described, followed by the information-theoretic analysis. Different encoding/decoding strategies are compared from the capacity point of view. A semi-analytical evolution technique is proposed to track the convergence behavior of iterative decoding. Analytical error-rate analysis is then conducted to predict the asymptotic performance. Numerous examples demonstrate that the analysis tools discussed in this paper can provide reasonably accurate performance prediction for SCM schemes.

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1. Introduction

Superposition coded modulation (SCM) is a high-rate transmission scheme in which the transmit signal $x_i$ is a weighted sum of $K$ coded symbols $\{x_{i,1}, x_{i,2}, \ldots, x_{i,K}\}$, i.e.,

$$x_i = \sum_{k=1}^{K} \beta_k x_{i,k},$$ 

(1)

where each $x_{i,k}$ is drawn from a binary phase-shift keying (BPSK) constellation [1–8] and $\{\beta_k\}$ are a set of weighting constants. We will call $\{x_{i,k} : i = 1, 2, \ldots\}$ collectively as the $k$th layer. A $K$-layer SCM is defined over a size-$2^K$ constellation.

A distinctive property of SCM is that it can be formed by naturally superimposing the signals from different users/antennas/relay nodes. This property is very useful for multi-user, multi-antenna and ad hoc environments. For this reason, SCM has been frequently used as a theoretical concept in the derivation of the channel capacity in such environments.

However, inter-layer interference constitutes a major obstacle in practice for SCM. Theoretically, such interference can be resolved through successive interference cancelation (SIC) [9,10], but SIC is effective only when ideal codes are involved and it may lead to considerable performance degradation when non-ideal codes are involved. For this reason, compared with other schemes such as trellis coded modulation (TCM) and bit-interleaved coded modulation (BICM), SCM has mostly remained as a theoretical concept in the past.

With the advent of turbo and low-density-parity-check (LDPC) codes, low-cost iterative detection techniques have been studied for SCM [3,5,8]. In particular, the layer-by-layer detector outlined in [3,8] (based on the detector for interleave-division multiple-access (IDMA) [11]) has complexity $O(K)$ per symbol for a $K$-layer SCM. As a comparison, the detection complexity is $O(2^K)$ per symbol for a TCM or BICM scheme defined over a $2^K$-ary constellation. Thus, SCM can offer an attractive performance-complexity tradeoff, especially for very high-rate applications.

Most existing work on SCM so far relies on Monte Carlo simulations for performance evaluation. In this paper, we will conduct a comprehensive performance analysis for
SCM. We broadly classify SCM into two classes: single-code SCM (SC-SCM) and multi-code SCM (MC-SCM), as shown in Fig. 1. With SC-SCM, the bits in different layers are generated using a single encoder, which can be seen as a special BICM scheme [12–14] over the constellation formed by the superposition operation in (1). With MC-SCM, the bits in different layers belong to different codewords, which can be seen as a special multi-level coding (MLC) scheme [15,16]. For simplicity, we will assume that the $K$ encoders in Fig. 1(b) are all based on the same forward error control (FEC) code.

We will analyze both SC-SCM and MC-SCM using three different approaches based on, respectively, capacity, signal-to-noise-ratio (SNR) evolution, and error bounds. These approaches are useful in different aspects: capacity analysis establishes theoretical limits; evolution analysis is a convenient tool in fast assessing the performance of iterative receivers; and error bounds sketch the error floor behavior. We will compare SC-SCM and MC-SCM using these different approaches. We will show that MC-SCM and SC-SCM have different advantages when practical codes are concerned. For example, SC-SCM may have a relatively lower error floor, but MC-SCM may achieve a faster convergence speed. We will use various examples to demonstrate that the performance of SCM can be predicted using the analysis tools outlined in this paper with reasonable accuracy.

2. System model and capacity

This section examines the performance limits of SCM with different sub-optimal encoding/decoding strategies. We will compare SC-SCM and MC-SCM from the viewpoint of capacity. We will also analyze the performance loss with a sub-optimal SIC receiver using the tools developed in [17] for mismatched decoding problems.

2.1. SC-SCM and MC-SCM

Consider a coded modulation system employing binary component codes. The information bits are encoded and interleaved. The resultant bits are packed into binary $K$-tuples \( \{c_1, c_2, \ldots, c_K\}\) with each $c_{i,k} \in \{0, 1\}$. With SCM [3], each $c_{i,k}$ is mapped to a BPSK symbol $x_{i,k}$ and then superposition is used to generate the transmit symbols as defined in (1). We distinguish the following two types of SCM schemes as illustrated in Fig. 1:

- **SC-SCM**: \( \{c_{i,k}, \forall k\} \) are from a common encoder, as in BICM [13,14].
- **MC-SCM**: \( \{c_{i,k}, \forall k\} \) are from $K$ independent encoders, as in MLC [15,16].

The parameters $\{\beta_k\}$ in (1) are crucial design parameters that determine the resultant constellation and performance. A special choice is setting $\{\beta_k\} = \{1, 2, 4, \ldots\}$, which results in the conventional $2^k$-ary PAM constellations with equispaced signal points.

For simplicity, we will focus on real-valued $\{\beta_k\}$ and an additive white Gaussian noise (AWGN) channel model

$$y_i = x_i + w_i$$

in this section. The results can be easily extended to more general cases.

2.2. Single-stage decoding

With single-stage decoding, we first produce a soft estimate $\lambda_{i,k}$ for every $x_{i,k}$, $\forall i, \forall k$, based on observation $y_i$ in (2). The results are delivered to the decoder to recover the information bits. In this way, each bit position $k$ can be treated as a sub-channel with binary input $c_{i,k}$ and output $\lambda_{i,k}$. There are $K$ such parallel sub-channels in total. The capacity of this system is

$$C = \sum_{k=1}^{K} I(y_i; c_{i,k}) = K \left( -\sum_{k=1}^{K} E_{y_i,c_{i,k}} \left[ \log \frac{\sum_{s \in X_{i,k}} p(Y_i|s)}{\sum_{s \in X_{i,k}} p(Y_i|s)} \right] \right),$$

where $I(\cdot; \cdot)$ denotes mutual information, $E_{y_i,c_{i,k}}[\cdot]$ denotes the expectation with respect to (w.r.t.) the joint distribution of $y_i$ and $c_{i,k}$, $Y_i$ and $C_{i,k}$ are, respectively, realizations of $y_i$ and $c_{i,k}$, $X$ the transmit signal constellation, and $X_{i,k}$ the subset of the signal points in $X$ whose $k$th label bit has value $c_{i,k}$. The above is similar to the capacity
analysis for BICM. It is known that such an approach cannot achieve the constrained capacity related to the transmit constellation. Note that (3) can be applied to both SC-SCM and MC-SCM.

2.3. MC-SCM with optimal successive interference cancelation (SIC)

For MC-SCM, we adopt the SIC decoding strategy. Assume that \( \{\beta_k\} \) are ordered such that \( 0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_K \). We first decode layer-\( K \) by treating all the signals from layers-1 to \( K - 1 \) as additive noises. After successfully decoding layer-\( K \), we subtract the results from the received signal and decode layer-(\( K - 1 \)) by treating all the signals from layers-1 to \( K - 2 \) as additive noises. This procedure continues until layer-1 is decoded.

It is well known that the SIC strategy can achieve the constrained capacity provided that the decoder for each individual layer is locally optimal w.r.t. the distribution of the distortion seen by this decoder.

2.4. MC-SCM with sub-optimal SIC

Next we consider a sub-optimal SIC approach for MC-SCM. We approximate the distortion component seen by the decoder for layer-\( K \) as an additive Gaussian noise. This is not true since the distortion term (after stripping off layers-(\( k + 1 \)) to \( K \)) is actually given in the following form

\[
\tilde{w}_{i,k} = \sum_{m=1}^{k-1} \beta_m (-1)^{c_{i,m}} + w_i, \quad i = 1, 2, \ldots,
\]

where \( \{c_{i,m}\} \) are binary. The decoder for layer-\( k \) based on this Gaussian approximation (GA) is no longer optimal. The performance limit of such an approach can be computed using the technique discussed in [17]. It is easy to verify that the channel seen for layer-\( k \) is a binary-input symmetric channel (BSC). Then the maximum achievable rate for layer-\( k \) with GA is given by [17]

\[
C^A_k = \max_{\alpha, \beta_k \geq 0} \sum_{c_{i,k} \in \{0, 1\}} \int_{-\infty}^{+\infty} \Pr(c_{i,k}) \Pr(\tilde{y}_{i,k} | c_{i,k}) \times \log\left( \frac{e^{-\alpha d(\tilde{y}_{i,k}, c_{i,k})}}{\sum_{z \in \{0, 1\}} \Pr(z) e^{-\alpha d(\tilde{y}_{i,k}, z)}} \right) d\tilde{y}_{i,k},
\]

where \( \Pr(c_{i,k}) = 1/2, \forall c_{i,k} \in \{0, 1\}, \alpha \) is a constant (obtained through optimization), \( \tilde{y}_{i,k} = \beta_k (-1)^{c_{i,k}} + \tilde{w}_{i,k} \) and \( d(\tilde{y}_{i,k}, c_{i,k}) = |\tilde{y}_{i,k} - \beta_k (-1)^{c_{i,k}}|^2 \). Assuming that the rate for each layer is fixed, we can compute the minimum required power based on (5).

2.5. Examples

In Fig. 2, we show the capacities achieved by SC-SCM and MC-SCM with \( K = 2 \) in an AWGN channel. The single-stage decoder is used for SC-SCM and the optimal SIC decoder is used for MC-SCM. We assume the same weighting factors \( \{\beta_k\} = \{1, 3/2\} \) for the two schemes. The Shannon limit is also included for reference. From Fig. 2, MC-SCM outperforms SC-SCM when the same SCM signaling is used. Fig. 3 shows the maximum achievable rate of layer-2 with the sub-optimal SIC decoder in MC-SCM with \( K = 2 \) and different \( \{\beta_k\} \). The capacity achieved by the optimal SIC decoder is also included. (Note that when decoding layer-1, only the AWGN is present and so the two SIC methods yield the same performance.) From Fig. 3, the loss in achievable rates with the sub-optimal SIC method depends on the values of \( \{\beta_k\} \). If \( \rho = \beta_2/\beta_1 \) is large, the loss is marginal, whereas if \( \rho \) is small, the loss becomes noticeable. Intuitively, this may be explained by that the interference-plus-noise \( \tilde{w}_{i,k} \) seen by layer-2 is “more Gaussian-like” when \( \beta_1 \) is smaller and so the GA becomes “more accurate” in this case.

3. Performance with iterative decoding

In this section, we will briefly outline the iterative receiver principles for SCM. We will then develop an SNR evolution technique to predict the performance of SCM receivers. Gaussian approximation is the key to the discussion in this section, which not only greatly reduces the operation complexity, but also greatly simplifies the analysis problem involved.

For MC-SCM over AWGN channels, the SNR evolution technique developed for IDMA [18] can be directly used. However, new treatments are required for more general cases such as SC-SCM over fading channels, as developed below. Again, we consider real-valued SCM. We also assume infinite interleaving lengths, following the extrinsic information transfer (EXIT) chart method [19].

3.1. Iterative receiver

Consider a memoryless, fading channel model

\[
y_i = h_i x_i + w_i, \quad i = 1, 2, \ldots,
\]

where \( x_i = \sum_{k=1}^{K} \beta_k (-1)^{c_{i,k}} \) \{\( h_i \)\} are the independent, identically distributed (i.i.d.) channel coefficients perfectly known at the receiver and \{\( w_i \)\} the AWGN with variance \( \sigma^2 = N_0/2 \) per dimension. We consider a sub-optimal receiver consisting of a channel de-mapper and a bank of decoder(s). In the first detection iteration, the de-mapper first generates the log-likelihood ratios (LLRs) of \( \{c_{i,k}\} \) from the channel outputs (6). Then the decoder performs a posteriori probability (APP) decoding based on the de-mapper outputs and constraints of the component codes. In the subsequent iterations, the decoder outputs are feedback to the de-mapper and used as a priori information to refine the de-mapping performance. This follows the celebrated “turbo” principle [20]. Since APP decoding is a standard function, we focus on de-mapping below.

3.2. GA de-mapping

Assume that an a priori LLR (obtained from the decoder) for coded bit \( c_{i,k} \) is available:

\[
\gamma_{i,k} = \ln\left( \frac{\Pr(c_{i,k} = 0)}{\Pr(c_{i,k} = 1)} \right), \quad \forall k, \forall i.
\]
The optimal APP de-mapping method \[14\] may involve excessive complexity \(O(2^K)\). Thus, we consider the following GA method (with complexity \(O(K)\)) for SCM. The key is to model (6) as a binary-input system with distortion \(\zeta_{i,k}\):

\[y_i = h_i \beta_k (-1)^{c_{i,k}} + \zeta_{i,k},\]

where

\[\zeta_{i,k} = h_i \sum_{m \neq k} \beta_m (-1)^{c_{i,m}} + w_i\]

is the interference-plus-noise w.r.t. \(c_{i,k}\). We approximately treat \(\zeta_{i,k}\) as a Gaussian variable. The de-mapper output is then computed as

\[\lambda_{i,k} = \ln \left( \frac{2 \pi V[\zeta_{i,k}]}{(2 \pi V[\zeta_{i,k}])^{1/2}} \exp \left( -\frac{(y_i - h_i \beta_k - E[\zeta_{i,k}])^2}{2V[\zeta_{i,k}]} \right) \right)\]

\[= 2 \beta_k h_i y_i - E[\zeta_{i,k}] V[\zeta_{i,k}],\]

where \(E[\zeta_{i,k}]\) and \(V[\zeta_{i,k}]\), respectively, denote the mean and variance of \(\zeta_{i,k}\), which can be found from the a priori LLRs \(\gamma_{i,k}\) \[8\].

3.3. EXIT function for the decoder

Consider an APP decoder with input LLRs \(\lambda_{i}\) and output LLRs \(\gamma_{i}\). Following the common treatment in \[19\], we model \(\lambda_{i}\) and \(\gamma_{i}\) as i.i.d. random variables with consistent, symmetric, Gaussian distributions. Then, we can characterize the decoder inputs by the mutual information \(I_{\lambda}\) between \(\lambda_{i}\) and the coded bits \(c_{i}\). Tosimplify the discussion in Section 3.4, we characterize the decoder outputs by \(\sigma_{c}^2\), the average of the conditional variance of \((-1)^{c_{i}}\) given \(\gamma_{i}\):

\[\sigma_{c}^2 = \mathbb{E}[V((-1)^{c_{i}})] = \mathbb{E}[1 - \tanh^2(\gamma_{i}/2)].\]

where \(\mathbb{E}[-\cdot]\) denotes expectation w.r.t. the distribution of \(\gamma_{i}\). (Note that with the above modeling, \(\sigma_{c}^2\) has a one-to-one correspondence with the distribution of \(\gamma_{i}\).) Then, we can characterize the input–output relationship of the decoder by an EXIT function

\[\sigma_{c}^2 = T_{DEC}(l_{i}).\]

The BER can be treated as a function of \(\sigma_{c}^2\) as

\[\text{BER} = g(\sigma_{c}^2).\]
We can obtain $T_\text{DEC} (\cdot)$ and $g (\cdot)$ by simply applying Monte Carlo simulation to a BPSK scheme over an AWGN channel.

### 3.4. EXIT function for an SC-SCM de-mapper

We focus on SC-SCM in this subsection. From (10),

$$\lambda_{i,k} = \frac{2 \beta_k h_i}{\mathbb{V}[\xi_{i,k}]} h_i \beta_k (-1)^{\xi_{i,k}} + h_i v_{i,k} + w_i,$$

where

$$v_{i,k} = \sum_{m \neq k} \beta_m ((-1)^{\xi_{i,m}} - \mathbb{E}(-1)^{\xi_{i,m}})$$

represents the inter-layer interference in $\lambda_{i,k}$. To simplify the problem, we will model $v_{i,k}$ as a zero-mean Gaussian random variable in what follows. Due to the overall interleaving in SC-SCM, we have

$$\sigma^2_{c_k} = \sigma^2_c, \quad \forall k,$$

i.e., the bit variance (feedback from the decoder) is independent of the layer index. Now, $v_{i,k}$ can be characterized by a zero-mean Gaussian interference with average power

$$\sigma^2_v = \sum_{m \neq k} \beta^2_m \sigma^2_{c_m},$$

From (14), given the distribution of the channel coefficient $h_i$, the de-mapping performance is jointly determined by the interference $v_{i,k}$ and channel noise $w_i$. For general channels (except the AWGN channel), $v_{i,k}$ and $w_i$ have different impact on performance as $v_{i,k}$ is an interference. To characterize the relative power of $v_{i,k}$ and $w_i$, we define the signal-to-interference ratio (SIR) and signal-to-noise ratio (SNR) w.r.t. layer $k$ as

$$\text{sir}_k = \frac{\beta^2_k}{\sigma^2_{c_k}}, \quad \text{snr}_k = \frac{\beta^2_k}{\sigma^2}.$$

Suppose that the distribution of $\{h_i\}$ is given and fixed. Then the average mutual information between $\lambda_{i,k}$ and $c_{i,k}$ is determined by $\text{sir}_k$ and $\text{snr}_k$:

$$l_{i,k} = I(\lambda_{i,k}; c_{i,k}) = \mathcal{T} (\text{sir}_k, \text{snr}_k),$$

where $\mathcal{T} (\cdot, \cdot)$ is a two-variable function to characterize the relationship between $l_{i,k}$ and $(\text{sir}_k, \text{snr}_k)$. To evaluate $\mathcal{T} (\cdot, \cdot)$, we adopt a binary-input testing channel characterized by the transition probability

$$p(y|x, h) = \frac{1}{\sqrt{2\pi (\text{sir}_k + \text{snr}_k)}} \exp \left(- \frac{|y - hx|^2}{2(\text{sir}_k + \text{snr}_k)} \right)$$

Then we can apply the Monte Carlo method to find $\mathcal{T} (\cdot, \cdot)$.

With SC-SCM, the de-mapper outputs are de-interleaved before input to the decoder. Assume that a random interleaver is applied. Then the decoding performance can be approximately characterized by the average mutual information contained in the decoder inputs, i.e.,

$$l_k = \frac{1}{K} \sum_{i,k=1}^K l_{i,k}.$$  

### 3.5. Evolution analysis for SC-SCM

We assume that the three functions $\mathcal{T} (\cdot, \cdot)$, $T_\text{DEC} (\cdot)$ and $g (\cdot)$ are known. (They all can be obtained by simulating a binary-input system.) Then the iterative decoding performance of SC-SCM can be tracked as follows:

**Initialization**: Set $\sigma^2_c = 1$.

**Recursion**: Update $\sigma^2_{c}$ as

$$\sigma^2_{c} = T_\text{DEC} \left( \frac{1}{K} \sum_{k=1}^K \mathcal{T} \left( \sum_{m \neq k} \beta^2_m \sigma^2_{c_m}, \frac{\beta^2_k}{\sigma^2} \right) \right).$$

**Termination**: After a preset number of recursions, estimate the BER by substituting the final value of $T_\text{DEC}$ into (13).

The above semi-analytical method has much lower complexity than Monte Carlo simulation in evaluating the overall performance. The conversion of SIRs and SNRs to mutual information in (19) greatly simplifies the problem. If $\mathcal{T} (\cdot, \cdot)$ is pre-calculated and stored in a look-up table, then the evolution method can be used to quickly predict the performance for SC-SCM with arbitrary parameters $\{\beta_k\}$. Alternatively, we can apply the conventional EXIT chart technique but the complexity can be high. This is because, with the conventional EXIT method, we need to apply the Monte Carlo method to generate the EXIT functions of the de-mapper for every set of $\{\beta_k\}$.

Note that, similar to the conventional EXIT chart technique, the above evolution method can only approximately predict the performance. The reasons are as follows. First, interleavers with infinite length are assumed, which is optimistic for performance. Second, the Gaussian assumption of the interference may be pessimistic.

### 3.6. EXIT analysis for MC-SCM

MC-SCM differs from SC-SCM in that each layer is independently encoded, as shown in Fig. 1. Different layers are protected unequally if unequal power allocations $\{|\beta_k|^2\}$ are used, implying that multiple variables need to be tracked. In this case, due to the unequal protection,

$$\sigma^2_{c_k} \neq \sigma^2_{c_m}, \quad \text{if } |\beta_k| \neq |\beta_m|.$$  

Given $\{\sigma^2_{c_k}\}$, the output mutual information of the de-mapper can be found using (18) and (19) in the same way as in SC-SCM. The key is to evaluate $\{\sigma^2_{c_k}\}$ separately for the different layers that are independently decoded:

$$\sigma^2_{c_k} = T_\text{DEC} (l_{i,k}).$$

The transfer functions involved can be obtained using the Monte Carlo method. The evolution process is similar to that for SC-SCM and the details are omitted here.

### 3.7. Examples

Fig. 4 compares the simulation and evolution results for SC- and MC-SCM schemes with $K = 8$ and equal power allocations $\{|\beta_k|^2\}$ over AWGN channels. (Examples of MC-SCM over fading channels will be presented later in Fig. 7.) The component code is the concatenation of a rate-1/2 convolutional code (23, 35)B with a rate-1/4 repetition code. System rate is $R = 1$ bit/dim. The information block length is 8192 for SC-SCM and 1024 per
layer for MC-SCM such that the frame length is the same in the two schemes. We can see that the evolution results are quite accurate for both SC-SCM and MC-SCM.

As the different layers in MC-SCM correspond to independent codewords, MC-SCM is decoded in a layer-by-layer manner. By contrast, the different layers in SC-SCM can only be decoded simultaneously since they are from one common codeword. From Fig. 4, we can see that MC-SCM has quicker convergence of iterative decoding (in terms of number of iterations) than SC-SCM, implying that MC-SCM may reduce the computational complexity.

4. Error-rate analysis

Capacity analysis is important in establishing the performance limits. However, in many cases, error bounding analysis provides more insights into performance with practical codes. In this section, we outline some simple analytical methods to predict the asymptotic performance. We also discuss the factors that affect the error floor behavior using these methods.

4.1. Union bounds for SC-SCM

We view SC-SCM as a special case of BICM. Thus the union bound [13] and error floor (EF) bound [14] derived for BICM can be applied. Let the underlying binary code have rate $k_c/n_c$ and free Hamming distance $d_H$. Following [13], the union bound on BER for SC-SCM writes

$$P_b = \frac{1}{k_c} \sum_{d=d_H}^{\infty} W_i(d) f(d, \beta),$$

where $W_i(d)$ is the total information weight of all error events at Hamming distance $d$; $\beta = [\beta_1, \beta_2, \ldots, \beta_K]$; $f(d, \beta)$ denotes the pairwise error probability (PEP):

$$f(d, \beta) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \psi_{ab}(s, \beta) \right|^2 ds;$$

$\delta$ is a positive real number in the convergence region of

$$\psi_{ab}(s, \beta) = \frac{1}{K^2} \sum_{k=1}^{K} \sum_{b=0}^{1} \sum_{x \in X_k^b} \Phi_{\Delta(x, z)}(s);$$

where $\Phi_{\Delta(x, z)}(s)$ is the Laplace transform of the probability density function (p.d.f.) of the metric difference

$$\Delta(x, z) = \ln(p(y_i|x)) - \ln(p(y_i|z))$$

with $y_i$ being the received signal. The computational details for (26) can be found in [21].

With iterative decoding, the decoder feedback is used as the a priori information for the de-mapper. From [14], the asymptotic error floor performance is achieved when the decoder feedback is perfect. It can be lower bounded by expurgating the irrelevant error events in (27) as

$$\psi_{ef}(s, \beta) = \frac{1}{K^2} \sum_{k=1}^{K} \sum_{b=0}^{1} \sum_{x \in X_k^b} \Phi_{\Delta(x, z)}(s),$$

where $\tilde{z}$ is the signal point whose label bits differ from those of $x$ only at the $k$th bit position.

The simulation results and error bounds for an SC-SCM scheme with and without iterative decoding are demonstrated in Fig. 5. The 4-state convolutional code $(5, 7)_8$ is used as the component code. $K = 2, R = 1$ bit/dim. It is shown that both the union bound and error floor bound are tight for high SNRs.

4.2. Approximate error floor analysis for MC-SCM

For MC-SCM, we adopt an approximation method based on the assumption that the iterative receiver has converged so that the inter-layer interference is negligible and the error rate of each single layer is dominated by the additive channel noise. With this view, the error rate for layer-$k$ is given by

$$p_{b}^{(k)} = \frac{1}{k_c} \sum_{d=d_H}^{\infty} W_i(d) f(d, \beta_k).$$
where \( f ( d, \beta_k) \) is the PEP for the coded BPSK system with input alphabet \(+\beta_k, -\beta_k\)

\[
f ( d, \beta_k) \leq \frac{1}{2\pi j} \int_{d-\infty}^{d+\infty} \left[ \Phi_{\Delta(\pm\beta_k, -\beta_k)} (s) \right]^d ds.
\] (31)

The overall BER of MC-SCM is approximated as

\[
P_b \approx \frac{1}{K} \sum_{k=1}^{K} P_b^{(k)}.
\] (32)

Note that (32) only gives an approximation but not a bound of the BER with iterative decoding. This is because the interference-free assumption underestimates the BER while the bounding in (31) overestimates the BER. However, as shown later in Fig. 7, (32) can provide a very accurate prediction of BER at high SNRs.

4.3. Discussions

We now consider the high-SNR regime and examine the factors that affect the asymptotic performance.

4.3.1. SC-SCM versus MC-SCM

With SCM, it is easy to show that

\[
|x - \tilde{x}| = 4|\beta_k|^2, \quad \forall x \in \mathcal{X}_k^b,
\] (33)

where the notations \( x \) and \( \tilde{x} \) follow from (29). Using Chernoff bounds \([13,22]\), the error floor of SC-SCM can be upper bounded by

\[
P_{b, \text{Che}}^{\text{SC-SCM}} = \frac{W_0 (d_H)}{k_c} \left( \frac{\sum_{k=1}^{K} \Phi_{\text{ef}, k} \left( \frac{1}{2N_0} \right)}{K} \right)^{d_H},
\] (34)

where

\[
\Phi_{\text{ef}, k} (s) = \exp \left( 4|\beta_k|^2 s (N_0 s - 1) \right)
\] (35)

for AWGN channels and

\[
\Phi_{\text{ef}, k} (s) = \frac{1}{1 + 4|\beta_k|^2 s (1 - N_0 s)}
\] (36)

for Rayleigh fading channels \([13]\).

Similarly, for MC-SCM, the Chernoff bound can be obtained from (30) and (32) as

\[
P_{b, \text{Che}}^{\text{MC-SCM}} \leq P_{b, \text{Che}}^{\text{MC-SCM}} = \frac{W_0 (d_H)}{k_c} \left( \frac{\sum_{k=1}^{K} \Phi_{\text{ef}, k} \left( \frac{1}{2N_0} \right)}{K} \right)^{d_H}.
\] (37)

By Jensen’s inequality, it can be shown that for both AWGN and Rayleigh fading channels

\[
P_{b, \text{Che}}^{\text{SC-SCM}} \leq P_{b, \text{Che}}^{\text{MC-SCM}},
\] (38)

where the equality holds when \(|\beta_k|\) are equal. This implies that SC-SCM may lead to a lower error floor than MC-SCM. This is illustrated in Fig. 6 where the convergence behavior of iterative decoding is also compared. Rayleigh fading channels are assumed. The component code is the concatenation of the convolutional code \((5,7)\) and the rate-1/4 repetition code. \( R = 1 \) bit/dim. The information block length is set to 16384 for SC-SCM and 2048 per layer for MC-SCM such that the two schemes have the same latency. We can see that SC-SCM achieves a lower error floor but it leads to a slower convergence of iterative decoding.

4.3.2. Component codes

For coded modulation systems, the minimum Hamming distance \( d_H \) of the component code serves as the maximum diversity order in fading channels. A simple approach to increasing \( d_H \) is to employ an extra repetition code. The increase of \( d_H \) is proportional to the length of the repetition code. In this approach, the rate of the component code is decreased. In order to maintain a given system rate, a larger constellation (with a larger \( K \)) should be used. This can increase the de-mapping complexity exponentially in conventional coded modulation schemes. However, with SCM and the GA method, the de-mapping complexity increases only linearly w.r.t. \( K \).

4.3.3. Weighting factors

Given the component code, the performance of SCM can be further optimized through adjusting the weighting factors. Let us take SC-SCM over AWGN channels as an example. From (34) and (35), the Chernoff bound on error

![Fig. 5. Error bounds for an SC-SCM scheme with \(|\beta_k| = \{1, 2\} \) over AWGN channels. \( R = 1 \) bit/dim.](image-url)
Fig. 6. Simulation results for SC-SCM and MC-SCM over fully-interleaved Rayleigh fading channels with $\{\beta_k\} = \{1 \times 6, 1.4 \times 2\}$ and different number of iterations.

Fig. 7. Simulation, evolution, and analytical results for MC-SCM over fully-interleaved Rayleigh fading channels. The above discussions focus on the asymptotic case with ideal decoder feedback. In practice, equal $|\beta_k|$ may lead to poor convergence of iterative decoding when rate is high. Hence, it is more useful to optimize $\{\beta_k\}$ for a given target BER and number of iterations. The evolution method discussed in Section 3 together with some searching methods can be used for this purpose. In high-rate cases, the optimized $\{\beta_k\}$ are usually unequal.

Fig. 7 compares the simulation results and the analytical results based on (32) for MC-SCM schemes with optimized weighting factors over fading channels. The results predicted by the evolution method outlined in Section 3 are also included. The component code is the same as that in Fig. 6. We consider $R = 1, 1.5$ and 2 bits/dim. The number of layers is $K = 8R$. The weighting factors are listed in Table 1. The number of iterations $I_t = 12$ for all curves. The information block length is 1024 per layer. We can observe good agreements among the performance results obtained using different methods.

5. Conclusions

We have presented a comprehensive study of two types of SCM schemes, namely, SC-SCM and MC-SCM. The basic features of SCM are reviewed. The information-theoretic
analysis shows that MC-SCM outperforms SC-SCM in terms of capacity. It also shows that the performance loss caused by sub-optimal SIC decoding is not serious for SCM with proper power allocations. The evolution analysis and error bounds, respectively, can be used to characterize the iterative decoding convergence and error floor behavior. Numerical examples demonstrate that the analysis methods developed in this paper can provide reasonably accurate prediction of the performance of SCM.

In this paper, we have focused on AWGN and fully-interleaved single-path fading channels where the channel states are fixed. In block-fading channels, the channel states change block-by-block and the overall performance cannot be directly predicted by the analysis methods presented here. However, in single-path cases, this can be tackled by applying those methods repeatedly to the different samples of channel states and then taking average. The extension is straightforward and the details are omitted. In multipath cases, a modified evolution method discussed in [23] can provide accurate prediction of the overall performance.

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References